Bhardwaj and Gupta *Journal of Inequalities and Applications* 2014, 2014:12
http://www.journalofinequalitiesandapplications.com/content/2014/1/12

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH | Open Access

On some generalizations of statistical boundedness

Vinod K Bhardwaj[1*] and Sandeep Gupta[2]

*Correspondence: vinodk_bhj@rediffmail.com
[1]Department of Mathematics, Kurukshetra University, Kurukshetra, 136119, India
Full list of author information is available at the end of the article

Abstract

Fridy and Orhan (Proc. Am. Math. Soc. 125(12):3625-3631, 1997) introduced the concepts of statistical boundedness, statistical limit superior, statistical limit inferior, and they established an analog of Knopp's Core Theorem. In the present paper, we examine the concept of statistical boundedness to establish statistical analogs of various well-known results concerning boundedness and generalize the concept of statistical boundedness by introducing the concepts of statistical boundedness of order α, λ-statistical boundedness, and λ-statistical boundedness of order α.

MSC: 40C05; 40A05; 46A45

Keywords: sequence space; statistical boundedness; statistical convergence; Köthe-Toeplitz duals

1 Introduction

The idea of statistical convergence was given by Zygmund [1] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was formally introduced by Steinhaus [2] and Fast [3] and later reintroduced by Schoenberg [4]. Over the years and under different names statistical convergence has been applied in the theory of Fourier analysis, ergodic theory, number theory, measure theory, trigonometric series, turnpike theory, and Banach spaces. Later on it was further investigated from the sequence space point of view and linked with summability theory by Šalàt [5], Fridy [6], Connor [7], Fridy and Orhan [8], Rath and Tripathy [9], Mòricz [10], Malkowsky and Savas [11], Mursaleen [12], Et [13], Bhardwaj and Bala [14, 15], Savas [16–19], Çolak [20], Savas and Mohiuddine [21], Mohiuddine *et al.* [22], and many others.

Statistical convergence depends on the density of subsets of the set $\mathbb{N} = \{1, 2, 3, \dots\}$. The natural density of a subset A of $\mathbb{N}$ is defined by

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} \bigl|\{k \le n : k \in A\}\bigr|,$$

where the vertical bars indicate the total number of elements in the enclosed set [23]. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta(A^c) = 1 - \delta(A)$, where $A^c = \mathbb{N} - A$. A set A is said to be statistically dense [24] if $\delta(A) = 1$. A subsequence of a sequence is said to be statistically dense if the set of all indices of its elements is statistically dense. A sequence (x_k) (of real or complex numbers) is said to be statistically convergent to some number L if for every $\epsilon > 0$, $\delta(\{k \in \mathbb{N} : |x_k - L| > \epsilon\}) = 0$. In this case, we write $st\text{-}\lim_k x_k = L$. We shall denote the set of all statistically convergent sequences by $[S]$.

Throughout this paper, we deal with sequences of scalars. We recall [6] that if $x = (x_k)$ is a sequence such that x_k satisfies property P for all k except a set of natural density zero, then we say that $x = (x_k)$ satisfies P for 'almost all k' and we abbreviate this by 'a.a.k'.

The following concept is due to Fridy [6]. A sequence (x_k) is said to be statistically Cauchy if for each $\epsilon > 0$, there exists a number $N = N(\epsilon)$ such that $|x_k - x_N| < \epsilon$, for a.a.k, that is, $\delta(\{k \in \mathbb{N} : |x_k - x_N| \geq \epsilon\}) = 0$. It was shown that a number sequence is statistically convergent if and only if it is statistically Cauchy.

Fridy [25] enriched the theory of statistical convergence by introducing the concept of statistical limit point and statistical cluster point as follows: If $\{x_{k(j)}\}$ is a subsequence of $x = (x_k)$ and $K = \{k(j) : j \in \mathbb{N}\}$, then abbreviate $(x_{k(j)})$ by $(x)_K$. In case $\delta(K) \neq 0$, $(x)_K$ is called a nonthin subsequence of x. The number ζ is a statistical limit point of the number sequence x provided that there is a nonthin subsequence of x that converges to ζ. The number γ is a statistical cluster point of the number sequence x provided that for every $\epsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - \gamma| < \epsilon\}$ does not have density zero.

In 2000, Mursaleen [12] generalized the concept of statistical convergence by introducing the concept of λ-statistical convergence as follows. A sequence $x = (x_k)$ is said to be λ-statistically convergent or S_λ-convergent to L if for every $\epsilon > 0$

$$\lim_n \frac{1}{\lambda_n} \left|\left\{k \in I_n : |x_k - L| > \epsilon\right\}\right| = 0,$$

where $I_n = [n - \lambda_n + 1, n]$ and $\lambda = (\lambda_n)$ is a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$. We shall denote the set of all such sequences (λ_n) by Γ and the set of all λ-statistically convergent sequences by S_λ.

