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Notes on Greub-Rheinboldt inequalities

Di Zhao[1], Hongyi Li[1*] and Zhiguo Gong[2]

*Correspondence:
zdhyl2010@126.com
[1]LMIB, School of Mathematics and System Science, Beihang University, Beijing, China
Full list of author information is available at the end of the article

Abstract

In this paper, we focus on matrix Greub-Rheinboldt inequalities for commutative positive definite Hermitian matrix pairs. Some improvements, which yield sharpened bounds compared with existing results, are presented.

1 Introduction and preliminaries

Let $M_{m,n}$ denote the space of $m \times n$ complex matrices and write $M_n \equiv M_{n,n}$. The identity matrix in M_n is denoted by I_n. As usual, $A^* = (\bar{A})^T$ denotes the conjugate transpose of the matrix A. A matrix $A \in M_n$ is an Hermite matrix if $A^* = A$. An Hermitian matrix A is said to be positive semi-definite or nonnegative definite, written as $A \geq 0$, if $x^* Ax \geq 0$, $\forall x \in \mathbb{C}^n$. A is further called positive definite, symbolized $A > 0$, if $x^* Ax > 0$ for all nonzero $x \in \mathbb{C}^n$. An equivalent condition for $A \in M_n$ to be positive definite is that A is an Hermitian matrix and all eigenvalues of A are positive.

Denote by $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ the eigenvalues of an Hermitian matrix A. The matrix version of the well-known Kantorovich inequality for a positive definite matrix A is stated as follows (see, *e.g.*, [1, 2]):

$$1 \leq \frac{x^* Axx^* A^{-1}x}{(x^*x)^2} \leq \frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1\lambda_n} \tag{1.1}$$

for any nonzero vector $x \in \mathbb{C}^n$.

An equivalent form of this result is the inequality

$$0 \leq \frac{x^* Axx^* A^{-1}x}{(x^*x)^2} - 1 \leq \frac{(\lambda_1 - \lambda_n)^2}{4\lambda_1\lambda_n} \tag{1.2}$$

valid for any nonzero vector $x \in \mathbb{C}^n$.

This famous inequality plays an important role in statistics (see [3, 4]; for the latest work on applications in statistics, we refer to Seddighin's work [3]) and numerical analysis, for example, studying the rates of convergence and error bounds of solving systems of equations (see in [5, 6]).

In 2008, Dragomir gave a refinement of the additive version of the operator Kantorovich inequality [7],

$$0 \leq K(A;x) - 1 \leq \frac{1}{4}\frac{(M-m)^2}{mM} - \left[\operatorname{Re}\langle C_{m,M}(A)x,x\rangle \operatorname{Re}\langle C_{\frac{1}{m},\frac{1}{M}}(A^{-1})x,x\rangle\right]^{1/2}, \tag{1.3}$$

where A is a self-adjoint bounded linear operator on a complex Hilbert space, $0 < m < M$, such that $mI \leq A \leq MI$ in the partial operator order, $K(A;x) := \langle Ax, x\rangle\langle A^{-1}x, x\rangle$, and $C_{\alpha,\beta}(A) := (A - \bar{\alpha}I)(\beta I - A)$.

A further improvement of the matrix version of (1.3) is proposed in [8], where the classical Kantorovich inequality (1.1) is modified to apply not only to positive definite, but also to all invertible Hermitian matrices.

We adopt the following transform for a positive definite Hermitian matrix $A \in M_n$ with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$:

$$C(A,x) = x^*(\lambda_n I - A)(A - \lambda_1 I)x, \tag{1.4}$$

and

$$C\left(A^{-1},x\right) = x^*\left(\frac{1}{\lambda_1}I - A^{-1}\right)\left(A^{-1} - \frac{1}{\lambda_n}I\right)x. \tag{1.5}$$

Then the following inequality holds [8]:

$$0 \leq x^*Ax \cdot x^*A^{-1}x - 1 \leq \frac{(\lambda_1 - \lambda_n)^2}{4\lambda_1\lambda_n} - \sqrt{C(A,x)\cdot C\left(A^{-1},x\right)} \leq \frac{(\lambda_1 - \lambda_n)^2}{4\lambda_1\lambda_n}. \tag{1.6}$$

The result above is an improvement of the Kantorovich inequality (1.1).

