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A new sum analogous to quadratic Gauss sums and its $2k$th power mean

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Abstract
The main purpose of this paper is using the analytic methods and the properties of Gauss sums to study the computational problem of a new sum analogous to quadratic Gauss sums, and to give an interesting asymptotic formula for its $2k$th power mean.
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Keywords: a sum analogous to quadratic Gauss sums; $2k$th power mean; asymptotic formula

1 Introduction

Let $q \geq 3$ be an integer, and let χ be a Dirichlet character $\bmod q$. Then for any integer n, the famous Gauss sums $G(\chi, n)$ and quadratic Gauss sums $G_2(\chi, n)$ are defined as follows:

$$G(\chi, n) = \sum_{a=1}^{q} \chi(a) \cdot e\left(\frac{na}{q}\right) \quad \text{and} \quad G_2(\chi, n) = \sum_{a=1}^{q} \chi(a) \cdot e\left(\frac{na^2}{q}\right),$$

where $e(y) = e^{2\pi iy}$.

These two sums play a very important role in the study of analytic number theory, and many famous number theoretic problems are closely related to them. The distribution of primes, the Goldbach problem, and the properties of Dirichlet L-functions are some good examples. The arithmetic properties of $G(\chi, n)$ and $G_2(\chi, n)$ can be found in [1, 2], and [3].

The upper bound estimate of $G_2(\chi, n)$ has been studied by some authors, and one obtained many important results. For example, if $q = p$ is a prime and $(p, m) = 1$, then from Weil's work [4] we can obtain the estimate

$$\left| \sum_{a=1}^{p} \chi(a) \cdot e\left(\frac{f(a)}{p}\right) \right| \ll p^{\frac{1}{2}},$$

where $f(x)$ is a polynomial. Related work can also be found in [5–7], and [8].

In this paper, we introduce a new sum $G(\chi, m, n, c; q)$, analogous to quadratic Gauss sums $G_2(\chi, n)$, as follows:

$$G(\chi, m, n, c; q) = \sum_{a=0}^{q-1} \sum_{b=0}^{q-1} \chi\left(a^2 + na - b^2 - nb + c\right) \cdot e\left(\frac{mb^2 - ma^2}{q}\right),$$

where c, m, and n are any integers, χ is a non-principal Dirichlet character $\bmod q$.

In this paper, we shall study the asymptotic properties of $G(\chi,m,n,c;q)$. As regards this problem, it seems that none has studied it yet, at least we have not seen any related results before. The problem is interesting, because it has a close relation with the Gauss sums, and it is also analogous to quadratic Gauss sums. Of course, it can also help us to further understand and study the quadratic Gauss sums.

The main purpose of this paper is using the analytic method and the properties of Gauss sums to study the $2k$th power mean of $G(\chi,m,n,c;p)$, and to give a sharp asymptotic formula for it. That is, we shall prove the following two conclusions.

Theorem 1 *Let p be an odd prime, χ be any non-principal character* $\bmod p$. *Then for any integers c, m and n with $(cmn,p)=1$, we have the estimate*

$$p-\sqrt{p}\le\left|\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\chi\left(a^2-b^2+na-nb+c\right)\cdot e\left(\frac{mb^2-ma^2}{p}\right)\right|\le p+\sqrt{p}.$$

Theorem 2 *Let p be an odd prime, χ be any non-principal character* $\bmod p$, *k be any fixed positive integer. Then for any integers m and n with $(mn,p)=1$, we have the asymptotic formula*

$$\sum_{c=1}^{p-1}\left|\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\chi\left(a^2-b^2+na-nb+c\right)\cdot e\left(\frac{mb^2-ma^2}{p}\right)\right|^{2k}$$
$$=\begin{cases}p^{2k+1}+\frac{k^2-3k-2}{2}\cdot p^{2k}+O(p^{2k-1}), & \text{if } \chi \text{ is the Legendre symbol} \bmod p;\\ p^{2k+1}+\frac{k^2-3k-2}{2}\cdot p^{2k}+O(p^{2k-\frac{1}{2}}), & \text{if } \chi \text{ is a complex character} \bmod p.\end{cases}$$

From Theorem 2 we can also deduce the following two corollaries.