Generalizing the concept of statistical convergence, Çolak [26] in 2010 introduced the concept of statistical convergence of order α by defining the α-density $\delta_\alpha(E)$ of a subset E of $\mathbb{N}$ as follows:

$$\delta_\alpha(E) = \lim_n \frac{1}{n^\alpha} \left|\{k \leq n : k \in E\}\right|,$$

provided the limit exists. It is clear that any finite subset of $\mathbb{N}$ has zero α-density. Note that the α-density of any set reduces to the natural density of the set in the case $\alpha = 1$.

Let $0 < \alpha \leq 1$ be given. A sequence $x = (x_k)$ is said to be statistically convergent of order α if there is a complex number L such that

$$\lim_n \frac{1}{n^\alpha} \left|\left\{k \leq n : |x_k - L| \geq \epsilon\right\}\right| = 0$$

for every $\epsilon > 0$. The set of all statistically convergent sequences of order α will be denoted by $[S]^\alpha$.

In 2011, Çolak and Bektaş [27] introduced the concept of λ-statistical convergence of order α as follows: A sequence (x_k) is said to be λ-statistically convergent of order α to L, if for given $\epsilon > 0$

$$\lim_{n \to \infty} \frac{1}{\lambda_n^\alpha} \left|\left\{k \in I_n : |x_k - L| > \epsilon\right\}\right| = 0,$$

where $I_n = [n - \lambda_n + 1, n]$, $\lambda = (\lambda_n) \in \Gamma$ and $0 < \alpha \leq 1$. The concept of λ-statistical convergence of order α is a generalization of the concepts of statistical convergence, λ-statistical convergence, and statistical convergence of order α.

Quite recently, the concepts of 'λ-statistical convergence of order α for the sequences of function' and 'the pointwise and uniform statistical convergence of order α' have been introduced by Et *et al.* [28] and Çinar *et al.* [29], respectively.

In 1997, a statistical analog of a very basic property of convergent sequences was given by Fridy and Orhan [30] by the formal introduction of the concept of statistical boundedness as follows: 'The real number sequence x is statistically bounded if there is a number B such that $\delta(\{k : |x_k| > B\}) = 0$'.

They developed the statistical analogs of properties of the ordinary limit superior and inferior, and a natural analog of Knopp's core. In the same year, Tripathy [31] proved the Decomposition Theorem for statistically bounded sequences and established the necessary and sufficient condition for a sequence to be statistically bounded.

In the present paper, we take the opportunity to contribute to the theory of statistical convergence, by introducing the concepts of λ-statistical boundedness, statistical boundedness of order α, and λ-statistical boundedness of order α. In the second section of this paper, we study and examine the concept of statistical boundedness to establish statistical analogs of various well-known results concerning bounded sequences. The third section is devoted to the introduction of the concept of statistical boundedness of order α where $0 < \alpha \leq 1$. The relationship between statistical boundedness of order α and statistical boundedness of order β for $\alpha \leq \beta$ has been studied. In the fourth section of this paper, we introduce the concept of λ-statistical boundedness and prove inclusion relations between $S(b)$, $S_\lambda(b)$ and $S_\mu(b)$ under different conditions. In the last section, we introduce the concept of λ-statistical boundedness of order α and establish the condition for a statistically bounded sequence to be λ-statistically bounded of order α. It is observed that the concept of λ-statistical boundedness of order α is more general than the concepts of statistical boundedness, λ-statistical boundedness, and statistical boundedness of order α as it includes these concepts for some particular choices of α and λ.

Before proceeding to establish the proposed results, we recall some more definitions, which may be found in [32, 33].

A sequence space X is called

(i) normal (or solid) if $y = (y_k) \in X$ whenever $|y_k| \leq |x_k|$, $k \geq 1$, for some $x = (x_k) \in X$,
(ii) monotone if it contains the canonical preimages of all its stepspaces,
(iii) sequence algebra if $xy = (x_k y_k) \in X$ whenever $x = (x_k), y = (y_k) \in X$,
(iv) symmetric if $(x_k) \in X$ implies $(x_{\pi(k)}) \in X$ where π is a permutation on $\mathbb{N}$.