A generalized form of the Kantorovich inequality presented by Greub and Rheinboldt [1] in 1959 is known as the Greub-Rheinboldt inequality in operator theoretic terms, which is also an important and early example of the so-called complementary inequality referred to in [9],

$$\langle Ax, Ax\rangle\langle Bx, Bx\rangle \leq \frac{(M_1M_2 + m_1m_2)^2}{4m_1m_2M_1M_2}\langle Ax, Bx\rangle^2, \tag{1.7}$$

where A and B are commuting positive definite self-adjoint operators on a Hilbert space, with upper and lower bounds M_i and m_i, $i = 1, 2$, respectively.

In 1997, Fujii *et al.* [10] generalized the Greub-Rheinboldt inequality to pairs of invertible operators that may not even commute,

$$\left\langle A^2\sharp B^2x, x\right\rangle \leq \left\langle A^2, x\right\rangle^{1/2}\left\langle B^2, x\right\rangle^{1/2} \leq \frac{m_1m_2 + M_2M_2}{2\sqrt{m_1m_2M_1}M_2}\left\langle A^2\sharp B^2x, x\right\rangle\langle Ax, Bx\rangle^2, \tag{1.8}$$

where A, B are invertible positive operators satisfying $0 < m_1 \leq A \leq M_1$ and $0 < m_2 \leq B \leq M_2$, and $A\sharp B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$. By using the viewpoint of interaction antieigenvalue, Gustafson [9] sharpened the Greub-Rheinboldt inequality (1.7) to obtain the following result:

$$\langle Ax, Ax\rangle\langle Bx, Bx\rangle \leq \frac{(m(AB^{-1}) + M(AB^{-1}))^2}{4m(AB^{-1})M(AB^{-1})}\langle Ax, Bx\rangle^2, \tag{1.9}$$

where A and B are commuting positive definite self-adjoint operators on a Hilbert space.

Let A and B be two positive definite Hermite matrices and $AB = BA$ with real eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$, respectively. Moreover, let $\langle Ax, Bx\rangle :=$

$(Ax)^*Bx = x^*A^*Bx$. Then a matrix version of (1.9) is

$$\frac{x^*A^2x \cdot x^*B^2x}{(x^*ABx)^2} \le \frac{(\lambda_1\mu_1 + \lambda_n\mu_n)^2}{4\lambda_1\lambda_n\mu_1\mu_n} \tag{1.10}$$

for any nonzero vector $x \in \mathbb{C}^n$.

In 2005, Seddighin [11] extended the Greub-Rheinboldt inequality (1.9) to pairs of normal operators and established for what vectors the Greub-Rheinboldt inequality becomes equality.

Let V be an $n \times r$ matrix such that $V^*V = I_r$, *i.e.*, V is suborthogonal. Another well-known matrix version of the Kantorovich inequality asserts that

$$V^*A^2V \le \frac{(m+M)^2}{4mM}\left(V^*AV\right)^2 \tag{1.11}$$

for any $A > 0$, $V^*V = I$, and $0 < mI < A < MI$.

Mond and Pečarić proved the following matrix version inequality (see (7) in [12]):

$$\left(V^*A^2V\right)^{1/2} - V^*AV \le \frac{(M-m)^2}{4(M-m)}I \tag{1.12}$$

for $A > 0$ and $V^*V = I$. For more related properties and applications, see, *e.g.*, [13–15].

In the next section, we propose some refinements about the matrix Kantorovich-type inequalities (1.2), the Greub-Rheinboldt inequality for commutative positive definite Hermitian matrix pairs, and (1.10) for positive definite matrices, yielding sharpened upper bounds compared with original results, together with an improvement to (1.12).

2 Main results

In this section, we first introduce some lemmas.

Lemma 2.1 (in [8], Lemma 2.2) *Let $A \in M_n$ be a positive definite Hermitian matrix. The following inequalities hold*:

$$\lambda_1\|x\|^2 \le x^*Ax \le \lambda_n\|x\|^2, \qquad 0 \le \left(\lambda_n\|x\|^2 - x^*Ax\right)\left(x^*Ax - \lambda_1\|x\|^2\right) \le \frac{1}{4}(\lambda_n - \lambda_1)^2\|x\|^4,$$

and

$$\begin{aligned} &\frac{1}{\lambda_n}\|x\|^2 \le x^*A^{-1}x \le \frac{1}{\lambda_1}\|x\|^2, \\ &0 \le \left(\frac{1}{\lambda_1}\|x\|^2 - x^*A^{-1}x\right)\left(x^*A^{-1}x - \frac{1}{\lambda_n}\|x\|^2\right) \le \frac{(\lambda_n - \lambda_1)^2}{4(\lambda_1\lambda_n)^2}\|x\|^4 \end{aligned} \tag{2.1}$$

for any $x \in \mathbb{C}^n$.

