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Some inequalities for (h,m)-convex functions

Bo-Yan Xi[1*], Shu-Hong Wang[1] and Feng Qi[1,2]

*Correspondence: baoyintu78@qq.com; baoyintu68@sohu.com
[1]College of Mathematics, Inner Mongolia University for Nationalities, Tongliao, Inner Mongolia Autonomous Region 028043, China
Full list of author information is available at the end of the article

Abstract
In the paper, the authors give some inequalities of Jensen type and Popoviciu type for (h,m)-convex functions and apply these inequalities to special means.

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1 Introduction

The following definition is well known in the literature.

Definition 1 A function $f : I \subseteq \mathbb{R} = (-\infty,\infty) \to \mathbb{R}$ is said to be convex if

$$f\big(tx + (1-t)y\big) \le tf(x) + (1-t)f(y) \tag{1}$$

holds for all $x, y \in I$ and $t \in [0,1]$.

We cite the following inequalities for convex functions.

Theorem 1 ([1, p.6]) *If f is a convex function on I and $x_1, x_2, x_3 \in I$, then*

$$\begin{aligned} &f(x_1) + f(x_2) + f(x_3) + f\left(\frac{x_1+x_2+x_3}{3}\right) \\ &\quad \ge \frac{4}{3}\left[f\left(\frac{x_1+x_2}{2}\right) + f\left(\frac{x_2+x_3}{2}\right) + f\left(\frac{x_3+x_1}{2}\right)\right]. \end{aligned} \tag{2}$$

Theorem 2 ([2, Popoviciu inequality]) *If f is a convex function on I and $x_1, x_2, \dots, x_n \in I$ with $n \ge 3$, then*

$$\sum_{i=1}^{n} f(x_i) + \frac{n}{n-2} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \ge \frac{2}{n-2}\sum_{i<j} f\left(\frac{x_i+x_j}{2}\right). \tag{3}$$

Theorem 3 ([2, Generalized Popoviciu inequality]) *If f is a convex function on I and $a_1, a_2, \dots, a_n \in I$ for $n \ge 3$, then*

$$(n-1)\sum_{i=1}^{n} f(b_i) \le n(n-2)f(a) + \sum_{i=1}^{n} f(a_i), \tag{4}$$

where $a = \frac{1}{n}\sum_{i=1}^{n} a_i$ and $b_i = \frac{na-a_i}{n-1}$ for $i = 1, 2, \dots, n$.

The above inequalities were generalized as follows.

Theorem 4 ([3]) *If f is a convex function on I and $x_1, x_2, \dots, x_n \in I$ for $n \geq 3$, then*

$$\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) \geq \frac{n-1}{n} \sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1}}{2}\right) \tag{5}$$

and

$$(n-1)\sum_{i=1}^{n} f(b_i) \leq n\left[\sum_{i=1}^{n} f(a_i) - f(a)\right], \tag{6}$$

where $x_{n+1} = x_1$, $a = \frac{1}{n}\sum_{i=1}^{n} a_i$, and $b_i = \frac{na-a_i}{n-1}$ for $i = 1, 2, \dots, n$.

Definition 2 ([4]) Let $s \in (0,1]$. A function $f : \mathbb{R}_0 = [0,\infty) \to \mathbb{R}_0$ is said to be s-convex in the second sense if

$$f\big(\lambda x + (1-\lambda)y\big) \leq \lambda^s f(x) + (1-\lambda)^s f(y) \tag{7}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

The following inequalities for s-convex functions were established.

Theorem 5 ([5, Theorem 4.2]) *If f is nonnegative and s-convex in the second sense on I and if $x_1, x_2, \dots, x_n \in I$ for $n \geq 3$, then*

$$\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \geq \frac{2^{s-1}(n^s - 1)}{n} \sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1}}{2}\right), \tag{8}$$

where $x_1 = x_{n+1}$.

Theorem 6 ([5, Theorem 4.4]) *If f is nonnegative and s-convex in the second sense on I and $a_1, a_2, \dots, a_n \in I$ for $n \geq 3$, then*

$$\big(n^s - 1\big)\sum_{i=1}^{n} b_i \leq n^s\left[\sum_{i=1}^{n} f(a_i) - f(a)\right], \tag{9}$$

where $a = \frac{1}{n}\sum_{i=1}^{n} a_i$ and $b_i = \frac{na-a_i}{n-1}$ for $i = 1, 2, \dots, n$.