The idea of dual sequence spaces was introduced by Köthe and Toeplitz [34] whose main results concerned α-duals; the α-dual of a sequence space X being defined as

$$X^\alpha = \left\{ a = (a_k) : \sum_k |a_k x_k| < \infty \text{ for all } x = (x_k) \in X \right\}.$$

In the same paper [34], they also introduced another kind of dual, namely, the β-dual (see [35] also, where it is called the g-dual by Chillingworth) defined as

$$X^\beta = \left\{ a = (a_k) : \sum_k a_k x_k \text{ converges for all } x = (x_k) \in X \right\}.$$

Obviously $\phi \subset X^\alpha \subset X^\beta$, where ϕ is the well-known sequence space of finitely non-zero scalar sequences. Also if $X \subset Y$, then $Y^\eta \subset X^\eta$ for $\eta = \alpha$ or β. For any sequence space X, we denote $(X^\delta)^\eta$ by $X^{\delta\eta}$ where $\delta, \eta = \alpha$ or β. It is clear that $X \subset X^{\eta\eta}$ where $\eta = \alpha$ or β.

For a sequence space X, if $X = X^{\alpha\alpha}$ then X is called a Köthe space or a perfect sequence space.

2 Some more results on statistical boundedness

In the present section, we examine the concept of statistical boundedness in greater detail by giving statistical analogs of some of the results concerning boundedness.

Definition 2.1 [30] A sequence $x = (x_k)$ is said to be statistically bounded if there exists some $L \geq 0$ such that

$$\delta\bigl(\{k : |x_k| > L\}\bigr) = 0, \quad \text{i.e., } |x_k| \leq L \text{ a.a.k.}$$

By $S(b)$, we shall denote the linear space of all statistically bounded sequences.

Bounded sequences are obviously statistically bounded as the empty set has zero natural density. However, the converse is not true, as the following example demonstrates.

Example 2.2 Consider the sequence $x = (x_k)$ where

$$x_k = \begin{cases} k, & \text{if } k \text{ is a square;} \\ 0, & \text{if } k \text{ is not a square.} \end{cases}$$

Clearly (x_k) is not a bounded sequence. However, $\delta(\{k : |x_k| > \frac{1}{2}\}) = 0$, as the set of squares has zero natural density and hence (x_k) is statistically bounded.

Proposition 2.3 *Every convergent sequence is statistically bounded.*

Although a statistically convergent sequence does not need to be bounded, the following proposition shows that every statistical convergent sequence is statistically bounded.

Proposition 2.4 *Every statistical convergent sequence is statistically bounded.*

Proof Let (x_k) converges to L statistically. So for each $\epsilon > 0$, $\delta(\{k : |x_k - L| > \epsilon\}) = 0$. As $\{k : |x_k| > |L| + \epsilon\} \subset \{k : |x_k - L| > \epsilon\}$, so $|x_k| \leq |L| + \epsilon$ a.a.k. □

The following example illustrates that not every sequence is statistically bounded.

Example 2.5 Consider the sequence $x = (x_k) = (1, 2, 3, \ldots)$. Then for any $L \geq 0$, we have $\{k : |x_k| > L\} = \mathbf{N} - S$, where S is a finite subset of $\mathbf{N}$ and so $\delta(\{k : |x_k| > L\}) = 1$. Thus (x_k) is not statistically bounded.

It is well known that every subsequence of a bounded sequence is bounded. However, for statistically bounded sequences this is no longer true. For example, the sequence $(y_k) = (1, 4, 9, \ldots)$ is a subsequence of the statistically bounded sequence (x_k), as given in Example 2.2 but (y_k) is not statistically bounded.

A useful characterization of a statistically bounded sequence in terms of its bounded subsequence has been given by Tripathy [31] as follows.

Theorem 2.6 [31] *(ξ_k) is statistically bounded if and only there exists such a set $K = \{k_1 < k_2 < k_3 < \cdots < k_n < \cdots\} \subset \mathbb{N}$ such that $\delta(K) = 1$ and $\{\xi_{k_n}\}_{n\in\mathbb{N}} \in \ell_\infty$, the space of all bounded scalar sequences.*

We give another characterization of a statistically bounded sequence in terms of its statistically dense subsequences as follows.

Theorem 2.7 *A sequence is statistically bounded if and only if every statistically dense subsequence of it is statistically bounded.*

The proof is similar to that of the corresponding result for statistical convergence in [24, Theorem 2.1] and hence is omitted.

We now give a statistical analog of the well-known result that 'If (x_k) is a null sequence and (y_k) is a bounded sequence, then $(x_k y_k)$ is a null sequence.'

Proposition 2.8 *If (x_k) converges statistically to 0 and $(y_k) \in S(b)$, then $(x_k y_k)$ converges statistically to 0.*

Fridy [6] has shown that a sequence $x = (x_k)$ is statistically convergent if and only if there exists a convergent sequence $y = (y_k)$ such that $x_k = y_k$ a.a.k. We now establish a similar result for statistically bounded sequences. Our proof is quite different from that of Fridy.