Corollary 1 *Let p be an odd prime, χ be any non-real character* $\bmod p$. *Then for any integers m and n with $(mn,p)=1$, we have the asymptotic formula*

$$\sum_{c=1}^{p-1}\left|\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\chi\left(a^2-b^2+na-nb+c\right)\cdot e\left(\frac{mb^2-ma^2}{p}\right)\right|^{6}=p^7-p^6+O\left(p^{\frac{11}{2}}\right).$$

Corollary 2 *Let p be an odd prime, χ be any non-real character* $\bmod p$. *Then for any integers m and n with $(mn,p)=1$, we have the asymptotic formula*

$$\sum_{c=1}^{p-1}\left|\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\chi\left(a^2-b^2+na-nb+c\right)\cdot e\left(\frac{mb^2-ma^2}{p}\right)\right|^{8}=p^9+p^8+O\left(p^{\frac{15}{2}}\right).$$

For general integer $q\ge 3$, whether there exists an asymptotic formula for the $2k$th power mean

$$\sum_{c=0}^{q-1}\left|\sum_{a=0}^{q-1}\sum_{b=0}^{q-1}\chi\left(a^2+na-b^2-nb+c\right)\cdot e\left(\frac{mb^2-ma^2}{q}\right)\right|^{2k}$$

is an interesting open problem, where m and n are any integers with $(mn,q)=1$.

2 Proof of the theorems

To complete the proofs of our theorems, we need the following simple conclusion.

Lemma *Let p be an odd prime, χ be any non-principal character* $\bmod p$. *Then for any integers c, m, and n with $(cmn,p)=1$, we have the identity*

$$\left|\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\chi\left(a^2-b^2+na-nb+c\right)\cdot e\left(\frac{mb^2-ma^2}{p}\right)\right|=p\cdot\left|1-\frac{\overline{\chi}(mc)\cdot e(\frac{mc}{p})}{\tau(\overline{\chi})}\right|.$$

Proof If $(n,p)=1$, then from the properties of Gauss sums and quadratic residue $\bmod p$ we have

$$\begin{aligned}\sum_{a=0}^{p-1}e\left(\frac{na^2}{p}\right)&=1+\sum_{a=1}^{p-1}e\left(\frac{na^2}{p}\right)=1+\sum_{a=1}^{p-1}\left(1+\left(\frac{a}{p}\right)\right)\cdot e\left(\frac{na}{p}\right)\\&=\sum_{a=0}^{p-1}e\left(\frac{na}{p}\right)+\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\cdot e\left(\frac{na}{p}\right)\\&=\left(\frac{n}{p}\right)\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\cdot e\left(\frac{a}{p}\right)=\left(\frac{n}{p}\right)\cdot\tau(\chi_2),\end{aligned}\tag{1}$$

where $\chi_2=(\frac{*}{p})$ denotes the Legendre symbol.

Since χ is a non-principal Dirichlet character $\bmod p$, from (1), for the properties of Gauss sums and a complete residue system $\bmod p$ we have