The concept of h-convex functions below was innovated as follows.

Definition 3 ([6, Definition 4]) Let $I, J \subseteq \mathbb{R}$ be intervals, $(0,1) \subseteq J$, and $h : J \to \mathbb{R}_0$ be a nonnegative function. A function $f : I \to \mathbb{R}_0$ is called h-convex, or as we say, f belongs to the class $\mathrm{SX}(h, I)$, if f is nonnegative and

$$f\big(tx + (1-t)y\big) \leq h(t)f(x) + h(1-t)f(y) \tag{10}$$

for all $x, y \in I$ and $t \in [0,1]$.

Definition 4 ([6, Section 3]) A function $h : J \subseteq \mathbb{R}$ is said to be a super-multiplicative on an interval J if

$$h(xy) \geq h(x)h(y) \tag{11}$$

is valid for all $x, y \in J$. If the inequality (11) reverses, then f is said to be a sub-multiplicative function on J.

The following inequalities were established for $f \in \mathrm{SX}(h, I)$.

Theorem 7 ([7, Theorem 6]) *Let $w_1, \dots, w_n$ for $n \geq 2$ be positive real numbers. If h is a nonnegative and super-multiplicative function and if $f \in \mathrm{SX}(h, I)$ and $x_1, \dots, x_n \in I$, then*

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leq \sum_{i=1}^{n} h\left(\frac{w_i}{W_n}\right) f(x_i), \tag{12}$$

where $W_n = \sum_{i=1}^{n} w_i$. If h is sub-multiplicative and $f \in \mathrm{SV}(h, I)$, then the inequality (12) *is reversed.*

Theorem 8 ([8, Theorem 11]) *Let h be a nonnegative and super-multiplicative function. If $f \in \mathrm{SX}(h, I)$ and $x_1, \dots, x_n \in I$, then*

$$\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \geq \frac{1 - h(1/n)}{2h(1/2)} \sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1}}{2}\right), \tag{13}$$

where $x_{n+1} = x_1$. The inequality (13) *is reversed if $f \in \mathrm{SV}(h, I)$.*

Theorem 9 ([8, Theorem 12]) *Let h be a nonnegative and super-multiplicative function. If $f \in \mathrm{SX}(h, I)$ and $x_1, \dots, x_n \in I$, then*

$$\left[1 - h\left(\frac{1}{n}\right)\right] \sum_{i=1}^{n} f(b_i) \leq (n-1) h\left(\frac{1}{n-1}\right) \left[\sum_{i=1}^{n} f(a_i) - f(a)\right], \tag{14}$$

where $a = \frac{1}{n}\sum_{i=1}^{n} a_i$ and $b_i = \frac{na - a_i}{n-1}$ for $i = 1, 2, \dots, n$ and $n \geq 3$. The inequality (14) *is reversed if $f \in \mathrm{SV}(h, I)$.*

Two new kinds of convex functions were introduced as follows.

Definition 5 ([9]) For $f : [0, b] \to \mathbb{R}$ and $m \in (0, 1]$, if

$$f\big(tx + m(1-t)y\big) \leq tf(x) + m(1-t)f(y) \tag{15}$$

is valid for all $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that $f(x)$ is an m-convex function on $[0, b]$.

Definition 6 ([10]) Let $J \subseteq \mathbb{R}$ be an interval, $(0, 1) \subseteq J$, $h : J \to \mathbb{R}$ be a nonnegative function. We say that $f : [0, b] \to \mathbb{R}$ is an (h, m)-convex function, or say, f belongs to the class $\mathrm{SMX}((h, m), [0, b])$, if f is nonnegative and, for all $x, y \in [0, b]$ and $t \in [0, 1]$ and for some

$m \in (0,1]$, we have

$$f\bigl(tx + m(1-t)y\bigr) \le h(t)f(x) + mh(1-t)f(y). \tag{16}$$

If the inequality (16) is reversed, then f is said to be (h,m)-concave and denoted by $f \in \mathrm{SMV}((h,m),[0,b])$.

Recently the h- and (h,m)-convex functions were generalized and some properties and inequalities for them were obtained in [11, 12].

The aim of this paper is to find some inequalities of Jensen type and Popoviciu type for (h,m)-convex functions.

2 Inequalities of Jensen type and Popoviciu type

Now we are in a position to establish some inequalities of Jensen type and Popoviciu type for (h,m)-convex functions.