Theorem 2.9 *$x = (x_k)$ is a statistically bounded sequence if and only if there exists a bounded sequence $y = (y_k)$ such that $x_k = y_k$ a.a.k.*

Proof First suppose $x = (x_k)$ is a statistically bounded sequence. Then there exists $L \geq 0$ such that $\delta(A) = 0$ where $A = \{k \in \mathbb{N} : |x_k| > L\}$. Let

$$y_k = \begin{cases} x_k, & \text{if } k \notin A, \\ 0, & \text{otherwise.} \end{cases}$$

Then $y = (y_k) \in \ell_\infty$ and $y_k = x_k$ a.a.k.

Conversely, as $y = (y_k) \in \ell_\infty$ so there exists $L \geq 0$ such that $|y_k| \leq L$ for all $k \in \mathbb{N}$. Let $D = \{k \in \mathbb{N} : x_k \neq y_k\}$. Clearly $\delta(D) = 0$. This gives $|x_k| \leq L$ a.a.k because $\{k \in \mathbb{N} : |x_k| > L\} \subset D$. □

Remark 2.10 Tripathy [31] has shown that $S(b) \subset \ell_\infty + \overline{c_0}$ where $\overline{c_0}$ and ℓ_∞ denote the spaces of statistically null and bounded sequences, respectively. We observe that $\ell_\infty + \overline{c_0} = S(b)$. But the representation of every element of $S(b)$ as the sum of an element of ℓ_∞ and of $\overline{c_0}$ is not unique, *i.e.*, $S(b) \neq \ell_\infty \oplus \overline{c_0}$ as $\ell_\infty \cap \overline{c_0} \neq \{0\}$. In fact, $\ell_\infty \cap \overline{c_0} \supset c_0$ where c_0 is the space of null sequences.

Our next result is a statistical analog of a well-known result stating that every Cauchy sequence is bounded but not conversely.

Theorem 2.11 *Every statistically Cauchy sequence is statistically bounded, but the converse does not need to be true.*

Proof Let $x = (x_k)$ be a statistically Cauchy sequence. Then for each $\epsilon > 0$, there exists a number N $(= N(\epsilon))$ such that $|x_k - x_N| < \epsilon$ a.a.k. This implies that $|x_k| < L$ a.a.k, where $L = \epsilon + |x_N|$.

The sequence $x = (x_k) = (-1, 1, -1, 1, \dots)$ is bounded and hence statistically bounded but is not statistically Cauchy. □

Proposition 2.12 [25] *Suppose x is a number sequence that has a monotone subsequence $(x)_M$. If $\delta(M) = 1$ and x is bounded on M, then x is statistically convergent.*

Corollary 2.13 *Every monotone and statistically bounded sequence is statistically convergent.*

Theorem 3 of [25] can be restated in terms of statistical boundedness as follows.

Theorem 2.14 *Every statistically bounded sequence has a statistical cluster point.*

Theorem 2.15

(a) *$S(b)$ is not symmetric.*
(b) *$S(b)$ is normal and hence monotone.*
(c) *$S(b)$ is a sequence algebra.*

Proof

(i) Let $x = (1, 0, 0, 0, 4, 0, 0, 0, 0, 9, \dots) \in S(b)$ and $y = (0, 1, 4, 0, 9, 16, 25, 36, 0, \dots)$ be a rearrangement of terms of the sequence (x_k). Clearly for any $L \geq 0$, $\delta(\{k : |y_k| > L\}) \neq 0$.
(ii) Let $x = (x_k) \in S(b)$ and $y = (y_k)$ be such that $|y_k| \leq |x_k|$ for all $k \in \mathbf{N}$. As $(x_k) \in S(b)$, there exists $L \geq 0$ such that $\delta(\{k : |x_k| > L\}) = 0$. Clearly $y = (y_k) \in S(b)$ as $\{k : |y_k| > L\} \subset \{k : |x_k| > L\}$. In view of the fact [33] that every normal space is monotone, $S(b)$ is monotone.
(iii) Let $(x_k), (y_k) \in S(b)$. Then there exists $L, M \geq 0$ such that $\delta(\{k : |x_k| \geq L\}) = 0$ and $\delta(\{k : |y_k| \geq M\}) = 0$. The proof follows from the fact that $\{k : |x_k y_k| \geq LM\} \subset \{k : |x_k| \geq L\} \cup \{k : |y_k| \geq M\}$. □

Proposition 2.16 *$S(b)^\alpha = S(b)^\beta = \phi$, the space of finitely non-zero scalar sequences.*

