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# Precise asymptotics in the law of the iterated logarithm for $R/S$ statistic

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**Abstract**
Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables which is in the domain of attraction of the normal law with zero mean and possibly infinite variance, $Q(n) = R(n)/S(n)$ be the rescaled range statistic, where $R(n) = \max_{1\leq k\leq n}\{\sum_{j=1}^k (X_j - \bar{X}_n)\} - \min_{1\leq k\leq n}\{\sum_{j=1}^k (X_j - \bar{X}_n)\}$, $S^2(n) = \sum_{j=1}^n (X_j - \bar{X}_n)^2/n$ and $\bar{X}_n = \sum_{j=1}^n X_j/n$. Then two precise asymptotics related to probability convergence for $Q(n)$ statistic are established under some mild conditions in this paper. Moreover, the precise asymptotics related to almost surely convergence for $Q(n)$ statistic is also considered under some mild conditions.

**MSC:** 60F15; 60G50

**Keywords:** domain of attraction of the normal law; law of the iterated logarithm; precise asymptotics; $R/S$ statistic

## 1 Introduction and main results

Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables and set $S_n = \sum_{j=1}^n X_j$ for $n \geq 1$, $\log x = \ln(x \vee e)$ and $\log\log x = \log(\log x)$. Hsu and Robbins [1] and Erdős [2] established the well known complete convergence result: for any $\varepsilon > 0$, $\sum_{n=1}^\infty \mathsf{P}(|S_n| \geq \varepsilon n) < \infty$ if and only if $\mathsf{E}X = 0$ and $\mathsf{E}X^2 < \infty$. Baum and Katz [3] extended this result and proved that, for $1 \leq p < 2$, $\varepsilon > 0$ and $r \geq p$, $\sum_{n=1}^\infty n^{r-2}\mathsf{P}(|S_n| \geq \varepsilon n^{1/p}) < \infty$ holds if and only if $\mathsf{E}X = 0$ and $\mathsf{E}|X|^{rp} < \infty$. Since then, many authors considered various extensions of the results of Hsu-Robbins-Erdős and Baum-Katz. Some of them studied the precise asymptotics of the infinite sums as $\varepsilon \to 0$ (*cf.* Heyde [4], Chen [5] and Spătaru [6]). We note that the above results do not hold for $p = 2$, this is due to the fact that $\mathsf{P}(|S_n| \geq \varepsilon n^{1/2}) \to \mathsf{P}(|N(0,1)| \geq \varepsilon/\mathsf{E}X^2)$ by the central limit theorem when $\mathsf{E}X = 0$, where $N(0,1)$ denotes a standard normal random variable. It should be noted that $\mathsf{P}(|N(0,1)| \geq \varepsilon/\mathsf{E}X^2)$ is irrespective of $n$. However, if $n^{1/2}$ is replaced by some other functions of $n$, the results of precise asymptotics may still hold. For example, by replacing $n^{1/2}$ by $\sqrt{n \log\log n}$, Gut and Spătaru [7] established the following results called the precise asymptotics in the law of the iterated logarithm.

**Theorem A** *Suppose* $\{X, X_n, n \geq 1\}$ *is a sequence of i.i.d. random variables with* $\mathsf{E}X = 0$, $\mathsf{E}X^2 = \sigma^2$ *and* $\mathsf{E}X^2(\log\log|X|)^{1+\delta} < \infty$ *for some* $\delta > 0$, *and let* $a_n = O(\sqrt{n}/(\log\log n)^\gamma)$ *for some* $\gamma > 1/2$. *Then*

$$\lim_{\varepsilon \searrow 1} \sqrt{\varepsilon^2 - 1} \sum_{n=1}^\infty \frac{1}{n} \mathsf{P}\left(|S_n| \geq \varepsilon\sigma\sqrt{2n\log\log n} + a_n\right) = 1.$$

**Theorem B** *Suppose* $\{X, X_n, n \geq 1\}$ *is a sequence of i.i.d. random variables with* $\mathsf{E}X = 0$ *and* $\mathsf{E}X^2 = \sigma^2 < \infty$. *Then*

$$\lim_{\varepsilon \searrow 0} \varepsilon^2 \sum_{n=1}^{\infty} \frac{1}{n \log n} \mathsf{P}\big(|S_n| \geq \varepsilon\sigma\sqrt{n \log\log n}\big) = 1.$$

Of lately, by applying strong approximation method which is different from Gut and Spătaru's, Zhang [8] gave the sufficient and necessary conditions for this kind of results to be held. One of his results is stated as follows.

**Theorem C** *Let* $a > -1$ *and* $b > -1/2$ *and let* $a_n(\varepsilon)$ *be a function of* $\varepsilon$ *such that*

$$a_n(\varepsilon) \log\log n \to \tau \quad \text{as } n \to \infty \text{ and } \varepsilon \searrow \sqrt{a+1}.$$

*Suppose that*

$$\mathsf{E}X = 0, \qquad \mathsf{E}X^2 = \sigma^2 < \infty \quad \text{and} \quad \mathsf{E}X^2\big(\log|X|\big)^a\big(\log\log|X|\big)^{b-1} < \infty \tag{1.1}$$

*and*

$$\mathsf{E}X^2 I\big\{|X| \geq t\big\} = o\big((\log\log t)^{-1}\big) \quad \text{as } t \to \infty. \tag{1.2}$$

*Then*

$$\begin{aligned} &\lim_{\varepsilon \searrow \sqrt{a+1}} \big(\varepsilon^2 - (a+1)\big)^{b+1/2} \sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^b}{n} \mathsf{P}\big(M_n \geq \big(\varepsilon + a_n(\varepsilon)\big)\sqrt{2\sigma^2 n \log\log n}\big) \\ &\quad = 2\sqrt{\frac{1}{\pi(a+1)}} \exp(-2\tau\sqrt{a+1})\Gamma(b+1/2) \end{aligned} \tag{1.3}$$

*and*

$$\begin{aligned} &\lim_{\varepsilon \searrow \sqrt{a+1}} \big(\varepsilon^2 - (a+1)\big)^{b+1/2} \sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^b}{n} \mathsf{P}\big(|S_n| \geq \big(\varepsilon + a_n(\varepsilon)\big)\sqrt{2\sigma^2 n \log\log n}\big) \\ &\quad = \sqrt{\frac{1}{\pi(a+1)}} \exp(-2\tau\sqrt{a+1})\Gamma(b+1/2). \end{aligned} \tag{1.4}$$

*Here* $M_n = \max_{k \leq n} |S_k|$, *and here and in what follows* $\Gamma(\cdot)$ *is a gamma function. Conversely, if either* (1.3) *or* (1.4) *holds for* $a > -1$, $b > -1/2$ *and some* $0 < \sigma < \infty$, *then* (1.1) *holds and*

$$\liminf_{t \to \infty} (\log\log t)\mathsf{E}X^2 I\big\{|X| \geq t\big\} = 0.$$

It is worth mentioning that the precise asymptotics in a Chung-type law of the iterated logarithm, law of logarithm and Chung-type law of logarithm were also considered by Zhang [9], Zhang and Lin [10] and Zhang [11], respectively.

The above-mentioned results are all related to partial sums. This paper is devoted to the study of some precise asymptotics for the rescaled range statistic (or the $R/S$ statistic), defined by $Q(n) = R(n)/S(n)$, where

$$\begin{cases} R(n) = \max_{1\le k\le n}\{\sum_{j=1}^{k}(X_j - \bar{X}_n)\} - \min_{1\le k\le n}\{\sum_{j=1}^{k}(X_j - \bar{X}_n)\}, \\ S^2(n) = \frac{1}{n}\sum_{j=1}^{n}(X_j - \bar{X}_n)^2, \quad \bar{X}_n = \frac{1}{n}\sum_{j=1}^{n} X_j. \end{cases} \tag{1.5}$$

This statistic, introduced by Hurst [12] when he studied hydrology data of the Nile river and reservoir design, plays an important role in testing statistical dependence of a sequence of random variables and has been used in many practical subjects such as hydrology, geophysics and economics, *etc.* Because of the importance of this statistic, some people studied some limit theorems for $R/S$ statistic. Among them, Feller [13] established the limit distribution of $R(n)/\sqrt{n}$ for i.i.d. case, Mandelbrot [14] studied weak convergence of $Q(n)$ for a more general case, while Lin [15–17] and Lin and Lee [18] established the law of the iterated logarithm for $Q(n)$ under various assumptions. Among Lin's results, we notice that Lin [15] proved that

$$\limsup_{n\to\infty}\sqrt{\frac{2}{n\log\log n}}Q(n) = 1 \quad \text{a.s.} \tag{1.6}$$

holds only if $\{X, X_n, n \ge 1\}$ is a sequence of i.i.d. random variables which is in the domain of attraction of the normal law with zero mean.

Recently, based on applying a similar method to the one employed by Gut and Spătaru [7], a result related to the precise asymptotics in the law of the iterated logarithm for $R/S$ statistic was established by Wu and Wen [19], that is, we have the following.

**Theorem D** *Suppose* $\{X, X_n, n \ge 1\}$ *is a sequence of i.i.d. random variables with* $\mathsf{E}X = 0$, $\mathsf{E}X^2 < \infty$. *Then for* $b > -1$,

$$\lim_{\varepsilon\searrow 0}\varepsilon^{2(b+1)}\sum_{n=1}^{\infty}\frac{(\log\log n)^b}{n\log n}\mathsf{P}\big(Q(n) \ge \varepsilon\sqrt{2n\log\log n}\big) = \frac{\mathsf{E}Y^{2(b+1)}}{2^{b+1}(b+1)}. \tag{1.7}$$

*Here and in what follows, we denote* $Y = \sup_{0\le t\le 1} B(t) - \inf_{0\le t\le 1} B(t)$ *and* $B(t)$ *be a standard Brownian bridge.*

It is natural to ask whether there is a similar result for $R/S$ statistic when $\varepsilon$ tends to a constant which is not equal to zero. In the present paper, the positive answer will be partially given under some mild conditions with the help of strong approximation method, and, since $R/S$ statistic is defined in a self-normalized form, we will not restrict the finiteness of the second moment for $\{X, X_n, n \ge 1\}$. Moreover, a more strong result than Wu and Wen's is established in this paper, based on which, a precise asymptotics related to a.s. convergence for $Q(n)$ statistic is considered under some mild conditions. Throughout the paper, we denote $C$ a positive constant whose value can be different in different places. The following are our main results.

