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On some inequalities for functions with nondecreasing increments of higher order

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Abstract
We investigate a class of functions with nondecreasing increments of higher order. A generalization of Brunk's theorem is proved for that class of functions. Also, we consider functions with nondecreasing increments of order three, we obtain the Levinson-type inequality, a generalization of Burkill-Mirsky-Pečarić's results, and a result for the integral mean of a function with nondecreasing increments of higher order.

Keywords: function with nondecreasing increments of higher order; integral mean; Levinson's inequality; monotonicity in means

1 Introduction

Let $\mathbb{R}^k$ denote the k-dimensional vector lattice of points $\mathbf{x} = (x_1, \dots, x_k)$, x_i be real for $i = 1, \dots, k$, with the partial ordering $\mathbf{x} = (x_1, \dots, x_k) \leq \mathbf{y} = (y_1, \dots, y_k)$ if and only if $x_i \leq y_i$ for $i = 1, \dots, k$. We denote

$$a\mathbf{x} + b\mathbf{y} = (ax_1 + by_1, \dots, ax_k + by_k),$$

where $a, b \in \mathbb{R}$, and k-tuple $(0, \dots, 0)$ is denoted by $\mathbf{0}$.

For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^k$, $\mathbf{a} \leq \mathbf{b}$, a set $\{\mathbf{x} \in \mathbb{R}^k : \mathbf{a} \leq \mathbf{x} \leq \mathbf{b}\}$ is called an interval $[\mathbf{a}, \mathbf{b}]$. The following definition of a function with nondecreasing increments is given in [1].

Definition 1.1 A real-valued function f on an interval $\mathbf{I} \subset \mathbb{R}^k$ is said to have nondecreasing increments if

$$f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) \leq f(\mathbf{b} + \mathbf{h}) - f(\mathbf{b}), \tag{1}$$

whenever $\mathbf{a} \in \mathbf{I}$, $\mathbf{b} + \mathbf{h} \in \mathbf{I}$, $\mathbf{0} \leq \mathbf{h} \in \mathbb{R}^k$, $\mathbf{a} \leq \mathbf{b}$.

In the same paper [1], Brunk gave some properties of that family of functions. The most remarkable result for functions with nondecreasing increments is the following Brunk theorem (see also [2, p.266]).

Theorem 1.2 *Let $\mathbf{I}$ be an interval in $\mathbb{R}^k$; $\mathbf{X}(t) = (X_1(t), \dots, X_k(t))$ be a vector of functions where X_i's $(1 \leq i \leq k)$, are nondecreasing and continuous from the right on $[a, b)$. Let H be*

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continuous from the left and of bounded variation on $[a,b)$ *with* $H(a)=0$. *Then*

$$\int_{[a,b)} f\big(\mathbf{X}(t)\big)\, dH(t) \geq 0$$

holds for every continuous function $f : \mathbf{I} \to \mathbb{R}$ *with nondecreasing increments if and only if*

$$H(b) = 0,$$
$$\int_{[a,b)} H(u)\, d\mathbf{X}(u) = 0,$$

and

$$\int_{[a,t]} H(u)\, d\mathbf{X}(u) \geq 0 \quad \textit{for } [a,t] \subset [a,b],$$

where $\int H\, d\mathbf{X} = (\int H\, dX_1, \dots, \int H\, dX_k)$.

More results about functions with nondecreasing increments can be found in papers [3] and [4]. The following theorem is the Jensen-Steffensen type inequality for a function with nondecreasing increments and it is proved in [4].

Theorem 1.3 *Let* $G : [a,b] \to \mathbb{R}$ *be a function of bounded variation such that*

$$G(a) \leq G(x) \leq G(b), \quad G(b) > G(a), \tag{2}$$

and let $\mathbf{X}(t)$ *be a continuous nondecreasing map from the real interval* $[a,b]$ *to the interval* $\mathbf{I} \subset \mathbb{R}^k$. *If* $f : \mathbf{I} \to \mathbb{R}$ *is a continuous function with nondecreasing increments, then*

$$f\left(\frac{\int_a^b \mathbf{X}(t)\, dG(t)}{\int_a^b dG(t)}\right) \leq \frac{\int_a^b f(\mathbf{X}(t))\, dG(t)}{\int_a^b dG(t)}, \tag{3}$$

where $\int_a^b \mathbf{X}\, dG$ *is the vector* $(\int_a^b X_1\, dG, \dots, \int_a^b X_k\, dG)$.

The following theorem gives us a Jensen-type inequality for a function with nondecreasing increments when the finite sequence of k-tuples $(\mathbf{X_1}, \dots, \mathbf{X_n})$ is monotone in means [3]. It is a Pečarić's generalization of Burkill-Mirsky's result. Firstly, let us describe a monotonicity in means. Let p_i, $i = 1, \dots, n$, be positive numbers, $[\mathbf{a}, \mathbf{b}]$ be an interval in $\mathbb{R}^k$. A finite sequence $(\mathbf{X_1}, \dots, \mathbf{X_n}) \in [\mathbf{a}, \mathbf{b}]^n$ is said to be nondecreasing in means with respect to weights $\mathbf{p} = (p_1, \dots, p_n)$ if

$$\mathbf{X_1} \leq A_2(\mathbf{X}; \mathbf{p}) \leq \cdots \leq A_n(\mathbf{X}; \mathbf{p}), \tag{4}$$

where

$$A_j(\mathbf{X}; \mathbf{p}) = \frac{1}{P_j} \sum_{i=1}^{j} p_i \mathbf{X_i}, \quad P_j = \sum_{i=1}^{n} p_i.$$

If inequalities are reversed in (4), then $(\mathbf{X_1}, \dots, \mathbf{X_n})$ is nonincreasing in means.