Proof To show that $S(b)^\alpha = \phi$, it is sufficient to show that $S(b)^\alpha \subset \phi$ since $\phi \subset S(b)^\alpha$ obviously. Let $(a_k) \in S(b)^\alpha$. Then $\sum_k |a_k x_k| < \infty$ for all $x = (x_k) \in S(b)$. Let, if possible, $(a_k) \notin \phi$, *i.e.*, (a_k) has infinitely many non-zero terms. Following [6, p.7], we choose an increasing sequence (k_i) of positive integers such that for each i, $k_i > i^2$ and $a_{k_i} \neq 0$. Define $x = \{x_k\}$ as

$$x_{k_i} = \begin{cases} \frac{1}{a_{k_i}}, & i \in \mathbf{N}, \\ 0, & \text{otherwise}. \end{cases}$$

Then $|\{k \leq n : |x_k| > 0\}| \leq \sqrt{n}$ and so $(x_k) \in S(b)$. But $\sum_k |a_k x_k| = \sum_i 1 = \infty$, which is a contradiction to the fact that $(a_k) \in S(b)^\alpha$. Hence $(a_k) \in \phi$.

In view of the fact [33] that for a monotone sequence space α- and β-dual spaces coincide, we have $S(b)^\beta = S(b)^\alpha = \phi$. □

The proofs of the following Tauberian theorems are similar to the proofs of Theorem 3 and Theorem 5 in [6] and hence are omitted.

Theorem 2.17 *If $x = (x_k)$ is a sequence such that $(x_k) \in S(b)$ and $\Delta x_k = O(\frac{1}{k})$, then $(x_k) \in \ell_\infty$.*

Theorem 2.18 *Let $(k(i))_{i\in\mathbb{N}}$ be an increasing sequence of positive integers such that $\liminf_i \frac{k(i+1)}{k(i)} > 1$ and let x be a corresponding gap sequence: $\Delta x_k = 0$ if $k \neq k(i)$ for $i = 1, 2, \dots$; if $(x_k) \in S(b)$, then $(x_k) \in \ell_\infty$.*

3 Statistical boundedness of order α

In the present section, we introduce and examine the concept of statistical boundedness of order α, where α is any real number such that $0 < \alpha \leq 1$.

Definition 3.1 Let $0 < \alpha \leq 1$ be given. A sequence $x = (x_k)$ is said to be statistically bounded of order α if

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \left|\left\{k \leq n : |x_k| > M\right\}\right| = 0, \quad \text{for some } M > 0.$$

By $[S(b)]^\alpha$, we shall denote the linear space of all statistically bounded sequences of order α. For $\alpha = 1$, the statistical boundedness of order α is the same as the statistical boundedness. We do not talk of statistical boundedness of order α, $\alpha > 1$, since in this case $[S(b)]^\alpha = \omega$, the space of all scalar sequences, *i.e.*, statistical boundedness, becomes trivial.

Theorem 3.2 $\ell_\infty \subset [S(b)]^\alpha$ *for all α, $0 < \alpha \leq 1$ and the inclusion is strict.*

Proof The proof follows from the fact that the α-density of an empty set is zero. For strict inclusion, consider $x = (x_k)$, defined by

$$x_k = \begin{cases} k, & k = n^2, \\ 0, & k \neq n^2 \text{ for } k \in \mathbb{N}. \end{cases}$$

Then $(x_k) \notin \ell_\infty$. But $\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n : |x_k| > 0\}| = 0$, for $\alpha > \frac{1}{2}$ and so $(x_k) \in [S(b)]^\alpha$ for $\alpha > \frac{1}{2}$. □

Now, one may ask a natural question: Is there any unbounded sequence which is not statistically bounded of order α, for any α ($0 < \alpha \leq 1$)? The answer is yes. In fact the sequence $(x_k) = (1, 2, 3, \dots)$ is an unbounded sequence but there does not exist any α, $0 < \alpha \leq 1$ such that (x_k) is statistically bounded of order α.

Theorem 3.3 *Let $0 < \alpha \leq \beta \leq 1$. Then $[S(b)]^\alpha \subset [S(b)]^\beta$ and the inclusion is strict for $\alpha < \beta$.*

Proof If $0 < \alpha \leq \beta \leq 1$, then

$$\lim_{n\to\infty} \frac{1}{n^\beta} \left|\left\{k \leq n : |x_k| > M\right\}\right| \leq \lim_{n\to\infty} \frac{1}{n^\alpha} \left|\left\{k \leq n : |x_k| > M\right\}\right|$$

and this gives $[S(b)]^\alpha \subset [S(b)]^\beta$. The inclusion is strict in view of the example cited in Theorem 3.2 as $(x_k) \in [S(b)]^\beta$ for $\frac{1}{2} < \beta \leq 1$, but $(x_k) \notin [S(b)]^\alpha$ for $0 < \alpha \leq \frac{1}{2}$. □

Corollary 3.4

(a) $[S(b)]^\alpha = [S(b)]^\beta$ *if and only if* $\alpha = \beta$.