**Theorem 1.1** *Suppose* $\{X, X_n, n \ge 1\}$ *is a sequence of i.i.d. random variables which is in the domain of attraction of the normal law with* $\mathsf{E}X = 0$, *and the truncated second moment*

*$l(x) = \mathsf{E}X^2 I\{|X| \le x\}$ satisfies $l(x) \le c_1 \exp(c_2(\log x)^\beta)$ for some $c_1 > 0$, $c_2 > 0$ and $0 \le \beta < 1$. Let $-1 < a < 0$, $b > -2$ and $a_n(\varepsilon)$ be a function of $\varepsilon$ such that*

$$a_n(\varepsilon) \log\log n \to \tau \quad \text{as } n \to \infty \text{ and } \varepsilon \searrow \sqrt{a+1}/2. \tag{1.8}$$

*Then we have*

$$\begin{aligned}&\lim_{\varepsilon \searrow \sqrt{a+1}/2} \bigl(4\varepsilon^2 - (a+1)\bigr)^{b+2} \sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^b}{n} \mathsf{P}\bigl(Q(n) \ge \bigl(\varepsilon + a_n(\varepsilon)\bigr)\sqrt{2n \log\log n}\bigr)\\ &\quad = 4(a+1)\Gamma(b+2)\exp(-4\tau\sqrt{a+1}).\end{aligned} \tag{1.9}$$

**Theorem 1.2** *Suppose $\{X, X_n, n \ge 1\}$ is a sequence of i.i.d. random variables which is in the domain of attraction of the normal law with $\mathsf{E}X = 0$, and the truncated second moment $l(x) = \mathsf{E}X^2 I\{|X| \le x\}$ satisfies $l(x) \le c_1 \exp(c_2(\log x)^\beta)$ for some $c_1 > 0$, $c_2 > 0$ and $0 \le \beta < 1$. Then for $b > -1$, (1.7) is true.*

**Theorem 1.3** *Suppose $\{X, X_n, n \ge 1\}$ is a sequence of i.i.d. random variables which is in the domain of attraction of the normal law with $\mathsf{E}X = 0$, and $l(x)$ satisfies $l(x) \le c_1 \exp(c_2(\log x)^\beta)$ for some $c_1 > 0$, $c_2 > 0$ and $0 \le \beta < 1$. Then for any $b > -1$, we have*

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log\log n)^b}{n \log n} I\bigl\{Q(n) \ge \varepsilon\sqrt{2n \log\log n}\bigr\} = \frac{\mathsf{E}Y^{2(b+1)}}{2^{b+1}(b+1)} \quad \text{a.s.}$$

**Remark 1.1** Note that $X$ belonging to the domain of attraction of the normal law is equivalent to $l(x)$ being a slowly varying function at $\infty$. We note also that $l(x) \le c_1 \exp(c_2(\log x)^\beta)$ is a weak enough assumption, which is satisfied by a large class of slowly varying functions such as $(\log\log x)^\alpha$ and $(\log x)^\alpha$, for some $0 < \alpha < \infty$.

**Remark 1.2** When $\mathsf{E}X^2 = \sigma^2 < \infty$, the truncated second moment $l(x)$ automatically satisfies the condition $l(x) \le c_1 \exp(c_2(\log x)^\beta)$ for some $c_1 > 0$, $c_2 > 0$ and $0 \le \beta < 1$. Hence, Theorems 1.1-1.3 not only hold for the random variables with finite second moments, but they also hold for a class of random variables with infinite second moments. Especially, Theorem 1.2 includes Theorem D as a special case.

**Remark 1.3** From Theorem C, one can see that the finiteness of the second moment does not guarantee the results about precise asymptotics in LIL for partial sums when $a > 0$. Moreover, it is clear that $R/S$ statistic is more complicated than partial sums. Hence, it seems that it is not possible, at least not easy, to prove (1.9) for $a > 0$ under the conditions stated in Theorem 1.1 only. However, if we impose more strong moment conditions which are similar to (1.1) and (1.2) on $\{X, X_n, n \ge 1\}$, it would be possible to prove (1.9) for $a > 0$, by following the ideas in Zhang [8].

**Remark 1.4** Checking the proof of Theorem 1.1, one can find that

$$\begin{aligned}&\lim_{\varepsilon \searrow \sqrt{a+1}} \bigl(\varepsilon^2 - (a+1)\bigr)^{b+2} \sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^b}{n} \mathsf{P}\bigl(Q(n) \ge \bigl(\varepsilon + a_n(\varepsilon)\bigr)\sqrt{n \log\log n/2}\bigr)\\ &\quad = 4(a+1)\Gamma(b+2)\exp\bigl(-4\tau'\sqrt{a+1}\bigr)\end{aligned}$$

holds if $a_n(\varepsilon)\log\log n \to \tau'$ as $n \to \infty$ and $\varepsilon \searrow \sqrt{a+1}$, which seems maybe more natural due to (1.6).

The remaining of this paper is organized as follows. In Section 2, Theorem 1.1 will be proved when $\{X, X_n, n \ge 1\}$ is a sequence of normal variables with zero mean. In Section 3, truncation method and strong approximation method will be employed to approximate the probability related to $R(n)$ statistic. In Section 4, Theorem 1.1 and Theorem 1.2 will be proved, while in Section 5 the proof of Theorem 1.3 will be given, based on some preliminaries.

## 2 Normal case

In this section, Theorem 1.1 in the case that $\{X, X_n, n \ge 1\}$ is a sequence of normal random variables with zero mean is proved. In order to do it, we firstly recall that $B(t)$ is a standard Brownian bridge and $Y = \sup_{0\le t\le 1} B(t) - \inf_{0\le t\le 1} B(t)$. The distribution of $Y$ plays an important role in our first result, and, fortunately, it has been given by Kennedy [20]:

$$\mathsf{P}(Y \le x) = 1 - 2\sum_{n=1}^{\infty}\left(4x^2n^2 - 1\right)\exp\left(-2x^2n^2\right). \tag{2.1}$$

Now, the main results in this section are stated as follows.

**Proposition 2.1** *Let $a > -1$, $b > -2$ and $a_n(\varepsilon)$ be a function of $\varepsilon$ such that*

$$a_n(\varepsilon)\log\log n \to \tau \quad \text{as } n \to \infty \text{ and } \varepsilon \searrow \sqrt{a+1}/2. \tag{2.2}$$

*Then we have*

$$\begin{aligned}\lim_{\varepsilon\searrow\sqrt{a+1}/2}&\left(4\varepsilon^2 - (a+1)\right)^{b+2}\sum_{n=1}^{\infty}\frac{(\log n)^a(\log\log n)^b}{n}\\ &\cdot \mathsf{P}\left(Y \ge \left(\varepsilon + a_n(\varepsilon)\right)\sqrt{2\log\log n}\right)\\ &= 4(a+1)\Gamma(b+2)\exp(-4\tau\sqrt{a+1}).\end{aligned}$$

*Proof* Firstly, it follows easily from (2.1) that

$$\mathsf{P}(Y \ge x) \sim 8x^2\exp\left(-2x^2\right)$$

as $x \to +\infty$. Then, by condition (2.2), one has

$$\begin{aligned}&\mathsf{P}\left(Y \ge \left(\varepsilon + a_n(\varepsilon)\right)\sqrt{2\log\log n}\right)\\ &\quad\sim 16\left(\varepsilon + a_n(\varepsilon)\right)^2\log\log n\cdot\exp\left(-4\left(\varepsilon + a_n(\varepsilon)\right)^2\log\log n\right)\\ &\quad\sim 16\varepsilon^2\log\log n\cdot\exp\left(-4\varepsilon^2\log\log n\right)\exp\left(-8\varepsilon a_n(\varepsilon)\log\log n\right)\end{aligned}$$

as $n \to \infty$ uniformly in $\varepsilon \in (\sqrt{a+1}/2, \sqrt{a+1}/2 + \delta)$ for some $\delta > 0$. Hence, for abovementioned $\delta > 0$ and any $0 < \theta < 1$, there exists an integer $n_0$ such that, for all $n \ge n_0$ and

$\varepsilon \in (\sqrt{a+1}/2, \sqrt{a+1}/2 + \delta)$,

$$\begin{aligned} &4(a+1)\log\log n \cdot \exp\bigl(-4\varepsilon^2 \log\log n\bigr)\exp(-4\tau\sqrt{a+1}-\theta) \\ &\quad \le \mathsf{P}\bigl(Y \ge \bigl(\varepsilon + a_n(\varepsilon)\bigr)\sqrt{2\log\log n}\bigr) \\ &\quad \le 4(a+1)\log\log n \cdot \exp\bigl(-4\varepsilon^2 \log\log n\bigr)\exp(-4\tau\sqrt{a+1}+\theta). \end{aligned}$$

Obviously, it suffices to show

$$\begin{aligned} &\lim_{\varepsilon \searrow \sqrt{a+1}/2} \bigl(4\varepsilon^2 - (a+1)\bigr)^{b+2} \sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^{b+1}}{n} \\ &\quad \cdot \exp\bigl(-4\varepsilon^2 \log\log n\bigr) = \Gamma(b+2) \end{aligned} \tag{2.3}$$

for proving Proposition 2.1 by the arbitrariness of $\theta$. To this end, by noting that the limit in (2.3) does not depend on any finite terms of the infinite series, we have

$$\begin{aligned} &\lim_{\varepsilon \searrow \sqrt{a+1}/2} \bigl(4\varepsilon^2 - (a+1)\bigr)^{b+2} \sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^{b+1}}{n} \exp\bigl(-4\varepsilon^2 \log\log n\bigr) \\ &\quad = \lim_{\varepsilon \searrow \sqrt{a+1}/2} \bigl(4\varepsilon^2 - (a+1)\bigr)^{b+2} \int_{e^e}^{\infty} \frac{(\log x)^{a-4\varepsilon^2} (\log\log x)^{b+1}}{x}\, dx \\ &\quad = \lim_{\varepsilon \searrow \sqrt{a+1}/2} \bigl(4\varepsilon^2 - (a+1)\bigr)^{b+2} \int_{1}^{\infty} \exp\bigl(y\bigl(a+1-4\varepsilon^2\bigr)\bigr) y^{b+1}\, dy \\ &\qquad \text{(by letting } y = \log\log x) \\ &\quad = \lim_{\varepsilon \searrow \sqrt{a+1}/2} \int_{4\varepsilon^2-(a+1)}^{\infty} e^{-u} u^{b+1}\, du \quad \bigl(\text{by letting } u = y\bigl(4\varepsilon^2 - (a+1)\bigr)\bigr) \\ &\quad = \Gamma(b+2). \end{aligned}$$

The proposition is proved now. □

**Proposition 2.2** *For any* $b > -1$,

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log\log n)^b}{n \log n} \mathsf{P}(Y \ge \varepsilon\sqrt{2\log\log n}) = \frac{\mathsf{E}Y^{2(b+1)}}{2^{b+1}(b+1)}.$$

*Proof* The proof can be found in Wu and Wen [19]. □

## 3 Truncation and approximation

In this section, we will use the truncation method and strong approximation method to show that the probability related to $R(n)$ with suitable normalization can be approximated by that for $Y$. To do this, we first give some notations. Put $c = \inf\{x \ge 1 : l(x) > 0\}$ and

$$\eta_n = \inf\left\{s : s \ge c+1, \frac{l(s)}{s^2} \le \frac{(\log\log n)^4}{n}\right\}. \tag{3.1}$$