Let A, B be two invertible commuting Hermite matrices. Denote by $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ and $\mu_1 \le \mu_2 \le \cdots \le \mu_n$ the eigenvalues of A and B, respectively. Then there exists a unitary matrix $U \in M_n$ such that $A = U\Lambda U^*$, $B = UMU^*$, where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, $M = \operatorname{diag}(\hat{\mu}_1, \ldots, \hat{\mu}_n)$. Note that $\hat{\mu}_1, \hat{\mu}_2, \ldots, \hat{\mu}_n$ is a permutation of $\mu_1, \mu_2, \ldots, \mu_n$. Let $\sigma_k = \frac{\lambda_k}{\hat{\mu}_k}$ $(k = 1, \ldots, n)$, then it is easy to see that all eigenvalues of AB^{-1} are $\sigma_1, \sigma_2, \ldots, \sigma_n$. Without loss of generality, we may assume that $\sigma_1 = \min_k\{\frac{\lambda_k}{\hat{\mu}_k}\}$, $\sigma_n = \max_k\{\frac{\lambda_k}{\hat{\mu}_k}\}$ and $\sigma_1 \le \cdots \le \sigma_n$. For convenience, we introduce the notation

$$D(AB,x) = x^* A(\sigma_n I - AB^{-1})(AB^{-1} - \sigma_1 I)Bx. \tag{2.2}$$

If $\sigma_1\sigma_n > 0$, then we can define

$$D\bigl((AB)^{-1},x\bigr) = x^* A\left(\frac{1}{\sigma_1}I - A^{-1}B\right)\left(A^{-1}B - \frac{1}{\sigma_n}I\right)Bx. \tag{2.3}$$

Lemma 2.2 *Let A and B be two positive definite commuting matrices with eigenvalues* $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$, $0 < \mu_1 \le \mu_2 \le \cdots \le \mu_n$, *respectively.* $\sigma_1 \le \sigma_2 \le \cdots \le \sigma_n$, $D(AB,x)$ *and* $D((AB)^{-1},x)$ *are as before. Then for any* $x \in \mathbb{C}^n$,

$$\begin{aligned} &0 \le D(AB,x) \le \frac{1}{4}(\sigma_n - \sigma_1)^2\left|x^* ABx\right|, \\ &0 \le D\bigl((AB)^{-1},x\bigr) \le \frac{(\sigma_n - \sigma_1)^2}{4(\sigma_1\sigma_n)^2}\left|x^* ABx\right| \end{aligned} \tag{2.4}$$

for any $x \in \mathbb{C}^n$.

Proof From (2.2),

$$\begin{aligned} D(AB,x) &= x^* A(\sigma_n I - AB^{-1})(AB^{-1} - \sigma_1 I)Bx \\ &= x^* U\Lambda U^*(\sigma_n I - U\Lambda U^* UM^{-1}U^*)(U\Lambda U^* UM^{-1}U^* - \sigma_1 I)UMU^* x \\ &= x^* U\Lambda(\sigma_n I - \Lambda M^{-1})(\Lambda M^{-1} - \sigma_1 I)MU^* x. \end{aligned} \tag{2.5}$$

Let $z = (z_1, \ldots, z_n)^T = (\Lambda M)^{1/2} U^* x$. Thus, $\|z\|^2 = z^* z = x^* U(\Lambda M)U^* x = x^* ABx$. Then

$$D(AB,x) = z^*(\sigma_n I - \Lambda M^{-1})(\Lambda M^{-1} - \sigma_1 I)z = \sum_{i=1}^{n}(\sigma_n - \sigma_i)(\sigma_i - \sigma_1)z_i^2 \ge 0. \tag{2.6}$$

On the other hand,

$$\sum_{i=1}^{n}(\sigma_n - \sigma_i)(\sigma_i - \sigma_1)z_i^2 \le \frac{(\sigma_n - \sigma_1)^2}{4}\|z\|^2. \tag{2.7}$$

Thus,

$$D(AB,x) \le \frac{(\sigma_n - \sigma_1)^2}{4}\|z\|^2 = \frac{(\sigma_n - \sigma_1)^2}{4}\left|x^* ABx\right|. \tag{2.8}$$