Theorem 1.4 *Let* $\mathbf{I}$ *be an interval in* $\mathbb{R}^k$, $f:\mathbf{I}\to\mathbb{R}$ *be a continuous function with nondecreasing increments and let* $p_1,\ldots,p_n$ *be positive numbers. If*

$$(\mathbf{X_1},\ldots,\mathbf{X_n}) \quad (\mathbf{X_i}\in\mathbf{I}; i=1,\ldots,n)$$

is nondecreasing or nonincreasing in means with respect to weights $\mathbf{p}=(p_1,\ldots,p_n)$, *then the Jensen-type inequality*

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_i\mathbf{X_i}\right)\le\frac{1}{P_n}\sum_{i=1}^{n}p_if(\mathbf{X_i})$$

holds.

In this paper, we extend the idea of functions with nondecreasing increments. Namely, we define a new class of functions with nondecreasing increments of higher order and prove a result similar to the above-mentioned Brunk theorem. In the third section, we consider functions with nondecreasing increments of order three. Finally, in the last section, a result for an arithmetic integral mean of a function with nondecreasing increments of higher order is given.

2 Functions with nondecreasing increments of order *n*

Let $\mathbf{I}$ be an interval from $\mathbb{R}^k$. Let us write

$$\triangle_{\mathbf{h_1}}f(\mathbf{x})=f(\mathbf{x}+\mathbf{h_1})-f(\mathbf{x})$$

and inductively,

$$\triangle_{\mathbf{h_1}}\triangle_{\mathbf{h_2}}\cdots\triangle_{\mathbf{h_n}}f(\mathbf{x})=\triangle_{\mathbf{h_1}}\big(\triangle_{\mathbf{h_2}}\cdots\triangle_{\mathbf{h_n}}f(\mathbf{x})\big),$$

where $\mathbf{x},\mathbf{x}+\mathbf{h_1}+\cdots+\mathbf{h_n}\in\mathbf{I}$, $\mathbf{0}\le\mathbf{h_i}\in\mathbb{R}^k$ $(i=1,\ldots,n)$. Using this notation with $\mathbf{h}=\mathbf{h_1}$, $\mathbf{s}=\mathbf{h_2}$, $\mathbf{b}=\mathbf{a}+\mathbf{s}$, a condition (1) from the definition of a function with nondecreasing increments becomes

$$\triangle_{\mathbf{h_1}}\triangle_{\mathbf{h_2}}f(\mathbf{a})\ge 0.$$

Let us extend that definition to the following.

Definition 2.1 A real-valued function f on an interval $\mathbf{I}\subset\mathbb{R}^k$ is a function with nondecreasing increments of order n if

$$\triangle_{\mathbf{h_1}}\cdots\triangle_{\mathbf{h_n}}f(\mathbf{x})\ge 0,$$

whenever $\mathbf{x},\mathbf{x}+\mathbf{h_1}+\cdots+\mathbf{h_n}\in\mathbf{I}$, $\mathbf{0}\le\mathbf{h_i}\in\mathbb{R}^k$ $(i=1,\ldots,n)$.

Brunk observed that even if $k=1$ and $n=2$, this does not imply continuity (see [1]). Indeed, every solution of Cauchy's equation $f(\mathbf{x}+\mathbf{y})=f(\mathbf{x})+f(\mathbf{y})$ is a function with nondecreasing increments of order n with null increments, *i.e.*, $\triangle_{\mathbf{h_1}}\cdots\triangle_{\mathbf{h_n}}f(\mathbf{x})=0$. If the nth partial derivatives $f_{i_1\cdots i_n}(\mathbf{x})=\frac{\partial^n}{\partial x_{i_1}\cdots x_{i_n}}f(\mathbf{x})$ exist, they are nonnegative. If f is a continuous function with nondecreasing increments of order n, it may be approximated uniformly on **I** by polynomials having nonnegative nth partial derivatives. To see this, let us set, for convenience, $\mathbf{I} = [\mathbf{0}, \mathbf{1}]$, $\mathbf{1} = (1, \dots, 1)$. It is known that the Bernstein polynomials

$$\sum_{i_1=0}^{n_1} \cdots \sum_{i_k=0}^{n_k} f\left(\frac{i_1}{n_1}, \dots, \frac{i_k}{n_k}\right) \prod_{j=1}^{k} \binom{n_j}{i_j} x_j^{i_j} (1-x_j)^{n_j - i_j}$$

converge uniformly to f on **I** as $n_1 \to \infty, \dots, n_k \to \infty$, if f is continuous. Furthermore, if f is a function with nondecreasing increments of order n, these polynomials have nonnegative nth partial derivatives, as may be shown by repeated application of the formula (see [1])

$$\frac{d}{dx} \sum_{i=0}^{n} \binom{n}{i} a_i x^i (1-x)^{n-i} = n \sum_{i=0}^{n-1} \binom{n-1}{i} (a_{i+1} - a_i) x^i (1-x)^{n-1-i}.$$

The aim of the rest of this section is to prove a result similar to Theorem 1.2. Let us introduce some further notations.

Let $p_1, \dots, p_r$ be positive integers and let $p_1 + \cdots + p_r = w$. Let $(i_1^{p_1} \cdots i_r^{p_r})_p$ be a set of all permutations with repetitions whose elements are from the multiset

$$S = \{\underbrace{i_1, \dots, i_1}_{p_1\text{-times}}, \underbrace{i_2, \dots, i_2}_{p_2\text{-times}}, \dots, \underbrace{i_r, \dots, i_r}_{p_r\text{-times}}\}, \quad i_1 < \cdots < i_r, i_1, \dots, i_r \in \{1, \dots, k\}.$$

There are $\frac{w!}{p_1! p_2! \cdots p_r!}$ elements in the class $(i_1^{p_1} \cdots i_r^{p_r})_p$.

For $0 < p_1 \le p_2 \le \cdots \le p_r$, $p_1 + \cdots + p_r = w$, let $(p_1 \cdots p_r)_c$ be a set whose elements are described in the following way. We say that permutation $j_1 \cdots j_w$ belongs to the set $(p_1 \cdots p_r)_c$ iff there exist $i_1, i_2, \dots, i_r \in \{1, 2, \dots, k\}$, $i_1 < i_2 < \cdots < i_r$ and permutation σ of the multiset $\{p_1 \cdots p_r\}$ such that $j_1 \cdots j_w \in (i_1^{\sigma(p_1)} \cdots i_r^{\sigma(p_r)})_p$. Family of all classes $(p_1 \cdots p_r)_c$ is denoted with C_w^k.