For each $n$ and $1 \le i \le n$, we let

$$\begin{cases} X'_{ni} = X_i I\{|X_i| \le \eta_n\}, & X^*_{ni} = X'_{ni} - \mathsf{E}X'_{ni}, \\ S'_{ni} = \sum_{j=1}^{i} X'_{nj}, & S^*_{ni} = \sum_{j=1}^{i} X^*_{nj}, \bar{X}^*_n = \frac{1}{n} S^*_{nn}, \qquad D_n^2 = \sum_{j=1}^{n} \mathsf{Var}(X'_{nj}). \end{cases} \tag{3.2}$$

It follows easily that

$$D_n^2 \sim \sum_{j=1}^{n} \mathsf{E}X'^2_{nj} \sim nl(\eta_n) \sim \eta_n^2 (\log\log n)^4.$$

Furthermore, we denote $R^*(n)$ be the truncated $R$ statistic which is defined by the first expression of (1.5) with every $X_i$ being replaced by $X^*_{ni}$, $i = 1, \dots, n$. In addition, for any $0 \le \beta < 1$, all $j \ge k$ and $k$ large enough, following the lines of the proof of (2.4) in Pang, Zhang and Wang [21], we easily have

$$\begin{aligned} \frac{C}{l(\eta_k)(\log k)^{\beta}(\log\log k)^2} &\le \frac{\exp(c_2(\log k)^{\beta})}{2l(\eta_k)} \sum_{j=k}^{\infty} \frac{1}{j\exp(c_2(\log j)^{\beta}) \log j (\log\log j)^2} \\ &\le \sum_{j=k}^{\infty} \frac{1}{jl(\eta_j) \log j (\log\log j)^2}, \end{aligned} \tag{3.3}$$

despite a little difference for the definitions of $\eta_n$, which are from Pang, Zhang and Wang [21] and this paper, respectively.

Next, we will give the main result in this section as follows.

**Proposition 3.1** *For any $a < 0$, $b \in R$ and $1/2 < p < 2$, there exists a sequence of positive numbers $\{p_n, n \ge 1\}$ such that, for any $x > 0$,*

$$\begin{aligned} &\mathsf{P}\big(Y \ge x + 2/(\log\log n)^p\big) - p_n \\ &\quad \le \mathsf{P}\big(R(n) \ge xD_n\big) \\ &\quad \le \mathsf{P}\big(Y \ge x - 2/(\log\log n)^p\big) + p_n, \end{aligned}$$

*where $p_n \ge 0$ satisfies*

$$\sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^b}{n} p_n < \infty. \tag{3.4}$$

To show this proposition, the following lemmas are useful for the proof.

**Lemma 3.1** *For any sequence of independent random variables $\{\xi_n, n \ge 1\}$ with zero mean and finite variance, there exists a sequence of independent normal variables $\{Y_n, n \ge 1\}$ with $\mathsf{E}Y_n = 0$ and $\mathsf{E}Y_n^2 = \mathsf{E}\xi_n^2$ such that, for all $q > 2$ and $y > 0$,*

$$\mathsf{P}\left(\max_{k \le n} \left| \sum_{i=1}^{k} \xi_i - \sum_{i=1}^{k} Y_i \right| \ge y\right) \le (Aq)^q y^{-q} \sum_{i=1}^{n} \mathsf{E}|\xi_i|^q,$$

*whenever $\mathsf{E}|\xi_i|^q < \infty$, $i = 1, \dots, n$. Here, $A$ is an universal constant.*

*Proof* See Sakhanenko [22, 23]. □

**Lemma 3.2** *Let $\{W(t); t \geq 0\}$ be a standard Wiener process. For any $\varepsilon > 0$ there exists a constant $C = C(\varepsilon) > 0$ such that*

$$\mathsf{P}\Bigl(\sup_{0\leq s\leq 1-h}\sup_{0<t\leq h}\bigl|W(s+t)-W(s)\bigr| \geq x\sqrt{h}\Bigr) \leq \frac{C}{h}e^{-\frac{x^2}{2+\varepsilon}}$$

*for every positive $x$ and $0 < h < 1$.*

*Proof* It is Lemma 1.1.1 of Csörgő and Révész [24]. □

**Lemma 3.3** *For any $a < 0$, $b \in R$ and $1/2 < p < 2$, there exists a sequence of positive numbers $\{q_n, n \geq 1\}$ such that, for any $x > 0$,*

$$\begin{aligned}\mathsf{P}\bigl(Y \geq x + 1/(\log\log n)^p\bigr) - q_n &\leq \mathsf{P}\bigl(R^*(n) \geq xD_n\bigr)\\ &\leq \mathsf{P}\bigl(Y \geq x - 1/(\log\log n)^p\bigr) + q_n,\end{aligned} \tag{3.5}$$

*where $q_n \geq 0$ satisfies*

$$\sum_{n=1}^{\infty}\frac{(\log n)^a(\log\log n)^b}{n}q_n < \infty. \tag{3.6}$$

*Proof* Let $q_n = \mathsf{P}(|R^*(n)/D_n - Y| > 1/(\log\log n)^p)$, then obviously, $q_n$ satisfies (3.5). For each $n$, let $\{W_n(t), t \geq 0\}$ be a standard Wiener process, then we have $\{W_n(tD_n^2)/D_n, t \geq 0\} \stackrel{\mathcal{D}}{=} \{W_n(t), t \geq 0\}$ and

$$\begin{aligned} q_n &\leq 2\mathsf{P}\left(\sup_{0\leq s\leq 1}\left|\frac{\sum_{j=1}^{[ns]}(X_{nj}^* - \bar{X}_n^*)}{D_n} - \frac{W_n(sD_n^2) - sD_nW_n(1)}{D_n}\right| \geq \frac{1}{2(\log\log n)^p}\right)\\ &\leq 2\mathsf{P}\left(\max_{k\leq n}\left|\sum_{j=1}^{k}\bigl(X_{nj}^* - \bar{X}_n^*\bigr) - \left(W_n\left(\frac{k}{n}D_n^2\right) - \frac{k}{n}D_nW_n(1)\right)\right| \geq \frac{D_n}{4(\log\log n)^p}\right)\\ &\quad + 2\mathsf{P}\left(\sup_{0\leq s\leq 1}\left|\left(W_n\left(\frac{[ns]}{n}D_n^2\right) - \frac{[ns]}{n}D_nW_n(1)\right) - \bigl(W_n\bigl(sD_n^2\bigr) - sD_nW_n(1)\bigr)\right|\right.\\ &\quad \left.\geq \frac{D_n}{4(\log\log n)^p}\right)\\ &:= I_n + II_n. \end{aligned} \tag{3.7}$$

We consider $I_n$ first. Clearly,

$$\begin{aligned} I_n &\leq 2\mathsf{P}\left(\max_{k\leq n}\left|\sum_{j=1}^{k}X_{nj}^* - W_n\left(\frac{k}{n}D_n^2\right)\right| \geq \frac{D_n}{8(\log\log n)^p}\right)\\ &\quad + 2\mathsf{P}\left(\max_{k\leq n}\left|k\bar{X}_n^* - \frac{k}{n}D_nW_n(1)\right| \geq \frac{D_n}{8(\log\log n)^p}\right)\\ &\leq 4\mathsf{P}\left(\max_{k\leq n}\left|\sum_{j=1}^{k}X_{nj}^* - W_n\left(\frac{k}{n}D_n^2\right)\right| \geq \frac{D_n}{8(\log\log n)^p}\right). \end{aligned}$$

It follows from Lemma 3.1 and (3.3) that, for all $q > 2$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^b}{n} I_n \\
&\quad \le C \sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^b}{n} \left( \frac{(\log\log n)^p}{D_n} \right)^q \sum_{j=1}^{n} \mathsf{E}\left|X_{nj}^*\right|^q \\
&\quad \le C \sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^{b+pq}}{(nl(\eta_n))^{q/2}} \mathsf{E}|X|^q I\left\{|X| \le \eta_n\right\} \\
&\quad \le C \sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^{b+pq}}{(nl(\eta_n))^{q/2}} \sum_{k=1}^{n} \mathsf{E}|X|^q I\left\{\eta_{k-1} < |X| \le \eta_k\right\} \\
&\quad \le C \sum_{k=1}^{\infty} \mathsf{E}|X|^q I\left\{\eta_{k-1} < |X| \le \eta_k\right\} \sum_{n=k}^{\infty} \frac{(\log n)^a (\log\log n)^{b+pq}}{(nl(\eta_n))^{q/2}} \\
&\quad \le C \sum_{k=1}^{\infty} \eta_k^{q-2} \mathsf{E}X^2 I\left\{\eta_{k-1} < |X| \le \eta_k\right\} \frac{(\log k)^a (\log\log k)^{b+pq}}{k^{q/2-1}(l(\eta_k))^{q/2}} \\
&\quad \le C \sum_{k=1}^{\infty} \frac{(\log k)^a (\log\log k)^{b+pq-2q+4}}{l(\eta_k)} \mathsf{E}X^2 I\left\{\eta_{k-1} < |X| \le \eta_k\right\} \\
&\quad \le C \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} \frac{1}{jl(\eta_j) \log j (\log\log j)^2} \mathsf{E}X^2 I\left\{\eta_{k-1} < |X| \le \eta_k\right\} \\
&\quad \le C \sum_{j=1}^{\infty} \frac{1}{jl(\eta_j) \log j (\log\log j)^2} \sum_{k=1}^{j} \mathsf{E}X^2 I\left\{\eta_{k-1} < |X| \le \eta_k\right\} \\
&\quad \le C \sum_{j=1}^{\infty} \frac{1}{j \log j (\log\log j)^2} < \infty.
\end{aligned} \tag{3.8}$$

Next, we treat with $II_n$. Clearly, one has

$$\begin{aligned}
II_n &\le 2\mathsf{P}\left( \sup_{0\le s\le 1} \left| W_n\left( \frac{[ns]}{n} D_n^2 \right) - W_n\left(sD_n^2\right) \right| \ge \frac{D_n}{8(\log\log n)^p} \right) \\
&\quad + 2\mathsf{P}\left( \sup_{0\le s\le 1} \left| \frac{[ns]}{n} D_n W_n(1) - sD_n W_n(1) \right| \ge \frac{D_n}{8(\log\log n)^p} \right) \\
&:= II_n(1) + II_n(2).
\end{aligned} \tag{3.9}$$

It follows from Lemma 3.2 that

$$\begin{aligned}
II_n(1) &= 2\mathsf{P}\left( \sup_{0\le s\le 1} \left| W_n\left( \frac{[ns]}{n} \right) - W_n(s) \right| \ge \frac{1}{8(\log\log n)^p} \right) \\
&= 2\mathsf{P}\left( \sup_{0\le s\le 1} \left| W_n\left( \frac{[ns]}{n} \right) - W_n(s) \right| \ge \sqrt{\frac{1}{n}} \cdot \frac{\sqrt{n}}{8(\log\log n)^p} \right) \\
&\le Cn \exp\left( -\frac{n}{192(\log\log n)^{2p}} \right),
\end{aligned}$$

which obviously leads to

$$\sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^b}{n} II_n(1) < \infty. \tag{3.10}$$