The proof of $D((AB)^{-1},x)$ is similar. □

Theorem 2.3 *With the assumptions of Lemma* 2.2,

$$0 \le \frac{x^* A^2 x \cdot x^* B^2 x}{(x^* ABx)^2} - 1 \le \frac{(\sigma_n - \sigma_1)^2}{4\sigma_1\sigma_n} - \frac{1}{|x^* ABx|}\sqrt{D(AB,x) \cdot D\bigl((AB)^{-1},x\bigr)}. \tag{2.9}$$

Proof Let $z = (\Lambda M)^{1/2}U^* x$, $E = \Lambda M^{-1} = \operatorname{diag}(\frac{\lambda_n}{\hat{\mu}_n}, \dots, \frac{\lambda_1}{\hat{\mu}_1}) = \operatorname{diag}(\sigma_n, \dots, \sigma_1)$. Then

$$\frac{x^* A^2 x \cdot x^* B^2 x}{(x^* ABx)^2} = \frac{z^* Ez \cdot z^* E^{-1} z}{(z^* z)^2}. \tag{2.10}$$

From (1.2) and (1.6),

$$\begin{aligned} 0 &\le \frac{z^* Ez \cdot z^* E^{-1} z}{(z^* z)^2} - 1 \le \frac{(\sigma_n - \sigma_1)^2}{4\sigma_1\sigma_n} - \sqrt{C\left(E, \frac{z}{\|z\|}\right) \cdot C\left(E^{-1}, \frac{z}{\|z\|}\right)} \\ &= \frac{(\sigma_n - \sigma_1)^2}{4\sigma_1\sigma_n} - \frac{1}{\|z\|^2}\sqrt{C(E,z) \cdot C\left(E^{-1}, z\right)}. \end{aligned} \tag{2.11}$$

From (2.5) and (2.10), we have

$$z^* z = x^* ABx, \qquad C(E,z) = D(AB,x), \qquad C\left(E^{-1}, z\right) = D\left((AB)^{-1}, x\right). \tag{2.12}$$

By substituting (2.12) and (2.10) into (2.11), the inequality becomes

$$0 \le \frac{x^* A^2 x \cdot x^* B^2 x}{(x^* ABx)^2} - 1 \le \frac{(\sigma_n - \sigma_1)^2}{4\sigma_1\sigma_n} - \frac{1}{|x^* ABx|}\sqrt{D(AB,x) \cdot D\left((AB)^{-1}, x\right)}.$$

□

Corollary 2.4 *Let A and B be two positive definite commuting matrices with eigenvalues* $0 < \lambda_1 \le \cdots \le \lambda_n$, $0 < \mu_1 \le \cdots \le \mu_n$, *respectively. Then*

$$\frac{x^* A^2 x \cdot x^* B^2 x}{(x^* ABx)^2} \le \frac{(\lambda_1\mu_1 + \lambda_n\mu_n)^2}{4\lambda_1\mu_1\lambda_n\mu_n} - \frac{1}{|x^* ABx|}\sqrt{D(AB,x) \cdot D\left((AB)^{-1}, x\right)} \tag{2.13}$$

holds for any nonzero vector $x \in \mathbb{C}^n$.

Proof By Theorem 2.3, we have the following:

$$0 \le \frac{x^* A^2 x \cdot x^* B^2 x}{(x^* ABx)^2} \le \frac{(\sigma_1 + \sigma_n)^2}{4\sigma_1\sigma_n} - \frac{1}{|x^* ABx|}\sqrt{D(AB,x) \cdot D\left((AB)^{-1}, x\right)}. \tag{2.14}$$

Let $f(x) = \frac{(1+x)^2}{4x}$. It can be easily deduced that $f(x)$ is monotone increasing on $[1, +\infty)$. Let $\alpha_1 = \frac{\mu_1}{\lambda_n}$, $\alpha_n = \frac{\mu_n}{\lambda_1}$. From the definition of σ_1 and σ_n, we know that $\frac{\alpha_n}{\alpha_1} \ge \frac{\sigma_n}{\sigma_1} \ge 1$. Thus,

$$\frac{(\sigma_1 + \sigma_n)^2}{4\sigma_1\sigma_n} = f\left(\frac{\sigma_n}{\sigma_1}\right) \le f\left(\frac{\alpha_n}{\alpha_1}\right) = \frac{(\lambda_1\mu_1 + \lambda_1\mu_1)^2}{4\lambda_1\mu_1\lambda_1\mu_1}.$$

That is,

$$0 \le \frac{x^* A^2 x \cdot x^* B^2 x}{(x^* ABx)^2} \le \frac{(\lambda_1\mu_1 + \lambda_1\mu_1)^2}{4\lambda_1\mu_1\lambda_1\mu_1} - \frac{1}{|x^* ABx|}\sqrt{D(AB,x) \cdot D\left((AB)^{-1}, x\right)}. \tag{2.15}$$

□

Remark From Lemma 2.2 and (2.15), we can obtain a sharpened bound for the classical Kantorovich-type inequality, *i.e.*, the Greub-Rheinboldt inequality.