(b) $[S(b)]^\alpha \subset S(b)$ *i.e., every statistically bounded sequence of order* α $(0 < \alpha \leq 1)$ *is statistically bounded. In fact* $[S(b)]^\alpha = S(b)$ *if and only if* $\alpha = 1$.

The next theorem shows that a statistically convergent sequence of order α is statistically bounded of order α $(0 < \alpha \leq 1)$.

Theorem 3.5 *For* $0 < \alpha \leq 1$, $[S]^\alpha \subset [S(b)]^\alpha$, *the inclusion being strict.*

Proof Let $(x_k) \in [S]^\alpha$. Then there exists some $L \in \mathbb{C}$, such that

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \left|\left\{k \leq n : |x_k - L| \geq \epsilon\right\}\right| = 0, \quad \text{for every } \epsilon > 0.$$

Now the result follows from the fact that

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \left|\left\{k \leq n : |x_k| > |L| + \epsilon\right\}\right| \leq \lim_{n\to\infty} \frac{1}{n^\alpha} \left|\left\{k \leq n : |x_k - L| \geq \epsilon\right\}\right|.$$

For strict inclusion, we see $(x_k) \in [S(b)]^\alpha$ but $(x_k) \notin [S]^\alpha$ where

$$x_k = \begin{cases} 1, & k = 2n, \\ -1, & k \neq 2n; k, n \in \mathbb{N}. \end{cases}$$

□

4 λ-Statistical boundedness

In the present section, we introduce and examine the concept of λ-statistical boundedness and obtain the condition on $\lambda = (\lambda_n) \in \Gamma$ for which the concept of λ-statistical boundedness coincides with the concept of statistical boundedness.

Definition 4.1 Let $\lambda = (\lambda_n) \in \Gamma$. A sequence (x_k) is said to be λ-statistically bounded if there exists $M \geq 0$ such that

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left|\left\{k \in I_n : |x_k| > M\right\}\right| = 0,$$

where $I_n = [n - \lambda_n + 1, n]$. For $\lambda_n = n$, λ-statistical boundedness is the same as statistical boundedness. By $S_\lambda(b)$, we shall denote the linear space of all λ-statistically bounded sequences.

Theorem 4.2 $\ell_\infty \subset S_\lambda(b)$, *the inclusion being strict.*

Proof The inclusion is strict in view of the sequence $x = (x_k)$:

$$x_k = \begin{cases} k, & k = n^2, \\ 0, & k \neq n^2 \text{ for } k \in \mathbb{N} \end{cases}$$

and for $\lambda = (n)$. □

Theorem 4.3 *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Γ such that $\lambda_n \leq \mu_n$ for all $n \geq n_0$.*

(a) *If $\liminf_{n\to\infty} \frac{\lambda_n}{\mu_n} > 0$, then $S_\mu(b) \subset S_\lambda(b)$.*
(b) *If $\lim_{n\to\infty} \frac{\lambda_n}{\mu_n} = 1$, then $S_\lambda(b) \subset S_\mu(b)$.*
(c) *If $\lim_{n\to\infty} \frac{\lambda_n}{\mu_n} = 1$, then $S_\lambda(b) = S_\mu(b)$.*

Proof (a) Let $\liminf_{n\to\infty} \frac{\lambda_n}{\mu_n} > 0$ and $(x_k) \in S_\mu(b)$. Then there exists $M > 0$ such that

$$\lim_{n\to\infty} \frac{1}{\mu_n} \left|\left\{k \in I'_n : |x_k| > M\right\}\right| = 0,$$

where $I'_n = [n - \mu_n + 1, n]$. Now $\{k \in I'_n : |x_k| > M\} \supset \{k \in I_n : |x_k| > M\}$ for all $n \geq n_0$ and so $\frac{1}{\mu_n}|\{k \in I'_n : |x_k| > M\}| \geq \frac{\lambda_n}{\mu_n}\frac{1}{\lambda_n}|\{k \in I_n : |x_k| > M\}|$ which in turn implies that $\lim_{n\to\infty} \frac{1}{\lambda_n}|\{k \in I_n : |x_k| > M\}| = 0$.

(b) Let $(x_k) \in S_\lambda(b)$. Then

$$\begin{aligned}
\frac{1}{\mu_n}\left|\left\{k \in I'_n : |x_k| > M\right\}\right| &= \frac{1}{\mu_n}\left|\left\{k \in I'_n - I_n : |x_k| > M\right\}\right| + \frac{1}{\mu_n}\left|\left\{k \in I_n : |x_k| > M\right\}\right| \\
&\leq \frac{1}{\mu_n}(\mu_n - \lambda_n) + \frac{1}{\lambda_n}\left|\left\{k \in I_n : |x_k| > M\right\}\right| \\
&= \left(1 - \frac{\lambda_n}{\mu_n}\right) + \frac{1}{\lambda_n}\left|\left\{k \in I_n : |x_k| > M\right\}\right| \quad \text{for all } n \geq n_0 \\
&\to 0 \quad \text{as } n \to \infty
\end{aligned}$$

and so $(x_k) \in S_\mu(b)$.