On the other hand,

$$\begin{aligned} II_n(2) &= 2\mathsf{P}\left( \sup_{0 \le s \le 1} \left| \frac{[ns]}{n} - s \right| \cdot \left| W_n(1) \right| \ge \frac{1}{8(\log\log n)^p} \right) \\ &\le 2\mathsf{P}\left( \left| W_n(1) \right| \ge \frac{n}{8(\log\log n)^p} \right) \\ &\le \frac{C(\log\log n)^p}{n\sqrt{(1+o(1))l(\eta_n)}} \exp\left( -\frac{n^2}{128(1+o(1))l(\eta_n)(\log\log n)^{2p}} \right), \end{aligned}$$

which also obviously leads to

$$\sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^b}{n} II_n(2) < \infty. \tag{3.11}$$

Equations (3.7)-(3.11) yield (3.6). The proposition is proved now. □

**Lemma 3.4** *For any $a < 0$ and $b \in R$, one has*

$$\sum_{n=1}^{\infty} (\log n)^a (\log\log n)^b \mathsf{P}\left( |X| > \eta_n \right) < \infty.$$

*Proof* It follows from (3.3) that

$$\begin{aligned} &\sum_{n=1}^{\infty} (\log n)^a (\log\log n)^b \mathsf{P}\left( |X| > \eta_n \right) \\ &\quad \le C \sum_{n=1}^{\infty} (\log n)^a (\log\log n)^b \sum_{k=n}^{\infty} \mathsf{P}\left( \eta_k < |X| \le \eta_{k+1} \right) \\ &\quad \le C \sum_{k=1}^{\infty} \frac{1}{\eta_k^2} \mathsf{E}X^2 I\left\{ \eta_k < |X| \le \eta_{k+1} \right\} \sum_{n=1}^{k} (\log n)^a (\log\log n)^b \\ &\quad \le C \sum_{k=1}^{\infty} \frac{(\log k)^a (\log\log k)^{b+4}}{l(\eta_k)} \mathsf{E}X^2 I\left\{ \eta_k < |X| \le \eta_{k+1} \right\} \\ &\quad \le C \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} \frac{1}{j l(\eta_j) \log j (\log\log j)^2} \mathsf{E}X^2 I\left\{ \eta_k < |X| \le \eta_{k+1} \right\} \\ &\quad \le C \sum_{j=1}^{\infty} \frac{1}{j \log j (\log\log j)^2} < \infty. \end{aligned}$$

□

**Lemma 3.5** *Let $X$ be a random variable. Then the following statements are equivalent*:

(a) *$X$ is in the domain of attraction of the normal law,*

(b) $x^2 \mathsf{P}(|X| > x) = o(l(x))$,

(c) $x\mathsf{E}(|X|I\{|X| > x\}) = o(l(x))$,
(d) $\mathsf{E}(|X|^n I\{|X| \le x\}) = o(x^{n-2}l(x))$ *for* $n > 2$.

*Proof* It is Lemma 1 in Csörgő, Szyszkowicz and Wang [25]. □

**Lemma 3.6** *For any* $a < 0$ *and* $b \in R$, *one has, for* $\delta(n) = 1/(\log\log n \cdot \log\log\log n)$,

$$\sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^b}{n} \mathsf{P}\big(\big|S^2(n) - l(\eta_n)\big| > \delta(n) l(\eta_n)\big) < \infty.$$

*Proof* It is easy to see that, for large $n$,

$$\begin{aligned}
&\mathsf{P}\big(\big|S^2(n) - l(\eta_n)\big| > \delta(n) l(\eta_n)\big) \\
&\quad \le \mathsf{P}\left(\left|\frac{1}{n}\sum_{i=1}^n X_i^2 - l(\eta_n)\right| > \delta(n) l(\eta_n)/2\right) + \mathsf{P}\big(\bar{X}_n^2 > \delta(n) l(\eta_n)/2\big) \\
&\quad \le \mathsf{P}\left(\sum_{i=1}^n X_i^2 > \big(1 + \delta(n)/2\big) n l(\eta_n)\right) + \mathsf{P}\left(\sum_{i=1}^n X_i^2 < \big(1 - \delta(n)/2\big) n l(\eta_n)\right) \\
&\qquad + n\mathsf{P}\big(|X| > \eta_n\big) + \mathsf{P}\left(\sum_{i=1}^n X'_{ni} > n\sqrt{\delta(n) l(\eta_n)/2}\right) \\
&\quad \le \mathsf{P}\left(\sum_{i=1}^n X'^2_{ni} > \big(1 + \delta(n)/2\big) n l(\eta_n)\right) + \mathsf{P}\left(\sum_{i=1}^n X'^2_{ni} < \big(1 - \delta(n)/2\big) n l(\eta_n)\right) \\
&\qquad + 2n\mathsf{P}\big(|X| > \eta_n\big) + \mathsf{P}\left(\sum_{i=1}^n X^*_{ni} > n\sqrt{\delta(n) l(\eta_n)}/2\right),
\end{aligned} \tag{3.12}$$

since

$$\left|\mathsf{E}\left(\sum_{i=1}^n X'_{ni}\right)\right| \le n\mathsf{E}|X| I\big\{|X| > \eta_n\big\} = o\big(n l(\eta_n)/\eta_n\big) = o\big(n\sqrt{\delta(n) l(\eta_n)}\big)$$

by Lemma 3.5. Applying Lemma 3.4, we only need to show

$$\begin{cases}
\sum_{n=1}^\infty \frac{(\log n)^a (\log\log n)^b}{n} \mathsf{P}(\sum_{i=1}^n X'^2_{ni} > (1 + \delta(n)/2) n l(\eta_n)) < \infty, \\
\sum_{n=1}^\infty \frac{(\log n)^a (\log\log n)^b}{n} \mathsf{P}(\sum_{i=1}^n X'^2_{ni} < (1 - \delta(n)/2) n l(\eta_n)) < \infty, \\
\sum_{n=1}^\infty \frac{(\log n)^a (\log\log n)^b}{n} \mathsf{P}(\sum_{i=1}^n X^*_{ni} > n\sqrt{\delta(n) l(\eta_n)}/2) < \infty
\end{cases} \tag{3.13}$$

for proving Lemma 3.6. Consider the first part of (3.13) first. By employing Lemma 3.5 and Bernstein's inequality (*cf.* Lin and Bai [26]), we have for any fixed $\nu > 1$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^b}{n} \mathsf{P}\left(\sum_{i=1}^n X'^2_{ni} > \big(1 + \delta(n)/2\big) n l(\eta_n)\right) \\
&\quad = \sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^b}{n} \mathsf{P}\left(\sum_{i=1}^n X'^2_{ni} - n l(\eta_n) > \delta(n) n l(\eta_n)/2\right)
\end{aligned}$$

$$\begin{aligned}&\le C\sum_{n=1}^{\infty}\frac{(\log n)^a(\log\log n)^b}{n}\exp\left(-\frac{\delta^2(n)n^2l^2(\eta_n)/4}{2(n\mathsf{E}X^4I\{|X|\le\eta_n\}+\delta(n)\eta_n^2nl(\eta_n)/2)}\right)\\&\le C\sum_{n=1}^{\infty}\frac{(\log n)^a(\log\log n)^b}{n}\exp\left(-\frac{\delta^2(n)n^2l^2(\eta_n)/4}{o(1)\cdot\eta_n^2nl(\eta_n)}\right)\\&\le C\sum_{n=1}^{\infty}\frac{(\log n)^a(\log\log n)^b}{n}\exp(-\nu\log\log n)\\&<\infty.\end{aligned}\tag{3.14}$$

The second part of (3.13) can be proved by similar arguments. Now, let us consider the third part of (3.13). It follows from Markov's inequality that

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{(\log n)^a(\log\log n)^b}{n}\mathsf{P}\left(\sum_{i=1}^{n}X_{ni}^*>n\sqrt{\delta(n)l(\eta_n)}/2\right)\\&\quad\le C\sum_{n=1}^{\infty}\frac{(\log n)^a(\log\log n)^{b+1}}{n}\cdot\frac{nl(\eta_n)}{n^2\delta(n)l(\eta_n)}<\infty.\end{aligned}$$

The proof is completed now. □

**Lemma 3.7** *Define $\Delta_n=|R^*(n)-R(n)|$. Then for any $a<0$ and $b\in R$, one has*

$$\sum_{n=1}^{\infty}\frac{(\log n)^a(\log\log n)^b}{n}\mathsf{P}\left(\Delta_n>\frac{D_n}{(\log\log n)^2}\right)<\infty.$$

*Proof* Firstly, notice that $R(n)$ statistic has an equivalent expression

$$R(n)=\max_{1\le i<j\le n}\left|S_j-S_i-\frac{j-i}{n}S_n\right|\tag{3.15}$$

and so does $R^*(n)$ with $X_i$ being replaced by $X_{ni}^*$ in (3.15), $i=1,\ldots,n$. That is,

$$R^*(n)=\max_{1\le i<j\le n}\left|\left(S'_{nj}-S'_{ni}-\frac{j-i}{n}S'_{nn}\right)-\left(\mathsf{E}S'_{nj}-\mathsf{E}S'_{ni}-\frac{j-i}{n}\mathsf{E}S'_{nn}\right)\right|.$$

Let $\beta_n=2n\mathsf{E}|X|I\{|X|>\eta_n\}$, then

$$\max_{1\le i<j\le n}\left|\mathsf{E}S'_{nj}-\mathsf{E}S'_{ni}-\frac{j-i}{n}\mathsf{E}S'_{nn}\right|\le\beta_n.$$

Setting

$$\mathcal{L}=\left\{n:\beta_n\le\frac{\eta_n}{(\log\log n)^2}\right\},$$

then it is easily seen that, for $n\in\mathcal{L}$,

$$\left\{\Delta_n\ge\frac{D_n}{(\log\log n)^2}\right\}\subset\bigcup_{j=1}^{n}\{X_j\ne X'_{nj}\},$$

since $D_n \sim \eta_n (\log\log n)^2$. Hence, it follows from Lemma 3.4 that

$$\sum_{n\in\mathcal{L}} \frac{(\log n)^a(\log\log n)^b}{n} \mathsf{P}\left(\Delta_n > \frac{D_n}{(\log\log n)^2}\right) \\ \leq \sum_{n=1}^{\infty} (\log n)^a(\log\log n)^b \mathsf{P}\bigl(|X| > \eta_n\bigr) < \infty.$$

When $n \notin \mathcal{L}$, applying (3.3) yields

$$\begin{aligned}
&\sum_{n\notin\mathcal{L}} \frac{(\log n)^a(\log\log n)^b}{n} \mathsf{P}\left(\Delta_n > \frac{D_n}{(\log\log n)^2}\right) \\
&\quad \leq \sum_{n\notin\mathcal{L}} \frac{(\log n)^a(\log\log n)^b}{n} \\
&\quad \leq \sum_{n\notin\mathcal{L}} \frac{(\log n)^a(\log\log n)^b}{n} \cdot \frac{\beta_n(\log\log n)^2}{\eta_n} \\
&\quad \leq C\sum_{n=1}^{\infty} \frac{(\log n)^a(\log\log n)^{b+4}}{\sqrt{nl(\eta_n)}} \mathsf{E}|X| I\bigl\{|X| > \eta_n\bigr\} \\
&\quad \leq C\sum_{n=1}^{\infty} \frac{(\log n)^a(\log\log n)^{b+4}}{\sqrt{nl(\eta_n)}} \sum_{k=n}^{\infty} \mathsf{E}|X| I\bigl\{\eta_k < |X| \leq \eta_{k+1}\bigr\} \\
&\quad \leq C\sum_{k=1}^{\infty} \frac{\sqrt{k}(\log k)^a(\log\log k)^{b+4}}{\sqrt{l(\eta_k)}} \cdot \frac{\mathsf{E}X^2 I\{\eta_k < |X| \leq \eta_{k+1}\}}{\eta_k} \\
&\quad \leq C\sum_{k=1}^{\infty} \frac{(\log k)^a(\log\log k)^{b+6}}{l(\eta_k)} \mathsf{E}X^2 I\bigl\{\eta_k < |X| \leq \eta_{k+1}\bigr\} \\
&\quad \leq C\sum_{j=1}^{\infty} \frac{1}{j\log j(\log\log j)^2} < \infty.
\end{aligned}$$

□

Now, we turn to the proof of Proposition 3.1.