For illustration, we describe the above notation on one example. Let $k = 5$ and $w = 4$. Classes $(p_1 \cdots p_r)_c$ are the following: $(1,1,1,1)_c$, $(1,1,2)_c$, $(1,3)_c$, $(2,2)_c$ and $(4)_c$. Let us describe the elements of the set $(1,1,2)_c$. There are three different permutations of the multiset $\{1,1,2\}$. These are

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix}.$$

So, $(i_1^{\sigma(p_1)} \cdots i_r^{\sigma(p_r)})_p$ are $(i_1, i_2, i_3, i_3)_p$, $(i_1, i_2, i_2, i_3)_p$, $(i_1, i_1, i_2, i_3)_p$, where $i_1 < i_2 < i_3$ and $i_1, i_2, i_3 \in \{1,2,3,4,5\}$. If, for example, $(i_1, i_2, i_3, i_3)_p = (2,3,5,5)_p$, then it contains all permutations with repetitions of elements $2,3,5,5$, *i.e.*, $(2,3,5,5)_p = \{2355, 2535, 2533, \dots, 5532\}$ and it has $\frac{4!}{2!} = 12$ elements.

In the following text, H is a function of bounded variation on $[a,b]$ with $H(a) = 0$ and $i_1, i_2, \dots, i_n \in \{1, 2, \dots, k\}$. Let $K^n_{i_1 \cdots i_n}$ be a function such that

$$K^n_{i_1 \cdots i_n}(t) = \int_a^t K^{n-1}_{i_1 \cdots i_{n-1}}(x_n)\, dX_{i_n}(x_n) \quad \text{for } n \ge 2$$

and

$$K^1_{i_1}(t) = \int_a^t H(x_1)\, dX_{i_1}(x_1).$$

Further we write

$$\prod(S)(x) = \prod_{j\in S}\bigl(X_j(t) - X_j(x)\bigr),$$

$$\prod(\phi)(x) = 1,$$

where S is a multiset with elements from $\{1, 2, \ldots, k\}$.

It is obvious that

$$d\Bigl\{\prod(S)(x)\Bigr\} = -\sum_{j\in S} dX_j(x)\prod\bigl(S\setminus\{j\}\bigr)(x),$$

and

$$dK^n_{i_1\cdots i_n}(t) = K^{n-1}_{i_1\cdots i_{n-1}}(t)\,dX_{i_n}(t).$$

Now, the following result holds.

Lemma 2.2 *Let w be a fixed positive integer. Then*

$$\begin{aligned}&\int_a^t \prod\bigl(\{i_1, \ldots, i_w\}\bigr)(x)\,dH(x)\\&\quad = \sum_{j_1=1}^{w}\sum_{\substack{j_2=1\\ j_2\neq j_1}}^{w}\cdots\sum_{\substack{j_m=1\\ j_m\neq j_k\\ k<m}}^{w}\int_a^t \prod\bigl(\{i_1, \ldots, i_w\}\setminus\{i_{j_1}, \ldots, i_{j_m}\}\bigr)(x)\,dK^m_{i_{j_1}\cdots i_{j_m}}(x)\end{aligned}$$

holds for every $m \in \{1, 2, \ldots, w\}$.

Proof We prove it using induction by m. For $m = 1$, using integration by parts, we have

$$\begin{aligned}\int_a^t \prod\bigl(\{i_1, \ldots, i_w\}\bigr)(x)\,dH(x) &= -\int_a^t H(x)\,d\Bigl(\prod\bigl(\{i_1, \ldots, i_w\}\bigr)(x)\Bigr)\\&= \int_a^t H(x)\sum_{j_1=1}^{w} dX_{j_1}(x)\prod\bigl(\{i_1, \ldots, i_m\}\setminus\{i_{j_1}\}\bigr)(x)\\&= \sum_{j_1=1}^{w}\int_a^t \prod\bigl(\{i_1, \ldots, i_w\}\setminus\{i_{j_1}\}\bigr)(x)\,dK^1_{i_{j_1}}(x).\end{aligned}$$

Let us suppose that the statement holds for $m - 1$ and let us apply integration by parts on the right-hand side of the formula.

$$\begin{aligned}&\int_a^t \prod\bigl(\{i_1, \ldots, i_w\}\bigr)(x)\,dH(x)\\&\quad = \sum_{j_1=1}^{w}\cdots\sum_{\substack{j_{m-1}=1\\ j_{m-1}\neq j_k\\ k<m-1}}^{w}\int_a^t \prod\bigl(\{i_1, \ldots, i_w\}\setminus\{i_{j_1}, \ldots, i_{j_{m-1}}\}\bigr)(x)\,dK^{m-1}_{i_{j_1},\ldots,i_{j_{m-1}}}(x)\end{aligned}$$

$$= \sum_{j_1=1}^{w} \cdots \sum_{\substack{j_{m-1}=1 \\ j_{m-1}\neq j_k \\ k<m-1}}^{w} (-1) \int_a^t K^{m-1}_{i_{j_1},\dots,i_{j_{m-1}}}(x)\, d\Bigl(\prod\bigl(\{i_1,\dots,i_w\} \setminus \{i_{j_1},\dots,i_{j_{m-1}}\}\bigr)(x)\Bigr)$$

$$= \sum_{j_1=1}^{w} \cdots \sum_{\substack{j_{m-1}=1 \\ j_{m-1}\neq j_k \\ k<m-1}}^{w} (-1) \int_a^t K^{m-1}_{i_{j_1},\dots,i_{j_{m-1}}}(x)$$

$$\times (-1) \sum_{\substack{j_m=1 \\ j_m\neq j_k \\ k<m}}^{w} dX_{i_{j_m}}(x) \prod\bigl(\{i_1,\dots,i_w\} \setminus \{i_{j_1},\dots,i_{j_m}\}\bigr)(x)$$