(c) The proof follows from parts (a) and (b). □

Corollary 4.4 *Let $\lambda = (\lambda_n) \in \Gamma$.*

(a) *If $\liminf_{n\to\infty} \frac{\lambda_n}{n} > 0$, then $S(b) \subset S_\lambda(b)$.*
(b) *If $\lim_{n\to\infty} \frac{\lambda_n}{n} = 1$, then $S_\lambda(b) \subset S(b)$ and hence $S_\lambda(b) = S(b)$.*

Theorem 4.5 *Every λ-statistically convergent sequence is λ-statistically bounded.*

The proof is easy and therefore is omitted.

5 λ-Statistical boundedness of order α

In this last section, we generalize the concepts of statistical boundednes, statistical boundedness of order α, and λ-statistical boundedness by introducing the concept of λ-statistical boundedness of order α.

Definition 5.1 Let $\lambda = (\lambda_n) \in \Gamma$ and $0 < \alpha \leq 1$. A sequence (x_k) is said to be λ-statistically bounded of order α if there exists $M \geq 0$ such that

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} \left|\left\{k \in I_n : |x_k| > M\right\}\right| = 0,$$

where $I_n = [n - \lambda_n + 1, n]$. By $[S_\lambda(b)]^\alpha$, we denote the linear space of all λ-statistically bounded sequences of order α. We observe that λ-statistical boundedness of order α

(a) for $\lambda_n = n$ is the same as the statistical boundedness of order α,
(b) for $\alpha = 1$ is the same as the λ-statistical boundedness,
(c) for $\lambda_n = n$, $\alpha = 1$ is the same as the statistical boundedness.

Theorem 5.2 *$\ell_\infty \subset [S_\lambda(b)]^\alpha$, the inclusion being strict.*

The proof is easy and therefore is omitted.

The proof of the following theorem is similar to the proof of Theorem 3.3 in [28] and therefore is omitted.

Theorem 5.3 *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Γ such that $\lambda_n \leq \mu_n$ for all $n \geq n_0$ and $0 < \alpha \leq \beta \leq 1$.*
(a) *If $\liminf_{n\to\infty} \frac{\lambda_n^\alpha}{\mu_n^\beta} > 0$, then $[S_\mu(b)]^\beta \subset [S_\lambda(b)]^\alpha$.*
(b) *If $\lim_{n\to\infty} \frac{\lambda_n^\alpha}{\mu_n^\beta} = 1$ and $\lim_{n\to\infty} \frac{\mu_n}{\mu_n^\beta} = 1$, then $[S_\lambda(b)]^\alpha \subset [S_\mu(b)]^\beta$.*

Corollary 5.4 *Let $\lambda \in \Gamma$ and $0 < \alpha \leq 1$.*
(a) *If $\liminf_{n\to\infty} \frac{\lambda_n^\alpha}{n} > 0$, then $S(b) \subset [S_\lambda(b)]^\alpha$.*
(b) *If $\lim_{n\to\infty} \frac{\lambda_n^\alpha}{n} = 1$, then $[S_\lambda(b)]^\alpha \subset S(b)$ and so $[S_\lambda(b)]^\alpha = S(b)$.*

Theorem 5.5 *Every λ-statistically convergent sequence of order α is λ-statistically bounded of order α.*

The proof is easy and therefore is omitted.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
VKB and SG contributed equally. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Kurukshetra University, Kurukshetra, 136119, India. [2]Department of Mathematics, Arya P. G. College, Panipat, 132103, India.

Acknowledgements
The authors wish to thank the referees for their valuable suggestions, which have improved the presentation of the paper.