*Proof of Proposition* 3.1 Applying Lemma 3.3, one easily has

$$\begin{aligned}
&\mathsf{P}\bigl(R(n) \geq xD_n\bigr) \\
&\quad \leq \mathsf{P}\left(R(n) \geq xD_n, \Delta_n \leq \frac{D_n}{(\log\log n)^2}\right) + \mathsf{P}\left(\Delta_n > \frac{D_n}{(\log\log n)^2}\right) \\
&\quad \leq \mathsf{P}\left(R^*(n) \geq xD_n - \frac{D_n}{(\log\log n)^2}\right) + \mathsf{P}\left(\Delta_n > \frac{D_n}{(\log\log n)^2}\right) \\
&\quad \leq \mathsf{P}\left(Y \geq x - \frac{1}{(\log\log n)^2} - \frac{1}{(\log\log n)^p}\right) + q_n + \mathsf{P}\left(\Delta_n > \frac{D_n}{(\log\log n)^2}\right) \\
&\quad \leq \mathsf{P}\left(Y \geq x - \frac{2}{(\log\log n)^p}\right) + q_n + \mathsf{P}\left(\Delta_n > \frac{D_n}{(\log\log n)^2}\right).
\end{aligned}$$

Also, one has

$$\begin{aligned}
&\mathsf{P}\big(R(n) \ge xD_n\big) \\
&\quad \ge \mathsf{P}\biggl(R(n) \ge xD_n, \Delta_n \le \frac{D_n}{(\log\log n)^2}\biggr) \\
&\quad \ge \mathsf{P}\biggl(R^*(n) \ge xD_n + \frac{D_n}{(\log\log n)^2}\biggr) - \mathsf{P}\biggl(\Delta_n > \frac{D_n}{(\log\log n)^2}\biggr) \\
&\quad \ge \mathsf{P}\biggl(Y \ge x + \frac{1}{(\log\log n)^2} + \frac{1}{(\log\log n)^p}\biggr) - q_n - \mathsf{P}\biggl(\Delta_n > \frac{D_n}{(\log\log n)^2}\biggr) \\
&\quad \ge \mathsf{P}\biggl(Y \ge x + \frac{2}{(\log\log n)^p}\biggr) - q_n - \mathsf{P}\biggl(\Delta_n > \frac{D_n}{(\log\log n)^2}\biggr).
\end{aligned}$$

Letting $p_n = q_n + \mathsf{P}(\Delta_n > D_n/(\log\log n)^2)$ completes the proof by Lemmas 3.3 and 3.7. □

## 4 Proofs of Theorems 1.1 and 1.2

*Proof of Theorem* 1.1 For any $0 < \delta < \sqrt{a+1}/4$ and $\sqrt{a+1}/2 - \delta < \varepsilon < \sqrt{a+1}/2 + \delta$, we have

$$\begin{aligned}
&\mathsf{P}\big(Y \ge \big(\varepsilon + a_n''(\varepsilon)\big)\sqrt{2\log\log n}\big) - p_n - \mathsf{P}\big(\big|S^2(n) - l(\eta_n)\big| > \delta(n)l(\eta_n)\big) \\
&\quad = \mathsf{P}\big(Y \ge \big(\varepsilon + a_n'(\varepsilon)\big)\sqrt{2\log\log n} + 2/(\log\log n)^p\big) - p_n \\
&\qquad - \mathsf{P}\big(\big|S^2(n) - l(\eta_n)\big| > \delta(n)l(\eta_n)\big) \\
&\quad \le \mathsf{P}\big(R(n) \ge \big(\varepsilon + a_n'(\varepsilon)\big)\sqrt{2\log\log n}D_n\big) - \mathsf{P}\big(\big|S^2(n) - l(\eta_n)\big| > \delta(n)l(\eta_n)\big) \\
&\quad \le \mathsf{P}\Big(R(n) \ge \big(\varepsilon + a_n(\varepsilon)\big)\sqrt{2\big(1+\delta(n)\big)nl(\eta_n)\log\log n}\Big) \\
&\qquad - \mathsf{P}\big(\big|S^2(n) - l(\eta_n)\big| > \delta(n)l(\eta_n)\big) \\
&\quad \le \mathsf{P}\big(Q(n) \ge \big(\varepsilon + a_n(\varepsilon)\big)\sqrt{2n\log\log n}\big) \\
&\quad \le \mathsf{P}\Big(R(n) \ge \big(\varepsilon + a_n(\varepsilon)\big)\sqrt{2\big(1-\delta(n)\big)nl(\eta_n)\log\log n}\Big) \\
&\qquad + \mathsf{P}\big(\big|S^2(n) - l(\eta_n)\big| > \delta(n)l(\eta_n)\big) \\
&\quad \le \mathsf{P}\big(R(n) \ge \big(\varepsilon + a_n'''(\varepsilon)\big)\sqrt{2\log\log n}D_n\big) + \mathsf{P}\big(\big|S^2(n) - l(\eta_n)\big| > \delta(n)l(\eta_n)\big) \\
&\quad \le \mathsf{P}\big(Y \ge \big(\varepsilon + a_n'''(\varepsilon)\big)\sqrt{2\log\log n} - 2/(\log\log n)^p\big) + p_n \\
&\qquad + \mathsf{P}\big(\big|S^2(n) - l(\eta_n)\big| > \delta(n)l(\eta_n)\big) \\
&\quad = \mathsf{P}\big(Y \ge \big(\varepsilon + a_n''''(\varepsilon)\big)\sqrt{2\log\log n}\big) + p_n + \mathsf{P}\big(\big|S^2(n) - l(\eta_n)\big| > \delta(n)l(\eta_n)\big),
\end{aligned} \tag{4.1}$$

where

$$\begin{cases}
a_n'(\varepsilon) = \frac{\sqrt{nl(\eta_n)}}{D_n}(\varepsilon + a_n(\varepsilon))\sqrt{1+\delta(n)} - \varepsilon, \\
a_n''(\varepsilon) = \frac{\sqrt{nl(\eta_n)}}{D_n}(\varepsilon + a_n(\varepsilon))\sqrt{1+\delta(n)} - \varepsilon + \frac{\sqrt{2}}{(\log\log n)^{p+1/2}}, \\
a_n'''(\varepsilon) = \frac{\sqrt{nl(\eta_n)}}{D_n}(\varepsilon + a_n(\varepsilon))\sqrt{1-\delta(n)} - \varepsilon, \\
a_n''''(\varepsilon) = \frac{\sqrt{nl(\eta_n)}}{D_n}(\varepsilon + a_n(\varepsilon))\sqrt{1-\delta(n)} - \varepsilon - \frac{\sqrt{2}}{(\log\log n)^{p+1/2}}.
\end{cases}$$

Noting that $nl(\eta_n) \geq D_n^2 \sim nl(\eta_n)$, one easily has

$$\begin{aligned}
&\left(\frac{\sqrt{nl(\eta_n)}}{D_n}\big(\varepsilon + a_n(\varepsilon)\big)\sqrt{1 \pm \delta(n)} - \varepsilon\right)\log\log n \\
&\quad = \frac{\sqrt{nl(\eta_n)}}{D_n}\sqrt{1 \pm \delta(n)}a_n(\varepsilon)\log\log n + \left(\frac{\sqrt{nl(\eta_n)}}{D_n}\sqrt{1 \pm \delta(n)} - 1\right)\varepsilon\log\log n
\end{aligned} \tag{4.2}$$

and for large $n$,

$$\begin{aligned}
&\left|\left(\frac{\sqrt{nl(\eta_n)}}{D_n}\sqrt{1 \pm \delta(n)} - 1\right)\varepsilon\log\log n\right| \\
&\quad \leq \left|\big(\sqrt{1 \pm 2\delta(n)} - 1\big)\varepsilon\log\log n\right| \\
&\quad \leq 2\varepsilon\delta(n)\log\log n \\
&\quad = 2\varepsilon/\log\log\log n,
\end{aligned} \tag{4.3}$$

which tends to zero as $n \to \infty$ and $\varepsilon \searrow \sqrt{a+1}/2$. Hence, we have

$$a_n''(\varepsilon)\log\log n \to \tau \quad \text{and} \quad a_n''''(\varepsilon)\log\log n \to \tau \quad \text{as } n \to \infty \text{ and } \varepsilon \searrow \sqrt{a+1}/2,$$

since $p > 1/2$ and $a_n(\varepsilon)$ satisfies (1.8). Now, it follows from Proposition 2.1, (3.4) and Lemma 3.6 that Theorem 1.1 is true. □