$$= \sum_{j_1=1}^{w} \cdots \sum_{\substack{j_m=1 \\ j_m\neq j_k \\ k<m}}^{w} \int_a^t \prod\bigl(\{i_1,\dots,i_w\} \setminus \{i_{j_1},\dots,i_{j_m}\}\bigr)(x) K^{m-1}_{i_{j_1}\cdots i_{j_{m-1}}}(x)\, dX_{i_{j_m}}(x)$$

$$= \sum_{j_1=1}^{w} \cdots \sum_{\substack{j_m=1 \\ j_m\neq j_k \\ k<m}}^{w} \int_a^t \prod\bigl(\{i_1,\dots,i_w\} \setminus \{i_{j_1},\dots,i_{j_m}\}\bigr)(x)\, dK^{m}_{i_{j_1}\cdots i_{j_m}}(x).$$

□

Especially for $m = w$, we have

$$\begin{aligned} \int_a^t \prod\bigl(\{i_1,\dots,i_w\}\bigr)(x)\, dH(x) &= \sum_{j_1=1}^{w} \cdots \sum_{\substack{j_w=1 \\ j_w\neq j_k \\ k<w}}^{w} \int_a^t dK^{w}_{i_{j_1}\cdots i_{j_w}}(x) \\ &= \sum_{j_1=1}^{w} \cdots \sum_{\substack{j_w=1 \\ j_w\neq j_k \\ k<w}}^{w} K^{w}_{i_{j_1}\cdots i_{j_w}}(t) \\ &= p_1!p_2!\cdots p_r! \sum_{i_{j_1}\cdots i_{j_w}\in(i_1^{p_1}\cdots i_r^{p_r})_p} K^{w}_{i_{j_1}\cdots i_{j_w}}(t), \end{aligned} \tag{5}$$

where $\{i_{j_1},\dots,i_{j_w}\} = \{\underbrace{i_1,\dots,i_1}_{p_1\text{-times}},\dots,\underbrace{i_r,\dots,i_r}_{p_r\text{-times}}\}$, $i_1 < i_2 < \cdots < i_r$; $i_1, i_2, \dots, i_r \in \{1,2,\dots,k\}$, $p_1 + \cdots + p_r = w$.

Example 2.3 If $w = 3$, $i_1 = i_2 = 1$, $i_3 = 2$, then

$$\begin{aligned} \int_a^t \prod\bigl(\{1,1,2\}\bigr)(x)\, dH(x) &= \sum_{j_1=1}^{3} \sum_{\substack{j_2=1 \\ j_2\neq j_1}}^{3} \sum_{\substack{j_3=1 \\ j_3\neq j_1,j_2}}^{3} K^3_{i_{j_1} i_{j_2} i_{j_3}}(t) \\ &= 2!1!\bigl(K^3_{112} + K^3_{121} + K^3_{211}\bigr). \end{aligned}$$

Furthermore, if we suppose

$$\int_a^b X_{j_1}(u)\cdots X_{j_s}(u)\, dH(u) = 0 \quad \bigl(j_1,\dots,j_s \in \{1,\dots,k\}, s = 0,\dots,w\bigr),$$

then

$$p_1!\cdots p_r!\sum K^w_{i_{j_1}\cdots i_{j_w}}(b) = \int_a^b \prod\bigl(\{i_1,\ldots,i_w\}\bigr)(x)\,dH(x)$$
$$= \sum(-1)^s \int_a^b X_{j_1}(x)\cdots X_{j_s}(x)X_{j_{s+1}}(b)\cdots X_{j_w}(b)\,dH(x) = 0. \quad (6)$$

Theorem 2.4 *Let $\mathbf{X}:[a,b]\to\mathbf{I}\subset\mathbb{R}^k$ be a continuous function. Let H be a function of bounded variation on $[a,b]$ with $H(a)=H(b)=0$ and let f have continuous $(n-1)$th partial derivatives, $n\geq 2$. Then the following statement holds: if*

$$\int_a^b X_{i_1}(u)\cdots X_{i_m}(u)\,dH(u) = 0 \quad \bigl(i_1,\ldots,i_m\in\{1,\ldots,k\}, m=1,2,\ldots,n-1\bigr),$$

then

$$\int_a^b f\bigl(\mathbf{X}(t)\bigr)\,dH(t) = (-1)^{n-1}\sum_{(p_1\cdots p_r)_c\in C^k_{n-1}}\frac{1}{p_1!\cdots p_r!}$$
$$\times \sum_{(i_1^{p_1}\cdots i_r^{p_r})_p\subseteq(p_1\cdots p_r)_c}\int_a^b f_{\underbrace{i_1\cdots i_1}_{p_1\text{-times}}\cdots\underbrace{i_r\cdots i_r}_{p_r\text{-times}}}\bigl(\mathbf{X}(t)\bigr)$$
$$\times\, d\biggl(\int_a^t \prod\bigl(\{i_1^{p_1},\ldots,i_r^{p_r}\}\bigr)(x)\,dH(x)\biggr). \quad (7)$$

Proof For $n=2$, we have

$$\begin{aligned}
\int_a^b f\bigl(\mathbf{X}(t)\bigr)\,dH(t) &= -\sum_{i=1}^k\int_a^b f_i\bigl(\mathbf{X}(t)\bigr)H(t)\,dX_i(t)\\
&= -\sum_{i=1}^k\int_a^b f_i\bigl(\mathbf{X}(t)\bigr)\,dK_i^1(t)\\
&= -\sum_{i=1}^k\int_a^b f_i\bigl(\mathbf{X}(t)\bigr)\,d\biggl(\int_a^t H(x)\,dX_i(x)\biggr)\\
&= -\sum_{i=1}^k\int_a^b f_i\bigl(\mathbf{X}(t)\bigr)\,d\biggl(\int_a^t H(x)\,d\bigl(X_i(x)-X_i(t)\bigr)\biggr)\\
&= \sum_{i=1}^k\int_a^b f_i\bigl(\mathbf{X}(t)\bigr)\,d\biggl(\int_a^t H(x)\,d\bigl(X_i(t)-X_i(x)\bigr)\biggr)\\
&= -\sum_{i=1}^k\int_a^b f_i\bigl(\mathbf{X}(t)\bigr)\,d\biggl(\int_a^t \bigl(X_i(t)-X_i(x)\bigr)\,dH(x)\biggr)\\
&= -\sum_{i=1}^k\int_a^b f_i\bigl(\mathbf{X}(t)\bigr)\,d\biggl(\int_a^t \prod\bigl(\{i\}\bigr)(x)\,dH(x)\biggr).
\end{aligned}$$