Theorem 10 *Let $h:[0,1]\to\mathbb{R}_0$ be a super-multiplicative function and $m\in(0,1]$. If $f \in \mathrm{SMX}((h,m),[0,b])$, then for all $x_i\in[0,b]$ and $w_i>0$ with $i=1,2,\dots,n$ and $n\ge 2$, we have*

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} m^{i-1}w_i x_i\right) \le \sum_{i=1}^{n} m^{i-1}h\left(\frac{w_i}{W_n}\right)f(x_i), \tag{17}$$

where $W_n=\sum_{i=1}^{n} w_i$.

If h is sub-multiplicative and $f\in\mathrm{SMV}((h,m),[0,b])$, then the inequality (17) *is reversed.*

Proof Assume that $w_i' = \frac{w_i}{W_n}$ for $i=1,2,\dots,n$.

When $n=2$, taking $t=w_1'$ and $1-t=w_2'$ in Definition 6 gives the inequality (17) clearly.

Suppose that the inequality (17) holds for $n=k$, *i.e.*,

$$f\left(\sum_{i=1}^{k} m^{i-1}w_i' x_i\right) \le \sum_{i=1}^{k} m^{i-1}h\bigl(w_i'\bigr)f(x_i). \tag{18}$$

When $n=k+1$, letting $\Delta_k=\sum_{i=2}^{k+1} w_i'$ and making use of (18) result in

$$\begin{aligned} f\left(\sum_{i=1}^{k+1} m^{i-1}w_i' x_i\right) &= f\left(w_1'x_1 + m\Delta_k\sum_{i=2}^{k+1} m^{i-2}\frac{w_i'}{\Delta_k}x_i\right) \\ &\le h\bigl(w_1'\bigr)f(x_1) + mh(\Delta_k)f\left(\sum_{i=2}^{k+1} m^{i-2}\frac{w_i'}{\Delta_k}x_i\right) \\ &\le h\bigl(w_1'\bigr)f(x_1) + mh(\Delta_k)\sum_{i=2}^{k+1} m^{i-2}h\left(\frac{w_i'}{\Delta_k}\right)f(x_i). \end{aligned}$$

Since h is a super-multiplicative function, it follows that

$$h(\Delta_k)h\left(\frac{w_i'}{\Delta_k}\right) \le h\bigl(w_i'\bigr)$$

for $i=1,2,\dots,n$. Namely, when $n=k+1$, the inequality (17) holds. By induction, Theorem 10 is proved. □

Corollary 1 *Under the conditions of Theorem* 10,

1. *if* $W_n = 1$, *we have*

$$f\left(\sum_{i=1}^{n} m^{i-1} w_i x_i\right) \le \sum_{i=1}^{n} m^{i-1} h(w_i) f(x_i); \tag{19}$$

2. *if* $w_1 = w_2 = \cdots = w_n$, *we have*

$$f\left(\frac{1}{n}\sum_{i=1}^{n} m^{i-1} w_i x_i\right) \le h\left(\frac{1}{n}\right)\sum_{i=1}^{n} m^{i-1} f(x_i); \tag{20}$$

3. *if* h *is sub-multiplicative and* $f \in \mathrm{SMV}((h,m),[0,b])$, *then the inequalities* (19) *and* (20) *are reversed.*

Corollary 2 *For* $m \in (0,1]$ *and* $s \in (0,1]$, *the assertion* $f \in \mathrm{SMX}((t^s,m),[0,b])$ *is valid if and only if for all* $x_i \in [0,b]$ *and* $w_i > 0$ *with* $i = 1,2,\ldots,n$ *and* $n \ge 2$

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} m^{i-1} x_i\right) \le \sum_{i=1}^{n} m^{i-1}\left(\frac{w_i}{W_n}\right)^s f(x_i), \tag{21}$$

where $W_n = \sum_{i=1}^{n} w_i$.

Corollary 3 *Under the conditions of Corollary* 1, *if* $h(t) = t^s$ *for* $s \in (0,1]$, *then*

$$f\left(\frac{1}{n}\sum_{i=1}^{n} m^{i-1} x_i\right) \le \frac{1}{n^s}\sum_{i=1}^{n} m^{i-1} f(x_i). \tag{22}$$

If $f \in \mathrm{SMV}((h,m),[0,b])$, *then the inequality* (22) *is reversed.*

Theorem 11 *Let* $h : [0,1] \to \mathbb{R}_0$ *be a super-multiplicative function*, $m \in (0,1]$, *and* $n \ge 2$. *If* $f \in \mathrm{SMX}((h,m),[0,\frac{b}{m^{n-1}}])$, *then for all* $x_i \in [0,b]$ *and* $w_i > 0$ *with* $i = 1,2,\ldots,n$,

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \le \sum_{i=1}^{n} m^{i-1} h\left(\frac{w_i}{W_n}\right) f\left(\frac{x_i}{m^{i-1}}\right), \tag{23}$$

where $W_n = \sum_{i=1}^{n} w_i$.