*Proof of Theorem* 1.2 For any $0 < \gamma < 1$, applying similar arguments to those used in (4.1), we have for large $n$,

$$\begin{aligned}
&\mathsf{P}\big(Y \geq \varepsilon'\sqrt{2(1+\gamma)\log\log n}\big) - p_n - \mathsf{P}\big(\big|S^2(n) - l(\eta_n)\big| > \delta(n)l(\eta_n)\big) \\
&\quad = \mathsf{P}\big(Y \geq \varepsilon\sqrt{2(1+\gamma)\log\log n} + 2/(\log\log n)^p\big) - p_n - \mathsf{P}\big(\big|S^2(n) - l(\eta_n)\big| > \delta(n)l(\eta_n)\big) \\
&\quad \leq \mathsf{P}\big(R(n) \geq \varepsilon\sqrt{2(1+\gamma)\log\log n}D_n\big) - \mathsf{P}\big(\big|S^2(n) - l(\eta_n)\big| > \delta(n)l(\eta_n)\big) \\
&\quad \leq \mathsf{P}\Big(R(n) \geq \varepsilon\sqrt{2\big(1+\delta(n)\big)nl(\eta_n)\log\log n}\Big) - \mathsf{P}\big(\big|S^2(n) - l(\eta_n)\big| > \delta(n)l(\eta_n)\big) \\
&\quad \leq \mathsf{P}\big(Q(n) \geq \varepsilon\sqrt{2n\log\log n}\big) \\
&\quad \leq \mathsf{P}\Big(R(n) \geq \varepsilon\sqrt{2\big(1-\delta(n)\big)nl(\eta_n)\log\log n}\Big) + \mathsf{P}\big(\big|S^2(n) - l(\eta_n)\big| > \delta(n)l(\eta_n)\big) \\
&\quad \leq \mathsf{P}\big(R(n) \geq \varepsilon\sqrt{2(1-\gamma)\log\log n}D_n\big) + \mathsf{P}\big(\big|S^2(n) - l(\eta_n)\big| > \delta(n)l(\eta_n)\big) \\
&\quad \leq \mathsf{P}\big(Y \geq \varepsilon\sqrt{2(1-\gamma)\log\log n} - 2/(\log\log n)^p\big) + p_n + \mathsf{P}\big(\big|S^2(n) - l(\eta_n)\big| > \delta(n)l(\eta_n)\big) \\
&\quad = \mathsf{P}\big(Y \geq \varepsilon''\sqrt{2(1-\gamma)\log\log n}\big) + p_n + \mathsf{P}\big(\big|S^2(n) - l(\eta_n)\big| > \delta(n)l(\eta_n)\big),
\end{aligned}$$

where

$$\begin{cases}
\varepsilon' = \varepsilon + \frac{\sqrt{2}}{\sqrt{1+\gamma}(\log\log n)^{p+1/2}} \sim \varepsilon, \\
\varepsilon'' = \varepsilon - \frac{\sqrt{2}}{\sqrt{1-\gamma}(\log\log n)^{p+1/2}} \sim \varepsilon
\end{cases}$$

as $n \to \infty$. Hence, Proposition 2.2, (3.4) and Lemma 3.6 guarantee that

$$\begin{aligned}
&(1+\gamma)^{b+1}\frac{\mathsf{E}Y^{2(b+1)}}{2^{b+1}(b+1)} \\
&\quad\le \liminf_{\varepsilon\searrow 0}\varepsilon^{2(b+1)}\sum_{n=1}^{\infty}\frac{(\log\log n)^b}{n\log n}\mathsf{P}\big(Q(n)\ge\varepsilon\sqrt{2n\log\log n}\big) \\
&\quad\le \limsup_{\varepsilon\searrow 0}\varepsilon^{2(b+1)}\sum_{n=1}^{\infty}\frac{(\log\log n)^b}{n\log n}\mathsf{P}\big(Q(n)\ge\varepsilon\sqrt{2n\log\log n}\big) \\
&\quad\le (1-\gamma)^{b+1}\frac{\mathsf{E}Y^{2(b+1)}}{2^{b+1}(b+1)}.
\end{aligned}$$

Letting $\gamma \to 0$ completes the proof. □

## 5 Proof of Theorem 1.3

In this section, we first modify the definition in (3.1) as follows:

$$\tilde{\eta}_n = \inf\left\{s : s \ge c+1, \frac{l(s)}{s^2} \le \frac{\log\log n}{n}\right\}. \tag{5.1}$$

Then one easily has $nl(\tilde{\eta}_n) \sim \tilde{\eta}_n^2 \log\log n$. Moreover, we define for each $n$ and $1 \le i \le n$,

$$\begin{cases}\tilde{X}_{ni} = X_i I\{|X_i| \le \tilde{\eta}_n\}, & \tilde{X}^*_{ni} = \tilde{X}_{ni} - \mathsf{E}\tilde{X}_{ni}, \\ \tilde{S}_{ni} = \sum_{j=1}^{i}\tilde{X}_{nj}, & \tilde{S}^*_{ni} = \sum_{j=1}^{i}\tilde{X}^*_{nj}, \qquad D_n^{*2} = \mathsf{Var}(\tilde{S}^*_{nn}).\end{cases}$$

Secondly, we give two notations related to the truncated $R(n)$ statistic. That is,

$$\widetilde{R}(n) := \max_{1\le k\le n}\left\{\sum_{j=1}^{k}\left(\tilde{X}_{nj} - \frac{1}{n}\sum_{j=1}^{n}\tilde{X}_{nj}\right)\right\} - \min_{1\le k\le n}\left\{\sum_{j=1}^{k}\left(\tilde{X}_{nj} - \frac{1}{n}\sum_{j=1}^{n}\tilde{X}_{nj}\right)\right\}$$

and

$$\widetilde{R}^*(n) := \max_{1\le k\le n}\left\{\sum_{j=1}^{k}\left(\tilde{X}^*_{nj} - \frac{1}{n}\sum_{j=1}^{n}\tilde{X}^*_{nj}\right)\right\} - \min_{1\le k\le n}\left\{\sum_{j=1}^{k}\left(\tilde{X}^*_{nj} - \frac{1}{n}\sum_{j=1}^{n}\tilde{X}^*_{nj}\right)\right\}.$$

Then two lemmas which play key roles in the proof of Theorem 1.3 will be given, after which, we will finish the proof of Theorem 1.3.

**Lemma 5.1** *Suppose $\{X, X_n, n \ge 1\}$ is a sequence of i.i.d. random variables which is in the domain of attraction of the normal law with $\mathsf{E}X = 0$, and $l(x)$ satisfies $l(x) \le c_1\exp(c_2(\log x)^{\beta})$ for some $c_1 > 0$, $c_2 > 0$ and $0 \le \beta < 1$. Then, for any $b \in R$ and $1/2 < p < 2$, there exists a sequence of positive numbers $\{q'_n, n \ge 1\}$ such that, for any $x > 0$,*

$$\mathsf{P}\big(Y \ge x + 1/(\log\log n)^p\big) - q'_n \le \mathsf{P}\big(\widetilde{R}^*(n) \ge xD^*_n\big) \le \mathsf{P}\big(Y \ge x - 1/(\log\log n)^p\big) + q'_n,$$

*where $q'_n \ge 0$ satisfies*

$$\sum_{n=1}^{\infty}\frac{(\log\log n)^b}{n\log n}q'_n < \infty.$$

*Proof* The essential difference between this lemma and Lemma 3.3 is the different truncation levels are imposed on the random variables $\{X_n, n \geq 1\}$ in two lemmas. However, by checking the proof of Lemma 3.3 carefully, one can find that the proof of Lemma 3.3 is not sensitive to the powers of $\log\log n$. Hence, one can easily finish the proof by similar arguments to those used in Lemma 3.3. We omit the details here. □

**Lemma 5.2** *Suppose $\{X, X_n, n \geq 1\}$ is a sequence of i.i.d. random variables which is in the domain of attraction of the normal law with $\mathsf{E}X = 0$, and let $f(\cdot)$ be a real function such that $\sup_{x\in R}|f(x)| \leq C$ and $\sup_{x\in R}|f'(x)| \leq C$. Then for any $b \in R$, $0 < \varepsilon < 1/4$ and $l > m \geq 1$, we have*

$$\begin{cases} \mathsf{Var}(\sum_{n=m}^{l} \frac{(\log\log n)^b}{n\log n} f(\frac{\widetilde{R}^*(n)}{\rho(n,\varepsilon)})) = O(\frac{(\log\log m)^{2b-1/2}}{\varepsilon\log m}), \\ \mathsf{Var}(\sum_{n=m}^{l} \frac{(\log\log n)^b}{n\log n} f(\frac{\sum_{i=1}^n \tilde{X}_{ni}^2}{(1\pm\gamma/2)nl(\tilde{\eta}_n)})) = O(\frac{(\log\log m)^{2b}}{\log m}), \\ \mathsf{Var}(\sum_{n=m}^{l} \frac{(\log\log n)^b}{n\log n} f(\frac{\tilde{S}_{nn}^*}{n\sqrt{\gamma l(\tilde{\eta}_n)/2}})) = O(\frac{(\log\log m)^{2b}}{\sqrt{m}(\log m)^2}), \\ \mathsf{Var}(\sum_{n=m}^{l} \frac{(\log\log n)^b}{n\log n} \sum_{i=1}^n I\{|X_i| > \tilde{\eta}_n\}) = O(\frac{(\log\log m)^{2b+1}}{\log m}), \end{cases} \tag{5.2}$$

*where $\rho(n,\varepsilon) = \varepsilon\sqrt{2nl(\tilde{\eta}_n)\log\log n}$.*

*Proof* Firstly, we consider the first part of (5.2). For $j > i$, since $\widetilde{R}^*(i)$ is independent of

$$\widetilde{R}^*(i+1,j) := \max_{i<k\leq j}\left\{\sum_{l=i+1}^{k}\left(\tilde{X}_{jl}^* - \frac{1}{j}\sum_{l=i+1}^{j}\tilde{X}_{jl}^*\right)\right\} - \min_{i<k\leq j}\left\{\sum_{l=i+1}^{k}\left(\tilde{X}_{jl}^* - \frac{1}{j}\sum_{l=i+1}^{j}\tilde{X}_{jl}^*\right)\right\}.$$

It follows that

$$\begin{aligned} &\mathsf{Cov}\left(f\left(\frac{\widetilde{R}^*(i)}{\rho(i,\varepsilon)}\right), f\left(\frac{\widetilde{R}^*(j)}{\rho(j,\varepsilon)}\right)\right) \\ &= \mathsf{Cov}\left(f\left(\frac{\widetilde{R}^*(i)}{\rho(i,\varepsilon)}\right), f\left(\frac{\widetilde{R}^*(j)}{\rho(j,\varepsilon)}\right) - f\left(\frac{\widetilde{R}^*(i+1,j)}{\rho(j,\varepsilon)}\right)\right) \\ &\leq C\mathsf{E}\left|f\left(\frac{\widetilde{R}^*(j)}{\rho(j,\varepsilon)}\right) - f\left(\frac{\widetilde{R}^*(i+1,j)}{\rho(j,\varepsilon)}\right)\right| \\ &\leq C\frac{\mathsf{E}|\sum_{l=1}^{i}\tilde{X}_{jl}^*|}{\varepsilon\sqrt{2jl(\tilde{\eta}_j)\log\log j}} \\ &\leq C\frac{\sqrt{il(\tilde{\eta}_j)}}{\varepsilon\sqrt{2jl(\tilde{\eta}_j)\log\log j}} \\ &= O\left(\frac{\sqrt{i}}{\varepsilon\sqrt{j\log\log j}}\right). \end{aligned} \tag{5.3}$$