$$\begin{aligned}&\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\chi\left(a^2-b^2+na-nb+c\right)\cdot e\left(\frac{mb^2-ma^2}{p}\right)\\&=\frac{1}{\tau(\overline{\chi})}\cdot\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{r=1}^{p-1}\overline{\chi}(r)e\left(\frac{r(a^2+b^2+na-nb+c)}{p}\right)\cdot e\left(\frac{mb^2-ma^2}{p}\right)\\&=\frac{1}{\tau(\overline{\chi})}\cdot\sum_{r=1}^{p-1}\overline{\chi}(r)\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}e\left(\frac{(r-m)a^2+rna-(r-m)b^2-nrb+cr}{p}\right)\\&=\frac{1}{\tau(\overline{\chi})}\sum_{\substack{r=1\\(r-m,p)=1}}^{p-1}\overline{\chi}(r)\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}e\left(\frac{\overline{4}(r-m)((2a+\overline{r-m}nr)^2-(2b+\overline{r-m}nr)^2)+rc}{p}\right)\\&=\frac{1}{\tau(\overline{\chi})}\sum_{\substack{r=1\\(r-m,p)=1}}^{p-1}\overline{\chi}(r)\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}e\left(\frac{(r-m)(a^2-b^2)+rc}{p}\right)\\&=\frac{\tau^2(\chi_2)}{\tau(\overline{\chi})}\sum_{\substack{r=1\\(r-m,p)=1}}^{p-1}\overline{\chi}(r)\left(\frac{r-m}{p}\right)^2e\left(\frac{rc}{p}\right)=\frac{\tau^2(\chi_2)}{\tau(\overline{\chi})}\left(\sum_{r=1}^{p-1}\overline{\chi}(r)e\left(\frac{rc}{p}\right)-\overline{\chi}(m)e\left(\frac{mc}{p}\right)\right)\\&=\tau^2(\chi_2)\left(\chi(c)-\frac{\overline{\chi}(m)e(\frac{mc}{p})}{\tau(\overline{\chi})}\right),\end{aligned}\tag{2}$$

where $\overline{n}$ denotes the solution of the congruence equation $n\cdot x\equiv 1\bmod p$.

For any non-principal character $\chi \bmod p$, note that $|\tau(\chi)| = \sqrt{p}$; from (2) we may immediately complete the proof of our lemma. □

Now we use this lemma to prove our theorems. First we prove Theorem 1. In fact from the lemma and the absolute value inequality $|a| - |b| \le |a - b| \le |a| + |b|$ we have the estimate

$$\begin{aligned} p - \sqrt{p} = p - \frac{p}{|\tau(\overline{\chi})|} &\le \left| \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \chi\left(a^2 - b^2 + na - nb + c\right) \cdot e\left(\frac{mb^2 - ma^2}{p}\right) \right| \\ &\le p + \frac{|p \cdot \overline{\chi}(cm) \cdot e(\frac{mc}{p})|}{|\tau(\overline{\chi})|} = p + \sqrt{p}. \end{aligned}$$

This proves Theorem 1.

To prove Theorem 2, from the lemma we have

$$\begin{aligned} &\left| \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \chi\left(a^2 - b^2 + na - nb + c\right) \cdot e\left(\frac{mb^2 - ma^2}{p}\right) \right|^2 \\ &\quad = p^2\left(1 + \frac{1}{p}\right) - p^2\left(\frac{\overline{\chi}(mc) \cdot e(\frac{mc}{p})}{\tau(\overline{\chi})} - \frac{\chi(mc) \cdot e(\frac{-mc}{p})}{\overline{\tau(\overline{\chi})}}\right). \end{aligned} \tag{3}$$