If h *is sub-multiplicative and* $f \in \mathrm{SMV}((h,m),[0,\frac{b}{m^{n-1}}])$, *then the inequality* (23) *is reversed.*

Proof Putting $y_i = \frac{x_i}{m^{i-1}}$ for $i = 1,2,\ldots,n$, then from inequality (17), we have

$$\begin{aligned} f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) &= f\left(\frac{1}{W_n}\sum_{i=1}^{n} m^{i-1} w_i y_i\right) \\ &\le \sum_{i=1}^{n} m^{i-1} h\left(\frac{w_i}{W_n}\right) f(y_i) = \sum_{i=1}^{n} m^{i-1} h\left(\frac{w_i}{W_n}\right) f\left(\frac{x_i}{m^{i-1}}\right). \end{aligned}$$

The proof of Theorem 11 is complete. □

Corollary 4 *For $m \in (0,1]$, $s \in (0,1]$, and $n \geq 2$, the assertion $f \in \mathrm{SMX}((t^s, m), [0, \frac{b}{m^{n-1}}])$ is valid if and only if for all $x_i \in [0,b]$ and $w_i > 0$ with $i = 1,2,\dots,n$ the inequality*

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leq \sum_{i=1}^{n} m^{i-1}\left(\frac{w_i}{W_n}\right)^s f\left(\frac{x_i}{m^{i-1}}\right) \tag{24}$$

is valid, where $W_n = \sum_{i=1}^{n} w_i$.

Corollary 5 *Under the conditions of Theorem* 11*,*

1. *if $W_n = 1$, then*

$$f\left(\sum_{i=1}^{n} w_i x_i\right) \leq \sum_{i=1}^{n} m^{i-1} h(w_i) f\left(\frac{x_i}{m^{i-1}}\right); \tag{25}$$

2. *if $w_1 = w_2 = \cdots = w_n$, then*

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \leq h\left(\frac{1}{n}\right)\sum_{i=1}^{n} m^{i-1} f\left(\frac{x_i}{m^{i-1}}\right); \tag{26}$$

3. *if h is sub-multiplicative and $f \in \mathrm{SMV}((h,m), [0, \frac{b}{m^{n-1}}])$, then the inequalities* (25) *and* (26) *are reversed.*

Corollary 6 *Under the conditions of Corollary* 5*,*

1. *if $h(t) = t^s$ for $s \in (0,1]$, then*

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \leq \frac{1}{n^s}\sum_{i=1}^{n} m^{i-1} f\left(\frac{x_i}{m^{i-1}}\right); \tag{27}$$

2. *if $f \in \mathrm{SMV}((h,m), [0, \frac{b}{m^{n-1}}])$, then the inequality* (27) *is reversed.*

Theorem 12 *Let $h : [0,1] \to [0,1]$ be a super-multiplicative function and let $m \in (0,1]$ and $n \geq 3$. If $f \in \mathrm{SMX}((h,m), [0,b])$, then for all $x_i \in [0,b]$ with $i = 1,2,\dots,n$ and $2 \leq k \leq n$, we have*

$$\begin{aligned} &\sum_{i=1}^{n} f(x_i) - \left(\sum_{j=0}^{n-1} m^j\right)^{-1} \sum_{i=1}^{n} f\left(\frac{1}{n}\sum_{j=i}^{n+i-1} m^{j-i} x_j\right) \\ &\quad \geq \frac{1 - h(1/n)}{h(1/k)}\left(\sum_{j=0}^{k-1} m^j\right)^{-1} \sum_{i=1}^{n} f\left(\frac{1}{k}\sum_{j=i}^{k+i-1} m^{j-i} x_j\right), \end{aligned} \tag{28}$$

where $x_{n+1} = x_1, \dots, x_{2n-1} = x_{n-1}$.