If we have $\int_a^b X_{i_1}(u)\cdots X_{i_m}(u)\,dH(u) = 0$ for $m = 1,2,\ldots,n-2$, $i_1,\ldots,i_m \in \{1,\ldots,k\}$ and if we suppose that (7) holds for $(n-1)$, then

$$
\begin{aligned}
&\int_a^b f(\mathbf{X}(t))\,dH(t) \\
&= (-1)^{n-2} \sum_{(p_1\cdots p_r)_c\in C^k_{n-2}} \frac{1}{p_1!\cdots p_r!} \sum_{(i_1^{p_1}\cdots i_r^{p_r})_p\subseteq(p_1\cdots p_r)_c} \int_a^b f_{i_1^{p_1}\cdots i_r^{p_r}}(\mathbf{X}(t)) \\
&\quad\times d\biggl(\int_a^t \prod\bigl(\bigl\{i_1^{p_1},\ldots,i_r^{p_r}\bigr\}\bigr)(x)\,dH(x)\biggr) \\
&= (-1)^{n-2} \sum_{(p_1\cdots p_r)_c\in C^k_{n-2}} \frac{1}{p_1!\cdots p_r!} \sum_{(i_1^{p_1}\cdots i_r^{p_r})_p} \int_a^b f_{i_1^{p_1}\cdots i_r^{p_r}}(\mathbf{X}(t)) \\
&\quad\times d\biggl(p_1!\cdots p_r! \sum_{i_{j_1}\cdots i_{j_{n-2}}\in(i_1^{p_1}\cdots i_r^{p_r})_p} K^{n-2}_{i_{j_1}\cdots i_{j_{n-2}}}(t)\biggr) \\
&= (-1)^{n-1} \sum_{(p_1\cdots p_r)_c\in C^k_{n-2}} \sum_{(i_1^{p_1}\cdots i_r^{p_r})_p} \int_a^b df_{i_1^{p_1}\cdots i_r^{p_r}}(\mathbf{X}(t)) \\
&\quad\times \sum_{i_{j_1}\cdots i_{j_{n-2}}\in(i_1^{p_1}\cdots i_r^{p_r})_p} K^{n-2}_{i_{j_1}\cdots i_{j_{n-2}}}(t) \\
&= (-1)^{n-1} \sum_{(p_1\cdots p_r)_c\in C^k_{n-2}} \sum_{(i_1^{p_1}\cdots i_r^{p_r})_p} \int_a^b \sum_{i_{n-1}=1}^{k} f_{i_1^{p_1}\cdots i_r^{p_r}\,i_{n-1}}(\mathbf{X}(t)) \\
&\quad\times dX_{i_{n-1}}(t)\biggl(\sum_{i_{j_1}\cdots i_{j_{n-2}}} K^{n-2}_{i_{j_1}\cdots i_{j_{n-2}}}(t)\biggr) \\
&= (-1)^{n-1} \sum_{\substack{(s_1\cdots s_g)_c\in C^k_{n-1}\\ s_1+\cdots+s_g=n-1}} \sum_{(i_1^{s_1}\cdots i_g^{s_g})_p\subset(s_1\cdots s_g)_c} \int_a^b f_{i_1^{s_1}\cdots i_g^{s_g}}(\mathbf{X}(t)) \\
&\quad\times \biggl(\sum_{l_1\cdots l_{n-1}\in(i_1^{s_1}\cdots i_g^{s_g})_p} K^{n-2}_{l_1\cdots l_{n-2}}(t)\,dX_{l_{n-1}}(t)\biggr) \\
&= (-1)^{n-1} \sum_{(s_1\cdots s_g)_c\in C^k_{n-1}} \sum_{(i_1^{s_1}\cdots i_g^{s_g})_p} \int_a^b f_{i_1^{s_1}\cdots i_g^{s_g}}(\mathbf{X}(t))\,d\biggl(\sum_{l_1\cdots l_{n-1}} K^{n-1}_{l_1\cdots l_{n-1}}(t)\biggr) \\
&= (-1)^{n-1} \sum_{(s_1\cdots s_g)_c\in C^k_{n-1}} \sum_{(i_1^{s_1}\cdots i_g^{s_g})_p} \int_a^b f_{i_1^{s_1}\cdots i_g^{s_g}}(\mathbf{X}(t)) \\
&\quad\times d\biggl(\frac{1}{s_1!\cdots s_g!}\int_a^b \prod\bigl(\bigl\{i_1^{s_1}\cdots i_g^{s_g}\bigr\}\bigr)\,dH(x)\biggr)
\end{aligned}
$$

by (5) and (6). □

Theorem 2.5 *Let X be a nondecreasing continuous map from the real interval $[a,b]$ into an interval $\mathbf{I}\subset\mathbb{R}^k$, and let H be a function of bounded variation on $[a,b]$ with $H(a)=0$.*

Then

$$\int_a^b f\bigl(\mathbf{X}(t)\bigr)\,dH(t) \geq 0 \tag{8}$$

for every continuous function f with nondecreasing increments of order n on $\mathbf{I}$ if and only if

$$H(b) = 0, \tag{9}$$

$$\int_a^b X_{i_1}(t)\cdots X_{i_m}(t)\,dH(t) = 0 \tag{10}$$

for $i_1,\ldots,i_m \in \{1,\ldots k\}$, $m = 1,2,\ldots,n-1$ and

$$(-1)^n \int_a^t \prod\bigl(\{i_1,\ldots,i_{n-1}\}\bigr)(u)\,dH(u) \geq 0 \tag{11}$$

for all $t \in [a,b]$, $i_1,\ldots,i_{n-1} \in \{1,\ldots k\}$.