Hence, for any $0 < \varepsilon < 1/4$ and $l \geq m \geq 1$, we have

$$\begin{aligned} &\mathsf{Var}\left(\sum_{n=m}^{l}\frac{(\log\log n)^b}{n\log n} f\left(\frac{\widetilde{R}^*(n)}{\rho(n,\varepsilon)}\right)\right) \\ &\leq C\sum_{n=m}^{l}\frac{(\log\log n)^{2b}}{n^2(\log n)^2} + 2\sum_{j=m+1}^{l}\sum_{i=m}^{j-1}\frac{(\log\log i)^b}{i\log i}\frac{(\log\log j)^b}{j\log j}\cdot O\left(\frac{\sqrt{i}}{\varepsilon\sqrt{j\log\log j}}\right) \end{aligned}$$

$$\begin{aligned}&\le C\frac{(\log\log m)^{2b}}{m(\log m)^2}+O(1)\cdot\sum_{j=m+1}^{l}\frac{(\log\log j)^{2b-1/2}}{j(\log j)^2\varepsilon}\\&=O\biggl(\frac{(\log\log m)^{2b-1/2}}{\varepsilon\log m}\biggr).\end{aligned}$$

Consider the second part of (5.2). Similar arguments used in (5.3) leads easily to

$$\begin{aligned}&\mathsf{Cov}\biggl(f\biggl(\frac{\sum_{k=1}^{i}\tilde{X}_{ik}^2}{(1\pm\gamma/2)il(\tilde{\eta}_i)}\biggr),f\biggl(\frac{\sum_{k=1}^{j}\tilde{X}_{jk}^2}{(1\pm\gamma/2)jl(\tilde{\eta}_j)}\biggr)\biggr)\\&\quad\le\mathsf{Cov}\biggl(f\biggl(\frac{\sum_{k=1}^{i}\tilde{X}_{ik}^2}{(1\pm\gamma/2)il(\tilde{\eta}_i)}\biggr),f\biggl(\frac{\sum_{k=1}^{j}\tilde{X}_{jk}^2}{(1\pm\gamma/2)jl(\tilde{\eta}_j)}\biggr)-f\biggl(\frac{\sum_{k=i+1}^{j}\tilde{X}_{jk}^2}{(1\pm\gamma/2)jl(\tilde{\eta}_j)}\biggr)\biggr)\\&\quad\le C\cdot\frac{i}{j}.\end{aligned}$$

It follows that

$$\begin{aligned}&\mathsf{Var}\biggl(\sum_{n=m}^{l}\frac{(\log\log n)^b}{n\log n}f\biggl(\frac{\sum_{i=1}^{n}\tilde{X}_{ni}^2}{(1\pm\gamma/2)nl(\tilde{\eta}_n)}\biggr)\biggr)\\&\quad\le C\frac{(\log\log m)^{2b}}{m(\log m)^2}+2\sum_{j=m+1}^{l}\sum_{i=m}^{j-1}\frac{(\log\log i)^b}{i\log i}\frac{(\log\log j)^b}{j\log j}\cdot\frac{i}{j}\\&\quad=O\biggl(\frac{(\log\log m)^{2b}}{\log m}\biggr).\end{aligned}$$

Consider the third part of (5.2). The similar arguments used in (5.3) also lead easily to

$$\mathsf{Cov}\biggl(f\biggl(\frac{\tilde{S}_{ii}^*}{i\sqrt{\gamma l(\tilde{\eta}_i)}/2}\biggr),f\biggl(\frac{\tilde{S}_{jj}^*}{j\sqrt{\gamma l(\tilde{\eta}_j)}/2}\biggr)\biggr)=O\biggl(\frac{\sqrt{i}}{j}\biggr),$$

which implies that

$$\begin{aligned}&\mathsf{Var}\biggl(\sum_{n=m}^{l}\frac{(\log\log n)^b}{n\log n}f\biggl(\frac{S_{nn}^*}{n\sqrt{\gamma l(\tilde{\eta}_n)}/2}\biggr)\biggr)\\&\quad\le C\frac{(\log\log m)^{2b}}{m(\log m)^2}+2\sum_{j=m+1}^{l}\sum_{i=m}^{j-1}\frac{(\log\log i)^b}{i\log i}\frac{(\log\log j)^b}{j\log j}\cdot O\biggl(\frac{\sqrt{i}}{j}\biggr)\\&\quad=O\biggl(\frac{(\log\log m)^{2b}}{\sqrt{m}(\log m)^2}\biggr).\end{aligned}$$

Finally, we turn to handling the fourth part of (5.2). By employing Lemma 3.5 one has

$$\begin{aligned}&\mathsf{Var}\biggl(\sum_{n=m}^{l}\frac{(\log\log n)^b}{n\log n}\sum_{i=1}^{n}I\bigl\{|X_i|>\tilde{\eta}_n\bigr\}\biggr)\\&\quad\le C\sum_{n=m}^{l}\frac{(\log\log n)^{2b}}{n^2(\log n)^2}\cdot n\mathsf{P}\bigl(|X|>\tilde{\eta}_n\bigr)+2\sum_{j=m+1}^{l}\sum_{i=m}^{j-1}\frac{(\log\log i)^b}{i\log i}\frac{(\log\log j)^b}{j\log j}\end{aligned}$$

$$\cdot \operatorname{Cov}\left(\sum_{k=1}^{i} I\{|X_k| > \tilde{\eta}_i\}, \sum_{k=1}^{j} I\{|X_k| > \tilde{\eta}_j\}\right)$$
$$\le o(1) \cdot \sum_{n=m}^{l} \frac{(\log\log n)^{2b+1}}{n^2(\log n)^2} + 2\sum_{j=m+1}^{l}\sum_{i=m}^{j-1} \frac{(\log\log i)^b}{i\log i}\frac{(\log\log j)^b}{j\log j} \cdot i\mathsf{P}(|X| > \tilde{\eta}_j)$$
$$= o(1) \cdot \frac{(\log\log m)^{2b+1}}{m(\log m)^2} + C\sum_{j=m+1}^{l} \frac{(\log\log j)^{2b+1}}{j(\log j)^2} = O\left(\frac{(\log\log m)^{2b+1}}{\log m}\right).$$

The proof is completed. □

*Proof of Theorem* 1.3 At the beginning of the proof, we first give an upper bound and a lower bound for the indicator function of $R/S$ statistic. For any $x \ge \lambda\sqrt{n\log\log n}$ with $\lambda > 0$, $0 < \gamma < 1/2$ and large $n$, one has the following fact:

$$\begin{aligned}
I\left\{\frac{R(n)}{S(n)} \ge x\right\} &\le I\left\{\frac{R(n)}{\sqrt{(1-\gamma)l(\tilde{\eta}_n)}} \ge x\right\} + I\{|S^2(n) - l(\tilde{\eta}_n)| > \gamma l(\tilde{\eta}_n)\} \\
&\le I\left\{\frac{\widetilde{R}(n)}{\sqrt{(1-\gamma)l(\tilde{\eta}_n)}} \ge x\right\} + I\left\{\bigcup_{i=1}^{n} |X_i| > \tilde{\eta}_n\right\} \\
&\quad + I\left\{\sum_{i=1}^{n} X_i^2 > (1+\gamma/2)nl(\tilde{\eta}_n)\right\} + I\left\{\sum_{i=1}^{n} X_i^2 < (1-\gamma/2)nl(\tilde{\eta}_n)\right\} \\
&\quad + I\{|S_n| > n\sqrt{\gamma l(\tilde{\eta}_n)/2}\} \\
&\le I\left\{\frac{\widetilde{R}^*(n)}{\sqrt{(1-\gamma)l(\tilde{\eta}_n)}} \ge x + o(1)\right\} + 3I\left\{\bigcup_{i=1}^{n} |X_i| > \tilde{\eta}_n\right\} \\
&\quad + I\left\{\sum_{i=1}^{n} \tilde{X}_{ni}^2 > (1+\gamma/2)nl(\tilde{\eta}_n)\right\} + I\left\{\sum_{i=1}^{n} \tilde{X}_{ni}^2 < (1-\gamma/2)nl(\tilde{\eta}_n)\right\} \\
&\quad + I\{|\tilde{S}_{nn}^*| > n\sqrt{\gamma l(\tilde{\eta}_n)}/2\} \\
&\le I\left\{\frac{\widetilde{R}^*(n)}{\sqrt{(1-2\gamma)l(\tilde{\eta}_n)}} \ge x\right\} + 3I\left\{\bigcup_{i=1}^{n} |X_i| > \tilde{\eta}_n\right\} \\
&\quad + I\left\{\sum_{i=1}^{n} \tilde{X}_{ni}^2 > (1+\gamma/2)nl(\tilde{\eta}_n)\right\} + I\left\{\sum_{i=1}^{n} \tilde{X}_{ni}^2 < (1-\gamma/2)nl(\tilde{\eta}_n)\right\} \\
&\quad + I\{|\tilde{S}_{nn}^*| > n\sqrt{\gamma l(\tilde{\eta}_n)}/2\},
\end{aligned} \tag{5.4}$$

since one easily has

$$|\mathsf{E}\widetilde{R}(n)| = o\left(\sqrt{nl(\tilde{\eta}_n)\log\log n}\right).$$

Also, one has, for any $x \ge \lambda\sqrt{n\log\log n}$ with $\lambda > 0$, $0 < \gamma < 1/2$ and large $n$,

$$\begin{aligned}
I\left\{\frac{R(n)}{S(n)} \ge x\right\} &\ge I\left\{\frac{R(n)}{\sqrt{(1+\gamma)l(\tilde{\eta}_n)}} \ge x\right\} - I\{|S^2(n) - l(\tilde{\eta}_n)| > \gamma l(\tilde{\eta}_n)\} \\
&\ge I\left\{\frac{\widetilde{R}^*(n)}{\sqrt{(1+2\gamma)l(\tilde{\eta}_n)}} \ge x\right\} - 3I\left\{\bigcup_{i=1}^{n} |X_i| > \tilde{\eta}_n\right\}
\end{aligned}$$

$$- I\left\{\sum_{i=1}^{n}\tilde{X}_{ni}^2 > (1+\gamma/2)nl(\tilde{\eta}_n)\right\} - I\left\{\sum_{i=1}^{n}\tilde{X}_{ni}^2 < (1-\gamma/2)nl(\tilde{\eta}_n)\right\}$$
$$- I\bigl\{\bigl|\tilde{S}_{nn}^*\bigr| > n\sqrt{\gamma l(\tilde{\eta}_n)}/2\bigr\}.$$

Denote $K(\varepsilon) = \exp(\exp(1/(\varepsilon^2 M)))$ for any $0 < \varepsilon < 1/4$ and fixed $M > 0$. Let $\{f_i(\cdot), i = 1, \dots, 5\}$ be real functions such that $\sup_x |f_i'(x)| < \infty$ for $i = 1, \dots, 5$ and

$$\begin{cases} I\{|x| \ge \sqrt{1-2\gamma}\} \le f_{n,1}(x) := f_1(x) \le I\{|x| \ge 1-2\gamma\}, \\ I\{|x| \ge 1+\gamma/2\} \le f_{n,2}(x) := f_2(x) \le I\{|x| \ge 1+\gamma/4\}, \\ I\{|x| \le 1-\gamma/2\} \le f_{n,3}(x) := f_3(x) \le I\{|x| \le 1-\gamma/4\}, \\ I\{|x| > 1\} \le f_{n,4}(x) := f_4(x) \le I\{|x| > 1/2\}, \\ I\{|x| \ge \sqrt{\gamma}\} \le f_{n,5}(x) := f_5(x) \le I\{|x| \ge \sqrt{\gamma}/2\}. \end{cases} \tag{5.5}$$