Received: 19 September 2013 Accepted: 18 December 2013 Published: 09 Jan 2014

References
1. Zygmund, A: Trigonometric Series, 2nd edn. Cambridge University Press, Cambridge (1979)
2. Steinhaus, H: Sur la convergence ordinaire et la convergence asymptotique. Colloq. Math. **2**, 73-74 (1951)
3. Fast, H: Sur la convergence statistique. Colloq. Math. **2**, 241-244 (1951)
4. Schoenberg, IJ: The integrability of certain functions and related summability methods. Am. Math. Mon. **66**, 361-375 (1959)
5. Šalàt, T: On statistically convergent sequences of real numbers. Math. Slovaca **30**(2), 139-150 (1980)
6. Fridy, JA: On statistical convergence. Analysis **5**, 301-313 (1985)
7. Connor, JS: The statistical and strong *p*-Cesàro convergence of sequences. Analysis **8**, 47-63 (1988)
8. Fridy, JA, Orhan, C: Lacunary statistical convergence. Pac. J. Math. **160**, 43-51 (1993)
9. Rath, D, Tripathy, BC: On statistically convergent and statistically Cauchy sequences. Indian J. Pure Appl. Math. **25**(4), 381-386 (1994)
10. Mòricz, F: Statistical convergence of multiple sequences. Arch. Math. **81**, 82-89 (2003)
11. Malkowsky, E, Savas, E: Some λ-sequence spaces defined by a modulus. Arch. Math. **36**(3), 219-228 (2000)
12. Mursaleen, M: λ-Statistical convergence. Math. Slovaca **50**(1), 111-115 (2000)
13. Et, M: Strongly almost summable difference sequences of order *m* defined by a modulus. Studia Sci. Math. Hung. **40**(4), 463-476 (2003)
14. Bhardwaj, VK, Bala, I: On weak statistical convergence. Int. J. Math. Math. Sci. **2007**, Article ID 38530 (2007)
15. Bhardwaj, VK, Bala, I: On lacunary generalized difference sequence spaces defined by Orlicz functions in a seminormed space and Δ_q^m-lacunary statistical convergence. Demonstr. Math. **41**(2), 415-424 (2008)

16. Savas, E: Strong almost convergence and almost λ-statistical convergence. Hokkaido Math. J. **29**, 531-536 (2000)
17. Savas, E: On statistically convergent sequences of fuzzy numbers. Inf. Sci. **137**(1-4), 277-282 (2001)
18. Savas, E: On asymptotically λ-statistical equivalent sequences of fuzzy numbers. New Math. Nat. Comput. **3**(3), 301-306 (2007)
19. Savas, E: On $\overline{\lambda}$-statistically convergent double sequences of fuzzy numbers. J. Inequal. Appl. **2008**, Article ID 147827 (2008)
20. Çolak, R: On λ-statistical convergence. In: Conference on Summability and Applications, İstanbul, Turkey, May 12-13 (2011)
21. Savas, E, Mohiuddine, SA: $\overline{\lambda}$-Statistically convergent double sequences in probabilistic normed spaces. Math. Slovaca **62**(1), 99-108 (2012)
22. Mohiuddine, SA, Alotaibi, A, Mursaleen, M: A new variant of statistical convergence. J. Inequal. Appl. **2013**, Article ID 309 (2013)
23. Niven, I, Zuckerman, HS: An Introduction to the Theory of Numbers, 4th edn. Wiley, New York (1980)
24. Burgin, M, Duman, O: Statistical convergence and convergence in statistics. arXiv:math/0612179 [math.GM]
25. Fridy, JA: Statistical limit points. Proc. Am. Math. Soc. **118**, 1187-1192 (1993)
26. Çolak, R: Statistical convergence of order α. In: Modern Methods in Analysis and Its Applications, pp. 121-129. Anamaya Pub., New Delhi (2010)
27. Çolak, R, Bektaş, ÇA: λ-Statistical convergence of order α. Acta Math. Sin. Engl. Ser. **31**(3), 953-959 (2011)
28. Et, M, Çinar, M, Karakaş, M: On λ-statistical convergence of order α of sequences of function. J. Inequal. Appl. **2013**, Article ID 204 (2013)
29. Çinar, M, Karakaş, M, Et, M: On pointwise and uniform statistical convergence of order α for sequences of functions. Fixed Point Theory Appl. **2013**, Article ID 33 (2013)
30. Fridy, JA, Orhan, C: Statistical limit superior and limit inferior. Proc. Am. Math. Soc. **125**(12), 3625-3631 (1997)
31. Tripathy, BC: On statistically convergent and statistically bounded sequences. Bull. Malays. Math. Soc. **20**(1), 31-33 (1997)
32. Cooke, RG: Infinite Matrices and Sequence Spaces. Macmillan & Co., London (1950)
33. Kamthan, PK, Gupta, M: Sequence Spaces and Series. Dekker, New York (1981)
34. Köthe, G, Toeplitz, O: Lineare Räume mit unendlich vielen Koordinaten und Ringe unendlicher Matrizen. J. Reine Angew. Math. **171**, 193-226 (1934)
35. Chillingworth, HR: Generalized 'dual' sequence spaces. Nederl. Akad. Wetensch. Proc. Ser. A 61 - Indag. Math. **20**, 307-315 (1958)

10.1186/1029-242X-2014-12
Cite this article as: Bhardwaj and Gupta: On some generalizations of statistical boundedness. *Journal of Inequalities and Applications* 2014, 2014:12