So for any positive integer $k \ge 1$, from (3) and the binomial theorem we have

$$\begin{aligned} &\sum_{c=1}^{p-1} \left| \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \chi\left(a^2 - b^2 + na - nb + c\right) \cdot e\left(\frac{mb^2 - ma^2}{p}\right) \right|^{2k} \\ &\quad = \sum_{c=1}^{p-1} \left(p^2\left(1 + \frac{1}{p}\right) - p^2\left(\frac{\overline{\chi}(mc) \cdot e(\frac{mc}{p})}{\tau(\overline{\chi})} + \frac{\chi(mc) \cdot e(\frac{-mc}{p})}{\overline{\tau(\overline{\chi})}}\right)\right)^k \\ &\quad = p^{2k}\left(1 + \frac{1}{p}\right)^k (p-1) - \sum_{c=1}^{p-1} \left(\frac{\overline{\chi}(mc) \cdot e(\frac{mc}{p})}{\tau(\overline{\chi})} + \frac{\chi(mc) \cdot e(\frac{-mc}{p})}{\overline{\tau(\overline{\chi})}}\right) \\ &\quad \times kp^{2k}\left(1 + \frac{1}{p}\right)^{k-1} + \frac{k(k-1)}{2} p^{2k}\left(1 + \frac{1}{p}\right)^{k-2} \\ &\quad \times \sum_{c=1}^{p-1} \left(\frac{\overline{\chi}(mc) \cdot e(\frac{mc}{p})}{\tau(\overline{\chi})} + \frac{\chi(mc) \cdot e(\frac{-mc}{p})}{\overline{\tau(\overline{\chi})}}\right)^2 + \frac{k(k-1)(k-2)}{6} p^{2k} \\ &\quad \times \left(1 + \frac{1}{p}\right)^{k-3} \sum_{c=1}^{p-1} \left(\frac{\overline{\chi}(mc) \cdot e(\frac{mc}{p})}{\tau(\overline{\chi})} + \frac{\chi(mc) \cdot e(\frac{-mc}{p})}{\overline{\tau(\overline{\chi})}}\right)^3 + O\left(p^{2k-1}\right). \end{aligned} \tag{4}$$

If $\chi = \chi_2$ is the Legendre symbol, note that we have the trigonometric identities

$$\begin{aligned} &\sum_{c=1}^{p-1} \frac{\overline{\chi}^2(mc) \cdot e(\frac{2mc}{p})}{\tau^2(\overline{\chi})} = \sum_{c=1}^{p-1} \frac{e(\frac{2mc}{p})}{\tau^2(\overline{\chi})} = \frac{-1}{\tau^2(\overline{\chi})} = O\left(\frac{1}{p}\right), \\ &\sum_{c=1}^{p-1} \frac{\chi^2(mc) \cdot e(\frac{-2mc}{p})}{\overline{\tau^2(\overline{\chi})}} = \sum_{c=1}^{p-1} \frac{e(\frac{-2mc}{p})}{\overline{\tau^2(\overline{\chi})}} = \frac{-1}{\overline{\tau^2(\overline{\chi})}} = O\left(\frac{1}{p}\right), \end{aligned}$$

$$\sum_{c=1}^{p-1} \frac{\overline{\chi}(mc) \cdot e(\frac{mc}{p})}{\tau(\overline{\chi})} \cdot \frac{\chi(mc) \cdot e(\frac{-mc}{p})}{\overline{\tau(\overline{\chi})}} = \frac{p-1}{p}$$

and

$$\begin{aligned}
&\sum_{c=1}^{p-1} \left(\frac{\overline{\chi}(mc) \cdot e(\frac{mc}{p})}{\tau(\overline{\chi})} + \frac{\chi(mc) \cdot e(\frac{-mc}{p})}{\overline{\tau(\overline{\chi})}} \right)^3 \\
&\quad = \frac{1}{\tau^3(\overline{\chi})} \sum_{c=1}^{p-1} \overline{\chi}^3(c) e\left(\frac{3c}{p}\right) + \frac{3}{p \cdot \tau(\overline{\chi})} \sum_{c=1}^{p-1} \overline{\chi}(c) e\left(\frac{c}{p}\right) \\
&\qquad + \frac{1}{\overline{\tau(\overline{\chi})}^3} \sum_{c=1}^{p-1} \chi^3(c) e\left(\frac{-3c}{p}\right) + \frac{3}{p \cdot \overline{\tau(\overline{\chi})}} \sum_{c=1}^{p-1} \chi(c) e\left(\frac{-c}{p}\right) = O\left(\frac{1}{p}\right),
\end{aligned}$$