If h is sub-multiplicative and $f \in \mathrm{SMV}((h,m), [0,b])$, then the inequality (28) *is reversed.*

Proof By using the inequality (20), we have

$$\sum_{i=1}^{n} f\left(\frac{1}{k}\sum_{j=i}^{k+i-1} m^{j-i} x_j\right) \leq h\left(\frac{1}{k}\right)\sum_{i=1}^{n}\sum_{j=i}^{k+i-1} m^{j-i} f(x_j) = h\left(\frac{1}{k}\right)\left(\sum_{j=0}^{k-1} m^j\right)\sum_{i=1}^{n} f(x_i) \tag{29}$$

and

$$\sum_{i=1}^{n} f\left(\frac{1}{n}\sum_{j=i}^{n+i-1} m^{j-i}x_j\right) \le h\left(\frac{1}{n}\right)\sum_{i=1}^{n}\sum_{j=i}^{n+i-1} m^{j-i}f(x_j)$$
$$= h\left(\frac{1}{n}\right)\left(\sum_{j=0}^{n-1} m^j\right)\sum_{i=1}^{n} f(x_i). \tag{30}$$

If $h(\frac{1}{n}) = 1$, then, from the inequality (30), the inequality (28) holds. If $h(\frac{1}{n}) \le 1$, it is easy to see that

$$\sum_{i=1}^{n} f\left(\frac{1}{k}\sum_{j=i}^{k+i-1} m^{j-i}x_j\right)$$
$$\le h\left(\frac{1}{k}\right)\left(\sum_{j=0}^{k-1} m^j\right)\sum_{i=1}^{n} f(x_i)$$
$$= \frac{h(1/k)}{1-h(1/n)}\left(\sum_{j=0}^{k-1} m^j\right)\left[\sum_{i=1}^{n} f(x_i) - h\left(\frac{1}{n}\right)\sum_{i=1}^{n} f(x_i)\right]$$
$$\le \frac{h(1/k)}{1-h(1/n)}\left(\sum_{j=0}^{k-1} m^j\right)\left[\sum_{i=1}^{n} f(x_i) - \left(\sum_{j=0}^{n-1} m^j\right)^{-1}\sum_{i=1}^{n} f\left(\frac{1}{n}\sum_{j=i}^{n+i-1} m^{j-i}x_j\right)\right].$$

The proof of Theorem 12 is complete. □

Corollary 7 *Under the conditions of Theorem* 12, *let* $\bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i$.

1. *When* $m = 1$, *we have*

$$\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \ge \frac{1-h(1/n)}{kh(1/k)}\sum_{i=1}^{n} f\left(\frac{1}{k}\sum_{j=i}^{k+i-1} x_j\right). \tag{31}$$

2. *When* $m = 1$ *and* $k = 2$, *we have*

$$\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \ge \frac{1-h(1/n)}{2h(1/2)}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1}}{2}\right). \tag{32}$$

3. *When* $m = 1$ *and* $k = n-1$, *we have*

$$\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \ge \frac{1-h(1/n)}{(n-1)h(1/(n-1))}\sum_{i=1}^{n} f\left(\frac{n\bar{x}_n - x_i}{n-1}\right). \tag{33}$$

4. *If* h *is sub-multiplicative and* $f \in \mathrm{SMV}((h,m),[0,b])$, *then the inequalities* (31) *to* (33) *are reversed.*

Remark 1 The inequality (14) can be deduced from applying (33) to $a_i = x_i$ for $i = 1, 2, \dots, n$, $a = \frac{1}{n}\sum_{i=1}^{n} a_i$, and $b_i = \frac{na-a_i}{n-1}$ for $i = 1, 2, \dots, n$.

Corollary 8 *Under the conditions of Theorem* 12,

1. *if* $h(t) = t^s$ *for* $s \in (0,1]$, *then*

$$\sum_{i=1}^{n} f(x_i) - \left(\sum_{j=0}^{n-1} m^j\right)^{-1} \sum_{i=1}^{n} f\left(\frac{1}{n}\sum_{j=i}^{n+i-1} m^{j-i}x_j\right) \geq \frac{k^s(n^s-1)}{n^s}\left(\sum_{j=0}^{k-1} m^j\right)^{-1} \sum_{i=1}^{n} f\left(\frac{1}{k}\sum_{j=i}^{k+i-1} m^{j-i}x_j\right); \tag{34}$$