Define $\varepsilon_k = 1/k$, $k \ge M$. Then it follows from Lemma 5.2 that

$$\mathsf{Var}\left(\sum_{n>B(\varepsilon_k)} \frac{(\log\log n)^b}{n\log n} f\left(\frac{\widetilde{R}^*(n)}{\rho(n,\varepsilon_k)}\right)\right) = O\left(\frac{k(k^2/M)^{2b-1/2}}{\exp(k^2/M)}\right),$$

which together with the Borel-Cantelli lemma easily yield

$$\sum_{n>B(\varepsilon_k)} \frac{(\log\log n)^b}{n\log n}\left(f\left(\frac{\widetilde{R}^*(n)}{\rho(n,\varepsilon_k)}\right) - \mathsf{E}f\left(\frac{\widetilde{R}^*(n)}{\rho(n,\varepsilon_k)}\right)\right) \to 0 \quad \text{a.s.} \tag{5.6}$$

as $k \to \infty$. Similar arguments also yield

$$\begin{cases} \sum_{n>B(\varepsilon_k)} \frac{(\log\log n)^b}{n\log n}\bigl(f\bigl(\frac{\sum_{i=1}^n \tilde{X}_{ni}^2}{(1\pm\gamma/2)nl(\tilde{\eta}_n)}\bigr) - \mathsf{E}f\bigl(\frac{\sum_{i=1}^n \tilde{X}_{ni}^2}{(1\pm\gamma/2)nl(\tilde{\eta}_n)}\bigr)\bigr) \to 0 \quad \text{a.s.}, \\ \sum_{n>B(\varepsilon_k)} \frac{(\log\log n)^b}{n\log n} f\bigl(\bigl(\frac{\tilde{S}_{nn}^*}{n\sqrt{\gamma l(\tilde{\eta}_n)}/2)}\bigr) - \mathsf{E}f\bigl(\frac{\tilde{S}_{nn}^*}{n\sqrt{\gamma l(\tilde{\eta}_n)}/2)}\bigr)\bigr) \to 0 \quad \text{a.s.}, \\ \sum_{n>B(\varepsilon_k)} \frac{(\log\log n)^b}{n\log n}\bigl(\sum_{i=1}^n I\{|X_i| > \tilde{\eta}_n\} - n\mathsf{P}(|X| > \tilde{\eta}_n)\bigr) \to 0 \quad \text{a.s.} \end{cases} \tag{5.7}$$

as $k \to \infty$. Denote $\beta(n,\varepsilon) = \varepsilon\sqrt{2n\log\log n}$. Using the inequality (5.4), one has

$$\begin{aligned} &\limsup_{\varepsilon\searrow 0} \varepsilon^{2(b+1)} \sum_{n>K(\varepsilon)} \frac{(\log\log n)^b}{n\log n} I\bigl\{Q(n) \ge \varepsilon\sqrt{2n\log\log n}\bigr\} \\ &\quad\le \limsup_{k\to\infty} \varepsilon_{k-1}^{2(b+1)} \sum_{n>B(\varepsilon_{k-1})} \frac{(\log\log n)^b}{n\log n} I\bigl\{Q(n) \ge \beta(n,\varepsilon_k)\bigr\} \\ &\quad\le \limsup_{k\to\infty} \varepsilon_{k-1}^{2(b+1)} \sum_{n>B(\varepsilon_{k-1})} \frac{(\log\log n)^b}{n\log n} \left(I\left\{\frac{\widetilde{R}^*(n)}{\sqrt{(1-2\gamma)l(\tilde{\eta}_n)}} \ge \beta(n,\varepsilon_k)\right\}\right. \\ &\qquad + I\left\{\sum_{i=1}^n \tilde{X}_{ni}^2 > (1+\gamma/2)nl(\tilde{\eta}_n)\right\} + I\left\{\sum_{i=1}^n \tilde{X}_{ni}^2 < (1-\gamma/2)nl(\tilde{\eta}_n)\right\} \\ &\qquad \left. + 3I\left\{\bigcup_{i=1}^n |X_i| > \tilde{\eta}_n\right\} + I\bigl\{\bigl|\tilde{S}_{nn}^*\bigr| > n\sqrt{\gamma l(\tilde{\eta}_n)}/2\bigr\}\right) \\ &\quad := III + IV + V + VI + VII. \end{aligned} \tag{5.8}$$

We are going to treat the above terms, respectively. In view of (5.5), (5.6), Lemma 5.1 and Proposition 2.2, one has

$$\begin{aligned} III &\le \limsup_{k\to\infty} \varepsilon_{k-1}^{2(b+1)} \sum_{n>B(\varepsilon_{k-1})} \frac{(\log\log n)^b}{n\log n} f_1\left(\frac{\widetilde{R}^*(n)}{\sqrt{l(\tilde\eta_n)}\beta(n,\varepsilon_k)}\right) \\ &\le \limsup_{k\to\infty} \varepsilon_{k-1}^{2(b+1)} \sum_{n>B(\varepsilon_{k-1})} \frac{(\log\log n)^b}{n\log n} \mathsf{E} f_1\left(\frac{\widetilde{R}^*(n)}{\sqrt{l(\tilde\eta_n)}\beta(n,\varepsilon_k)}\right) \\ &\le \limsup_{k\to\infty} \varepsilon_{k-1}^{2(b+1)} \sum_{n>B(\varepsilon_{k-1})} \frac{(\log\log n)^b}{n\log n} \cdot \mathsf{P}\left(\frac{\widetilde{R}^*(n)}{\sqrt{l(\tilde\eta_n)}\beta(n,\varepsilon_k)} \ge 1-2\gamma\right) \\ &\le \limsup_{k\to\infty} \varepsilon_{k}^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log\log n)^b}{n\log n} \\ &\quad \cdot \mathsf{P}\left(Y \ge \varepsilon_k(1-2\gamma)\sqrt{2\log\log n} - 1/(\log\log n)^p\right) \\ &\le \frac{\mathsf{E}Y^{2(b+1)}}{2^{b+1}(b+1)(1-2\gamma)^{2(b+1)}}, \quad \text{a.s.} \end{aligned} \tag{5.9}$$

since

$$\varepsilon_k - \frac{1}{\sqrt{2}(1-2\gamma)(\log\log n)^{p+1/2}} \sim \varepsilon_k \quad \text{as } n\to\infty.$$

Applying (5.5), (5.7), and Bernstein's inequality, one has, for any $\nu' > 1$,

$$\begin{aligned} 0 \le IV &\le \limsup_{k\to\infty} \varepsilon_{k-1}^{2(b+1)} \sum_{n>B(\varepsilon_{k-1})} \frac{(\log\log n)^b}{n\log n} \\ &\quad \cdot \mathsf{P}\left(\sum_{i=1}^n \tilde X_{ni}^2 > (1+\gamma/4)nl(\tilde\eta_n)\right) \\ &\le \limsup_{k\to\infty} \varepsilon_{k-1}^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log\log n)^b}{n\log n} \cdot \mathsf{P}\left(\sum_{i=1}^n \tilde X_{ni}^2 > (1+\gamma/4)nl(\tilde\eta_n)\right) \\ &\le \limsup_{k\to\infty} \varepsilon_{k-1}^{2(b+1)} \sum_{n=1}^{\infty} \frac{C}{n\log n(\log\log n)^{1+\nu'}} \\ &= 0, \quad \text{a.s.} \end{aligned} \tag{5.10}$$

Similarly, one can prove

$$V = 0, \quad \text{a.s.} \tag{5.11}$$

For the fourth part of (5.8), by similar arguments to those used in (5.9) and Lemma 3.4, we have

$$\begin{aligned} 0 \le VI &\le 3\limsup_{k\to\infty} \varepsilon_{k-1}^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log\log n)^b}{\log n} \mathsf{P}\left(|X| > \tilde\eta_n/2\right) \\ &= 0, \quad \text{a.s.,} \end{aligned}$$

and the details are omitted here. As for the fifth part of (5.8), one can easily show that, for any fixed $\gamma > 0$,

$$\begin{aligned} 0 \le VII &\le \limsup_{k\to\infty} \varepsilon_{k-1}^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log\log n)^b}{n\log n} \mathsf{P}\big(\tilde{S}_{nn}^{*} > n\sqrt{\gamma l(\tilde{\eta}_n)}/4\big) \\ &\le C \limsup_{k\to\infty} \varepsilon_{k-1}^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log\log n)^b}{n\log n} \cdot \frac{nl(\tilde{\eta}_n)}{n^2 l(\tilde{\eta}_n)} \\ &= 0, \quad \text{a.s.} \end{aligned} \tag{5.12}$$

Hence, it follows from (5.8)-(5.12) that

$$\begin{aligned} &\limsup_{\varepsilon\searrow 0} \varepsilon^{2(b+1)} \sum_{n>K(\varepsilon)} \frac{(\log\log n)^b}{n\log n} I\big\{Q(n) \ge \varepsilon\sqrt{2n\log\log n}\big\} \\ &\quad \le \frac{\mathsf{E}Y^{2(b+1)}}{2^{b+1}(b+1)(1-2\gamma)^{2(b+1)}}, \quad \text{a.s.} \end{aligned} \tag{5.13}$$

On the other hand,

$$\begin{aligned} &\limsup_{\varepsilon\searrow 0} \varepsilon^{2(b+1)} \sum_{n\le K(\varepsilon)} \frac{(\log\log n)^b}{n\log n} I\big\{Q(n) \ge \varepsilon\sqrt{2n\log\log n}\big\} \\ &\quad \le \limsup_{\varepsilon\searrow 0} \varepsilon^{2(b+1)} \sum_{n\le K(\varepsilon)} \frac{(\log\log n)^b}{n\log n} \\ &\quad \le \frac{1}{M^{b+1}}. \end{aligned} \tag{5.14}$$

By (5.13), (5.14) and the arbitrarinesses of $M$ and $\gamma$, one has

$$\limsup_{\varepsilon\searrow 0} \varepsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log\log n)^b}{n\log n} I\big\{Q(n) \ge \varepsilon\sqrt{2n\log\log n}\big\} \le \frac{\mathsf{E}Y^{2(b+1)}}{2^{b+1}(b+1)}, \quad \text{a.s.}$$

Similarly, one has

$$\liminf_{\varepsilon\searrow 0} \varepsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log\log n)^b}{n\log n} I\big\{Q(n) \ge \varepsilon\sqrt{2n\log\log n}\big\} \ge \frac{\mathsf{E}Y^{2(b+1)}}{2^{b+1}(b+1)}, \quad \text{a.s.}$$

The proof is completed now. □

#### Competing interests
The authors declare that they have no competing interests.

#### Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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