Proof Necessity: The validity of (8) for constant functions $f = 1$ and $f = -1$ implies (9). From (8) for $f(x) = x_{i_1}\cdots x_{i_s}$ and $f(x) = -x_{i_1}\cdots x_{i_s}$ $(s = 1,\ldots,n-1)$, we have (10).

Inequality (11) is obtained from (8) on setting, for fixed $t \in [a,b]$ and fixed $i_1\cdots i_{n-1} \in \{1,\ldots,k\}$,

$$f(x) = -\bigl[x_{i_1} - X_{i_1}(t)\bigr]^- \cdots [x_{i_{n-1}} - X_{i_{n-1}(t)}]^-, \quad \text{where } c^- = \min\{c,0\}, (c \in \mathbb{R}).$$

Sufficiency: Since f may be approximated uniformly on $\mathbf{I}$ by functions with continuous nonnegative nth partial derivatives, we may assume that the nth partials $f_{i_1\cdots i_n}$ exist and are continuous and nonnegative. By Theorem 2.4 and (10), we have

$$\begin{aligned}
&\int_a^b f\bigl(\mathbf{X}(t)\bigr)\,dH(t)\\
&\quad = (-1)^n \sum_{(p_1\cdots p_r)_c \in C_{n-1}^k} \frac{1}{p_1!\cdots p_r!} \sum_{(i_1^{p_1}\cdots i_r^{p_r})_p \subseteq (p_1,\ldots,p_r)_c} \sum_{i_n=1}^k \int_a^b f_{i_1^{p_1}\cdots i_r^{p_r} i_n}\bigl(\mathbf{X}(t)\bigr)\\
&\qquad \times dX_{i_n}(t) \int_a^t \prod\bigl\{i_1^{p_1}\cdots i_r^{p_r}\bigr\}(x)\,dH(x).
\end{aligned}$$

By (11), each term in the sum is nonnegative so that (8) is verified. □

3 Functions with nondecreasing increments of order three

3.1 On inequalities of Levinson type

Levinson [5] proved that if a real-valued function f defined on $[0,2a] \subset \mathbb{R}$ has a nonnegative third derivative, then

$$\begin{aligned}
&\frac{1}{P_n}\sum_{k=1}^n p_k f(x_k) - f\left(\frac{1}{P_n}\sum_{k=1}^n p_k x_k\right)\\
&\quad \leq \frac{1}{P_n}\sum_{k=1}^n p_k f(y_k) - f\left(\frac{1}{P_n}\sum_{k=1}^n p_k y_k\right)
\end{aligned} \tag{12}$$

for $0 < x_k < a$, $y_k = 2a - x_k$, $p_k > 0$ $(1 \leq k \leq n)$, $P_n = \sum_{k=1}^n p_k$.

If $a = \frac{1}{2}$, $p_1 = \cdots = p_n = 1$ and $f(x) = \log x$, then Levinson's inequality (12) becomes the famous Ky-Fan inequality

$$\frac{G_n}{G'_n} \leq \frac{A_n}{A'_n},$$

where $A_n = \frac{1}{n}\sum_{k=1}^{n} x_k$, $A'_n = \frac{1}{n}\sum_{k=1}^{n}(1 - x_k)$, $G_n = (\prod_{k=1}^{n} x_k)^{1/n}$ and $G'_n = (\prod_{k=1}^{n}(1 - x_k))^{1/n}$.

In [6] Pečarić showed that instead of variables the sum of which is equal to $2a$, we can use variables the difference of which is constant, and that result becomes a source of some further generalizations [2, pp.74, 75]. In fact, he proved that if f is a real-valued 3-convex function on $[a, b]$ and x_k, y_k $(1 \leq k \leq n)$, $2n$ points on $[a, b]$ such that

$$y_1 - x_1 = y_2 - x_2 = \cdots = y_n - x_n > 0$$

and $p_k > 0$ $(1 \leq k \leq n)$, then (12) is valid.

The following theorem is a generalization of the Levinson inequality.

Theorem 3.1 *Let $G : [a, b] \to \mathbb{R}$ be a function of bounded variation such that* (2) *holds, and let $\mathbf{X}(t)$ be a continuous and nondecreasing map from $[a, b] \subset \mathbb{R}$ to an interval $\mathbf{I} = [\mathbf{0}, \mathbf{d}] \subset \mathbb{R}^k$, $\mathbf{d} > \mathbf{0}$. If f is a continuous function with nondecreasing increments of order three on $\mathbf{J} = [\mathbf{0}, 2\mathbf{d}]$, then*

$$\begin{aligned} &\frac{\int_a^b f(\mathbf{X}(t))\,dG(t)}{\int_a^b dG(t)} - f\left(\frac{\int_a^b \mathbf{X}(t)\,dG(t)}{\int_a^b dG(t)}\right) \\ &\quad \leq \frac{\int_a^b f(2\mathbf{d} - \mathbf{X}(t))\,dG(t)}{\int_a^b dG(t)} - f\left(\frac{\int_a^b (2\mathbf{d} - \mathbf{X}(t))\,dG(t)}{\int_a^b dG(t)}\right). \end{aligned}$$

Proof If f is a function with nondecreasing increments of order three on $\mathbf{J}$, then

$$\Delta_{\mathbf{h}}\Delta_{\mathbf{t}}\Delta_{\mathbf{s}} f(\mathbf{x}) \geq 0 \quad \bigl(\mathbf{x}, \mathbf{x} + \mathbf{h} + \mathbf{t} + \mathbf{s} \in \mathbf{J}, \mathbf{0} \leq \mathbf{h}, \mathbf{t}, \mathbf{s} \in \mathbb{R}^k\bigr),$$

i.e.,

$$\Delta_{\mathbf{h}}\Delta_{\mathbf{t}}\bigl(f(\mathbf{x} + \mathbf{s}) - f(\mathbf{x})\bigr) \geq 0. \tag{13}$$