2. *if* $h(t) = t^s$ *for* $s \in (0,1]$ *and* $m = 1$, *then*

$$\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \geq \frac{k^{s-1}(n^s-1)}{n^s} \sum_{i=1}^{n} f\left(\frac{1}{k}\sum_{j=i}^{k+i-1} x_j\right); \tag{35}$$

3. *if* $h(t) = t$ *and* $m = 1$, *then*

$$\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \geq \frac{n-1}{n} \sum_{i=1}^{n} f\left(\frac{1}{k}\sum_{j=i}^{k+i-1} x_j\right); \tag{36}$$

4. *if* $f \in \mathrm{SMV}((h,m),[0,b])$, *then the inequalities* (34) *to* (36) *are reversed.*

Theorem 13 *Let* $h:[0,1] \to [0,1]$ *be a super-multiplicative function and let* $m \in (0,1]$ *and* $n \geq 3$. *If* $f \in \mathrm{SMX}((h,m),[0,\frac{b}{m^{n-1}}])$, *then for all* $x_i \in [0,b]$ *with* $i = 1,2,\dots,n$ *and* $2 \leq k \leq n$ *and for* $\ell_1,\dots,\ell_k \in \mathbb{N}$, *we have*

$$\sum_{i=1}^{n} f(x_i) - \left(\sum_{j=0}^{n-1} m^j\right)^{-1} \sum_{i=1}^{n} f\left(\frac{1}{n}\sum_{j=i}^{n+i-1} m^{j-i}x_j\right) \geq \frac{1-h(1/n)}{\binom{n-1}{k-1}h(1/k)}\left(\sum_{j=0}^{k-1} m^j\right)^{-1} \sum_{1\leq \ell_1<\cdots<\ell_k\leq n} \sum_{i=1}^{k} f\left(\frac{1}{k}\sum_{j=i}^{k+i-1} m^{j-i}x_{\ell_j}\right), \tag{37}$$

where $\ell_{k+1} = \ell_1, \dots, \ell_{2k-1} = \ell_{k-1}$.

If h *is sub-multiplicative and* $f \in \mathrm{SMV}((h,m),[0,b])$, *then the inequality* (37) *is reversed.*

Proof By the inequality (20), we have

$$\begin{aligned}
&\sum_{1\leq \ell_1<\cdots<\ell_k\leq n} \sum_{i=1}^{k} f\left(\frac{1}{k}\sum_{j=i}^{k+i-1} m^{j-i}x_{\ell_j}\right) \\
&\quad\leq h\left(\frac{1}{k}\right) \sum_{1\leq \ell_1<\cdots<\ell_k\leq n} \sum_{i=1}^{k}\sum_{j=i}^{k+i-1} m^{j-i}f(x_{\ell_j}) \\
&\quad= h\left(\frac{1}{k}\right)\left(\sum_{j=0}^{k-1} m^j\right) \sum_{1\leq \ell_1<\cdots<\ell_k\leq n} \sum_{i=1}^{k} f(x_{\ell_j}) \\
&\quad= \binom{n-1}{k-1} h\left(\frac{1}{k}\right)\left(\sum_{j=0}^{k-1} m^j\right) \sum_{i=1}^{n} f(x_i).
\end{aligned} \tag{38}$$

If $h(\frac{1}{n}) = 1$, then, from the inequality (30), the inequality (28) holds. If $h(\frac{1}{n}) \le 1$, using (38) and (30), we have

$$\begin{aligned}
&\sum_{1\le\ell_1<\cdots<\ell_k\le n}\sum_{i=1}^{k} f\left(\frac{1}{k}\sum_{j=i}^{k+i-1} m^{j-i}x_{\ell_j}\right) \\
&\le \binom{n-1}{k-1} h\left(\frac{1}{k}\right)\left(\sum_{j=0}^{k-1} m^j\right)\sum_{i=1}^{n} f(x_i) \\
&= \frac{\binom{n-1}{k-1}h(1/k)}{1-h(1/n)}\left(\sum_{j=0}^{k-1} m^j\right)\left[\sum_{i=1}^{n} f(x_i) - h\left(\frac{1}{n}\right)\sum_{i=1}^{n} f(x_i)\right] \\
&\le \frac{\binom{n-1}{k-1}h(1/k)}{1-h(1/n)}\left(\sum_{j=0}^{k-1} m^j\right)\left[\sum_{i=1}^{n} f(x_i) - \left(\sum_{j=0}^{n-1} m^j\right)^{-1}\sum_{i=1}^{n} f\left(\frac{1}{n}\sum_{j=i}^{n+i-1} m^{j-i}x_j\right)\right].
\end{aligned}$$

The proof of Theorem 13 is complete. □

Corollary 9 *Under the conditions of Theorem* 13, *let* $\bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i$.