and from (4) we may immediately get the asymptotic formula

$$\begin{aligned}
&\sum_{c=1}^{p-1} \left| \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \chi\left(a^2 - b^2 + na - nb + c\right) \cdot e\left(\frac{mb^2 - ma^2}{p}\right) \right|^{2k} \\
&\quad = p^{2k} \left(1 + \frac{1}{p}\right)^k (p-1) - 2kp^{2k} \left(1 + \frac{1}{p}\right)^{k-1} \\
&\qquad + \frac{k(k-1)}{2} p^{2k} \left(1 + \frac{1}{p}\right)^{k-2} \frac{p-1}{p} + O\left(p^{2k-1}\right) \\
&\quad = p^{2k+1} + \frac{k^2 - 3k - 2}{2} \cdot p^{2k} + O\left(p^{2k-1}\right).
\end{aligned} \tag{5}$$

If χ is any non-real character $\bmod p$, then note that the identities

$$\begin{aligned}
&\sum_{c=1}^{p-1} \frac{\overline{\chi}^2(mc) \cdot e(\frac{2mc}{p})}{\tau^2(\overline{\chi})} = \frac{\chi^2(2)\tau(\overline{\chi}^2)}{\tau^2(\overline{\chi})} = O\left(\frac{1}{\sqrt{p}}\right), \\
&\sum_{c=1}^{p-1} \frac{\chi^2(mc) \cdot e(\frac{-2mc}{p})}{\overline{\tau^2(\overline{\chi})}} = \frac{\overline{\chi}^2(2)\tau(\chi^2)}{\overline{\tau^2(\overline{\chi})}} = O\left(\frac{1}{\sqrt{p}}\right), \\
&\sum_{c=1}^{p-1} \frac{\overline{\chi}(mc) \cdot e(\frac{mc}{p})}{\tau(\overline{\chi})} \cdot \frac{\chi(mc) \cdot e(\frac{-mc}{p})}{\overline{\tau(\overline{\chi})}} = \frac{p-1}{p},
\end{aligned}$$

and

$$\sum_{c=1}^{p-1} \left(\frac{\overline{\chi}(mc) \cdot e(\frac{mc}{p})}{\tau(\overline{\chi})} + \frac{\chi(mc) \cdot e(\frac{-mc}{p})}{\overline{\tau(\overline{\chi})}} \right)^3 = O\left(\frac{1}{p^{\frac{3}{2}}} \cdot \sum_{c=1}^{p-1} 1 \right) = O\left(\frac{1}{\sqrt{p}}\right).$$

From (4) we may immediately get the asymptotic formula

$$\begin{aligned}
&\sum_{c=1}^{p-1} \left| \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \chi\left(a^2 - b^2 + na - nb + c\right) \cdot e\left(\frac{mb^2 - ma^2}{p}\right) \right|^{2k} \\
&\quad = p^{2k+1} + \frac{k^2 - 3k - 2}{2} \cdot p^{2k} + O\left(p^{2k-\frac{1}{2}}\right).
\end{aligned} \tag{6}$$

Now combining (5) and (6) we have the asymptotic formula

$$\sum_{c=1}^{p-1}\left|\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\chi\left(a^2-b^2+na-nb+c\right)\cdot e\left(\frac{mb^2-ma^2}{p}\right)\right|^{2k}$$
$$=\begin{cases} p^{2k+1}+\frac{k^2-3k-2}{2}\cdot p^{2k}+O(p^{2k-1}), & \text{if } \chi \text{ is the Legendre symbol } \bmod p; \\ p^{2k+1}+\frac{k^2-3k-2}{2}\cdot p^{2k}+O(p^{2k-\frac{1}{2}}), & \text{if } \chi \text{ is a complex character } \bmod p. \end{cases}$$

This completes the proof of our theorems.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
DX obtained the theorems and completed the proof. LX corrected and improved the final version. Both authors read and approved the final manuscript.

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