If $\mathbf{x} \in \mathbf{I}$ and $\mathbf{s} = 2\mathbf{d} - 2\mathbf{x}$, we have

$$\Delta_{\mathbf{h}}\Delta_{\mathbf{t}}\bigl(f(2\mathbf{d} - \mathbf{x}) - f(\mathbf{x})\bigr) \geq 0,$$

i.e., the function $\mathbf{x} \mapsto f(2\mathbf{d} - \mathbf{x}) - f(\mathbf{x})$ is a function with nondecreasing increments of order two, *i.e.*, it is a function with nondecreasing increments. Now, using Theorem 1.3, we obtain Theorem 3.1. □

Theorem 3.2 *Let $G : [a, b] \to \mathbb{R}$ be a function of bounded variation such that* (2) *holds, and let f be a continuous function with nondecreasing increments of order three on*

$[\mathbf{c},\mathbf{d}] \subset \mathbb{R}^k$. *Let* $\mathbf{0} < \mathbf{a} < \mathbf{d} - \mathbf{c}$. *If* $\mathbf{X}(t) : [a,b] \to [\mathbf{c}, \mathbf{d} - \mathbf{a}]$ *is a continuous and nondecreasing map, then*

$$\frac{\int_a^b f(\mathbf{X}(t))\,dG(t)}{\int_a^b dG(t)} - f\left(\frac{\int_a^b \mathbf{X}(t)\,dG(t)}{\int_a^b dG(t)}\right) \le \frac{\int_a^b f(\mathbf{a} + \mathbf{X}(t))\,dG(t)}{\int_a^b dG(t)} - f\left(\frac{\int_a^b (\mathbf{a} + \mathbf{X}(t))\,dG(t)}{\int_a^b dG(t)}\right).$$

Proof Using (13) for $\mathbf{s} = \mathbf{a} = \text{constant} \in \mathbb{R}^k$, we have that the function $\mathbf{x} \mapsto f(\mathbf{a} + \mathbf{x}) - f(\mathbf{x})$ is a function with nondecreasing increments, so from Theorem 1.3, we obtain Theorem 3.2. For $k = 1$, we have a result from [6]. □

Corollary 3.3 (i) *Let* $\mathbf{X}$ *satisfy the assumptions of Theorem* 3.1. *Then*

$$\begin{aligned} 0 &\le \left(\int_a^b dG(t)\right)^{k-1} \int_a^b \prod_{i=1}^k X_i(t)\,dG(t) - \prod_{i=1}^k \int_a^b X_i(t)\,dG(t) \\ &\le \left(\int_a^b dG(t)\right)^{k-1} \int_a^b \prod_{i=1}^k \bigl(2d_i - X_i(t)\bigr)\,dG(t) - \prod_{i=1}^k \int_a^b \bigl(2d_i - X_i(t)\bigr)\,dG(t). \end{aligned}$$

(ii) *If* $\mathbf{X}$ *satisfies the assumptions of Theorem* 3.2, *then*

$$\begin{aligned} 0 &\le \left(\int_a^b dG(t)\right)^{k-1} \int_a^b \prod_{i=1}^k X_i(t)\,dG(t) - \prod_{i=1}^k \int_a^b X_i(t)\,dG(t) \\ &\le \left(\int_a^b dG(t)\right)^{k-1} \int_a^b \prod_{i=1}^k \bigl(a_i + X_i(t)\bigr)\,dG(t) - \prod_{i=1}^k \int_a^b \bigl(a_i + X_i(t)\bigr)\,dG(t), \end{aligned}$$

where all components of $\mathbf{X}$ *are nonnegative.*

Proof The function $f(\mathbf{x}) = x_1 \cdots x_k$ is a function with nondecreasing increments of orders two and three for $x_i \ge 0$ $(i = 1, \ldots, k)$. So, using Theorems 1.3, 3.1, and 3.2, we obtain Corollary 3.3. □

3.2 Generalization of Burkill-Mirsky-Pečarić result

In this subsection, we consider a sequence of k-tuples $(\mathbf{X_1}, \ldots, \mathbf{X_n})$ which is monotone in means.

Theorem 3.4 *Let* f *be a continuous function with nondecreasing increments of order three on* $\mathbf{J} = [\mathbf{0}, 2\mathbf{d}]$, $\mathbf{d} > \mathbf{0}$, *and let* $p_1, \ldots, p_n$ *be positive numbers. If*

$$(\mathbf{X_1}, \ldots, \mathbf{X_n}) \quad \bigl(\mathbf{X_i} \in \mathbf{I} = [\mathbf{0}, \mathbf{d}]\bigr)$$

is nondecreasing or nonincreasing in means with respect to positive weights p_i $(i = 1, \ldots, n)$, *then*

$$\frac{1}{P_n}\sum_{i=1}^n p_i f(\mathbf{X_i}) - f\left(\frac{1}{P_n}\sum_{i=1}^n p_i \mathbf{X_i}\right) \le \frac{1}{P_n}\sum_{i=1}^n p_i f(2\mathbf{d} - \mathbf{X_i}) - f\left(\frac{1}{P_n}\sum_{i=1}^n p_i (2\mathbf{d} - \mathbf{X_i})\right)$$

holds.