1. *When* $m = 1$, *we have*

$$\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \ge \frac{1-h(1/n)}{\binom{n-1}{k-1}h(1/k)}\sum_{1\le\ell_1<\cdots<\ell_k\le n} f\left(\frac{1}{k}\sum_{j=1}^{k} x_{\ell_j}\right). \tag{39}$$

2. *When* $m = 1$ *and* $k = 2$, *we have*

$$\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \ge \frac{1-h(1/n)}{(n-1)h(1/2)}\sum_{1\le i<j\le n} f\left(\frac{x_i+x_j}{2}\right). \tag{40}$$

3. *When* $m = 1$ *and* $k = n-1$, *we have*

$$\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \ge \frac{1-h(1/n)}{(n-1)h(1/(n-1))}\sum_{i=1}^{n}\left(\frac{n\bar{x}_n - x_i}{n-1}\right). \tag{41}$$

4. *If* h *is sub-multiplicative and* $f \in \mathrm{SMV}((h,m),[0,b])$, *then the inequalities* (39) *to* (41) *are reversed.*

Corollary 10 *Under the conditions of Theorem* 13,

1. *if* $h(t) = t^s$ *for* $s \in (0,1]$, *then*

$$\begin{aligned}
&\sum_{i=1}^{n} f(x_i) - \left(\sum_{j=0}^{n-1} m^j\right)^{-1}\sum_{i=1}^{n} f\left(\frac{1}{n}\sum_{j=i}^{n+i-1} m^{j-i}x_j\right) \\
&\quad\ge \frac{k^s(n^s-1)}{\binom{n-1}{k-1}n^s}\left(\sum_{j=0}^{k-1} m^j\right)^{-1}\sum_{1\le\ell_1<\cdots<\ell_k\le n}\sum_{i=1}^{k} f\left(\frac{1}{k}\sum_{j=i}^{k+i-1} m^{j-i}x_{\ell_j}\right);
\end{aligned} \tag{42}$$

2. *if $m = 1$ and $h(t) = t^s$ for $s \in (0,1]$, we have*

$$\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \geq \frac{k^s(n^s-1)}{\binom{n-1}{k-1}n^s} \sum_{1\leq \ell_1<\cdots<\ell_k\leq n} f\left(\frac{1}{k}\sum_{j=1}^{k} x_{\ell_j}\right); \tag{43}$$

3. *if $m = 1$ and $h(t) = t$, then*

$$\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \geq \frac{k(n-1)}{\binom{n-1}{k-1}n} \sum_{1\leq \ell_1<\cdots<\ell_k\leq n} f\left(\frac{1}{k}\sum_{j=1}^{k} x_{\ell_j}\right); \tag{44}$$

4. *if $f \in \mathrm{SMV}((h,m),[0,b])$, then the inequalities* (42) *to* (44) *are reversed.*

3 Applications to means

In what follows we will apply the theorems and corollaries in the above section to establish inequalities for some special means.

For $r \in \mathbb{R}$, $r \neq 0$, and $m,s \in (0,1]$, let $f(x) = x^r$ for $x \in \mathbb{R}_+$ and $h(t) = t^s$ for $t \in [0,1]$. Then

1. if $r \geq 1$ and $0 < m \leq 1$, or if $r < 0$ and $m = 1$, we have

$$\big(tx + m(1-t)y\big)^r \leq tx^r + (1-t)(my)^r \leq t^s x^r + m(1-t)^s y^r$$

for $x, y \in \mathbb{R}_+$;

2. if $0 < r \leq 1$, $0 < m \leq 1$, and $s = 1$, we have

$$\big(tx + m(1-t)y\big)^r \geq tx^r + (1-t)(my)^r \geq tx^r + m(1-t)y^r$$

for $x, y \in \mathbb{R}_+$.

Using Definition 6 yields the following:

1. if $r \geq 1$ and $0 < m \leq 1$, or if $r < 0$ and $m = 1$, the function $f(x) = x^r \in \mathrm{SMX}((t^s,m),\mathbb{R}_+)$;
2. if $0 < r \leq 1$, $0 < m \leq 1$, and $s = 1$, the function $f(x) = x^r \in \mathrm{SMV}((t,m),\mathbb{R}_+)$.

By virtue of Corollary 10, we obtain the following results.

Theorem 14 *Let $n \geq 3$ and $x_i \in \mathbb{R}_+$ for $i = 1,2,\ldots,n$, let $r \in \mathbb{R}$ with $r \neq 0$ and $m,s \in (0,1]$, and let $\ell_1,\ldots,\ell_k \in \mathbb{N}$ for $2 \leq k \leq n$ and $\ell_{k+1} = \ell_1,\ldots,\ell_{2k-1} = \ell_{k-1}$.*

1. *If $r \geq 1$ and $0 < m \leq 1$, or if $r < 0$ and $m = 1$, then we have*

$$\begin{aligned} &\sum_{i=1}^{n} x_i^r - \left(\sum_{j=0}^{n-1} m^j\right)^{-1} \sum_{i=1}^{n}\left(\frac{1}{n}\sum_{j=i}^{n+i-1} m^{j-i}x_j\right)^r \\ &\quad \geq \frac{k^s(n^s-1)}{\binom{n-1}{k-1}n^s}\left(\sum_{j=0}^{k-1} m^j\right)^{-1} \sum_{1\leq \ell_1<\cdots<\ell_k\leq n}\sum_{i=1}^{k}\left(\frac{1}{k}\sum_{j=i}^{k+i-1} m^{j-i}x_{\ell_j}\right)^r; \end{aligned} \tag{45}$$

2. *if $r \geq 1$ or $r < 0$ and if $m = 1$, we have*

$$\sum_{i=1}^{n} x_i^r - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^r \geq \frac{k^s(n^s-1)}{\binom{n-1}{k-1}n^s} \sum_{1\leq \ell_1<\cdots<\ell_k\leq n}\left(\frac{1}{k}\sum_{j=1}^{k} x_{\ell_j}\right)^r; \tag{46}$$

3. *if* $r \geq 1$ *or* $r < 0$ *and if* $m = s = 1$, *then*

$$\sum_{i=1}^{n} x_i^r - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^r \geq \frac{k(n-1)}{\binom{n-1}{k-1}n} \sum_{1\leq \ell_1 < \cdots < \ell_k \leq n} \left(\frac{1}{k}\sum_{j=1}^{k} x_{\ell_j}\right)^r; \tag{47}$$

4. *if* $0 < r \leq 1$, $0 < m \leq 1$, *and* $s = 1$, *then the inequality* (47) *are reversed*.

Corollary 11 *Under the conditions of Theorem* 14, *when* $\ell_{k+1} = \ell_1, \ldots, \ell_{2k-1} = \ell_{k-1}$, *we have the following conclusions*.

1. *If* $r = 2$, *we have*

$$\begin{aligned} &\sum_{i=1}^{n} x_i^2 - \left(\sum_{j=0}^{n-1} m^j\right)^{-1} \sum_{i=1}^{n}\left(\frac{1}{n}\sum_{j=i}^{n+i-1} m^{j-i}x_j\right)^2 \\ &\quad \geq \frac{k^s(n^s-1)}{\binom{n-1}{k-1}n^s}\left(\sum_{j=0}^{k-1} m^j\right)^{-1} \sum_{1\leq \ell_1 < \cdots < \ell_k \leq n} \sum_{i=1}^{k}\left(\frac{1}{k}\sum_{j=i}^{k+i-1} m^{j-i}x_{\ell_j}\right)^2; \end{aligned} \tag{48}$$

2. *if* $r = 2$ *and* $m = 1$, *we have*

$$\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2 \geq \frac{k^s(n^s-1)}{\binom{n-1}{k-1}n^s} \sum_{1\leq \ell_1 < \cdots < \ell_k \leq n} \left(\frac{1}{k}\sum_{j=1}^{k} x_{\ell_j}\right)^2; \tag{49}$$

3. *if* $r = 2$ *and* $m = s = 1$, *then*

$$\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2 \geq \frac{k(n-1)}{\binom{n-1}{k-1}n} \sum_{1\leq \ell_1 < \cdots < \ell_k \leq n} \left(\frac{1}{k}\sum_{j=1}^{k} x_{\ell_j}\right)^2. \tag{50}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

Author details
[1]College of Mathematics, Inner Mongolia University for Nationalities, Tongliao, Inner Mongolia Autonomous Region 028043, China. [2]Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin, 300387, China.

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