Proof Since f is a function with nondecreasing increments of order three on $\mathbf{J}$, so a function $\mathbf{x} \mapsto f(2\mathbf{d} - \mathbf{x}) - f(\mathbf{x})$ is a function with nondecreasing increments. Then by Theorem 1.4, we obtain the required result. □

Theorem 3.5 *Let f be a continuous function with nondecreasing increments of order three on $\mathbf{J} = [\mathbf{c}, \mathbf{d}]$ and let $p_1, \ldots, p_n$ be positive numbers. Let $\mathbf{0} < \mathbf{a} < \mathbf{d} - \mathbf{c}$. If*

$$(\mathbf{X_1}, \ldots, \mathbf{X_n}) \quad \bigl(\mathbf{X_i} \in \mathbf{I} = [\mathbf{c}, \mathbf{d} - \mathbf{a}]\bigr)$$

is nondecreasing or nonincreasing in means with respect to positive weights p_i $(i = 1, \ldots, n)$, then

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f(\mathbf{X_i}) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i \mathbf{X_i}\right) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i f(\mathbf{a} + \mathbf{X_i}) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i (\mathbf{a} + \mathbf{X_i})\right)$$

holds.

Proof By following the proof of Theorem 3.2, we obtain Theorem 3.5 by simply replacing 'Theorem 1.3' by 'Theorem 1.4' in the proof of Theorem 3.2. □

Corollary 3.6 (i) *Let* $\mathbf{X}$ *satisfy the assumptions of Theorem* 3.4. *Then*

$$\begin{aligned} 0 &\le P_n^{k-1}\sum_{i=1}^{n} p_i^k \left(\prod_{j=1}^{k} x_{ij}\right) - \prod_{j=1}^{k}\left(\sum_{i=1}^{n} p_i x_{ij}\right) \\ &\le P_n^{k-1}\sum_{i=1}^{n} p_i^k \left(\prod_{j=1}^{k} (2d_j - x_{ij})\right) - \prod_{j=1}^{k}\left(\sum_{i=1}^{n} p_i (2d_j - x_{ij})\right). \end{aligned}$$

(ii) *If* $\mathbf{X}$ *satisfies the assumptions of Theorem* 3.5. *Then*

$$\begin{aligned} 0 &\le P_n^{k-1}\sum_{i=1}^{n} p_i^k \left(\prod_{j=1}^{k} x_{ij}\right) - \prod_{j=1}^{k}\left(\sum_{i=1}^{n} p_i x_{ij}\right) \\ &\le P_n^{k-1}\sum_{i=1}^{n} p_i^k \left(\prod_{j=1}^{k} (a_j + x_{ij})\right) - \prod_{j=1}^{k}\left(\sum_{i=1}^{n} p_i (a_j + x_{ij})\right), \end{aligned}$$

where all components of $\mathbf{X}$ *are nonnegative.*

Proof We consider again the function $f(\mathbf{x}) = x_1 \cdots x_k$ which is a function with nondecreasing increments of orders two and three for $x_i \ge 0$ $(i = 1, \ldots, k)$. So, using Theorems 1.4, 3.4, and 3.5, we obtain Corollary 3.6. □

4 Arithmetic integral mean

It is known that if $f : [0, a] \to \mathbb{R}$, $a > 0$, is nonnegative and nondecreasing, then the function F,

$$F(x) = \frac{1}{x}\int_0^x f(u)\,du,$$

is also a nondecreasing function on $[0, a]$. Let us observe that F is an arithmetic integral mean of a function f on an interval $[0, a]$. This result was generalized in [7] considering a real-valued function f for which $\Delta_h^m f(x) \geq 0$ holds for any $h > 0$. Δ_h^m is defined as follows: $\Delta_h^0 f(x) = f(x)$, $\Delta_h^m f(x) = \Delta_h^{m-1} f(x+h) - \Delta_h^{m-1} f(x)$.

Here, we extend the above-mentioned result to functions with nondecreasing increments of higher order.

Theorem 4.1 *Let the function $f : [\mathbf{a}, \mathbf{b}] \to \mathbb{R}$ be continuous and with nondecreasing increments of order n. Then the function*

$$F(\mathbf{x}) = \left(\prod_{i=1}^{k}(x_i - a_i)\right)^{-1} \int_{a_1}^{x_1} \cdots \int_{a_k}^{x_k} f(\mathbf{u})\, \mathbf{du},$$

where $\mathbf{u} = (u_1, \ldots, u_k)$ and $\mathbf{du} = du_1 \cdots du_k$, is a function with nondecreasing increments of order n on $[\mathbf{a}, \mathbf{b}]$.

Proof Let $\mathbf{x} > \mathbf{a} = (a_1, \ldots, a_k)$. Then

$$F(\mathbf{x}) = \int_0^1 \cdots \int_0^1 f\big(\mathbf{a} + \mathbf{s}(\mathbf{x} - \mathbf{a})\big)\, \mathbf{ds},$$

where we used the substitutions $u_i = a_i + s_i(x_i - a_i)$ $(1 \leq i \leq k,\ 0 \leq s_i \leq 1)$, and where $\mathbf{a} + \mathbf{s}(\mathbf{x} - \mathbf{a}) = (a_1 + s_1(x_1 - a_1), \ldots, a_k + s_k(x_k - a_k))$, $\mathbf{ds} = ds_1 \cdots ds_k$. Now, we have

$$\begin{aligned} \Delta_{\mathbf{h_1}} \cdots \Delta_{\mathbf{h_n}} F(\mathbf{x}) &= \Delta_{\mathbf{h_1}} \cdots \Delta_{\mathbf{h_n}} \int_0^1 \cdots \int_0^1 f\big(\mathbf{a} + \mathbf{s}(\mathbf{x} - \mathbf{a})\big)\, \mathbf{ds} \\ &= \int_0^1 \cdots \int_0^1 \Delta_{\mathbf{h_1}} \cdots \Delta_{\mathbf{h_n}} f\big(\mathbf{a} + \mathbf{s}(\mathbf{x} - \mathbf{a})\big)\, \mathbf{ds} \geq 0 \end{aligned}$$

because if $f(\mathbf{x})$ is a function with nondecreasing increments of order n, then the function $f(\mathbf{a} + \mathbf{s}(\mathbf{x} - \mathbf{a}))$ is also a function with nondecreasing increments of order n. □

Competing interests
The authors declare that they have no competing interest.

Authors' contributions
JP made the main contribution in conceiving the presented research. JP and SV worked jointly on each section while ARK worked on first and third section and drafted the manuscript. All authors read and approved the final manuscript.

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