Besides the discussion on the Greub-Rheinboldt inequality (1.9), we are also interested in another form of Kantorovich-type inequality aforementioned. We turn our attention to the inequalities (1.11) and (1.12) in the remainder of this paper.

Let A be an $n \times n$ positive (semi-) definite Hermitian matrix with (nonzero) eigenvalues contained in the interval $[m, M]$, where $0 < m < M$. Let V be $n \times r$ matrices.

As is declared in (1.11), for $A > 0$, $V^* V = I$, and m, M mentioned above, the following inequality holds:

$$V^* A^2 V \leq \frac{(m+M)^2}{4mM} \left(V^* A V\right)^2.$$

It is not difficult to see that as $V^* V = I$, then $VV^* = VV^+ \leq I$, where + indicates the Moore-Penrose inverse. Multiplying from the right and from the left by $V^* A$ and AV respectively, we have $V^* A^2 V \geq (V^* AV)^2$ for $A > 0$. From the well-known Löwner-Heinz inequality, we have $(V^* A^2 V)^{1/2} \geq V^* AV$ and the following inequality (see in [16]):

$$\left(V^* A^2 V\right)^{1/2} \leq \frac{m+M}{2\sqrt{mM}} V^* AV.$$

For $z \in [m, M]$, $m > 0$, the convexity of $(z^{-1} + z/mM)$ implies that

$$z^{-1} \leq \frac{m+M}{mM} - \frac{z}{mM}. \tag{2.16}$$

If A has the representation $A = \Gamma D_\alpha \Gamma^*$, where Γ is unitary and $D_\alpha = \operatorname{diag}(\alpha_1, \ldots, \alpha_n)$, and if $0 < m \leq \alpha_i \leq M$, $i = 1, \ldots, n$, then from (2.16) it follows that

$$D_\alpha^{-1} \leq \frac{m+M}{mM} I - \frac{D_\alpha}{mM}. \tag{2.17}$$

After multiplying from the right and from the left by Γ and Γ^*, it is not difficult to see that (2.17) yields the following [17]:

$$A^{-1} \leq \frac{m+M}{mM} I - \frac{A}{mM}. \tag{2.18}$$

Based on (2.18), we derive several results on the inequality (1.12).

Theorem 2.5 *For any $A > 0$ and $V^* V = I$,*

$$\left(V^* A^2 V\right)^{1/2} - V^* AV \leq \frac{(M-m)^2}{4(M+m)} I - D^2(A, V), \tag{2.19}$$

where $D(A, V) = (\frac{1}{m+M} V^* A^2 V)^{1/2} - \frac{(M+m)^{1/2}}{2} I$.

Proof From (2.18) and $A > 0$, we can get

$$-A \leq -\frac{mM}{(M+m)} I - \frac{1}{(M+m)} A^2. \tag{2.20}$$

Since $V^*V = I$, (2.20) can be turned into

$$-V^*AV \leq -\frac{mM}{(M+m)}I - \frac{1}{(M+m)}V^*A^2V. \tag{2.21}$$

By adding $(V^*A^2V)^{1/2} \geq 0$ to both sides of the inequality (2.21), we obtain that

$$\left(V^*A^2V\right)^{1/2} - V^*AV \leq \left(V^*A^2V\right)^{1/2} - \frac{mM}{(M+m)}I - \frac{1}{(M+m)}V^*A^2V, \tag{2.22}$$

i.e.,

$$\begin{aligned}\left(V^*A^2V\right)^{1/2} - V^*AV &\leq \frac{(M-m)^2}{4(M+m)}I - \frac{1}{(M+m)}V^*A^2V + \left(V^*A^2V\right)^{1/2} - \frac{(M+m)}{4}I \\ &= \frac{(M-m)^2}{4(M+m)}I - \left[\left(\frac{1}{M+m}V^*A^2V\right)^{1/2} - \frac{(M+m)^{1/2}}{2}I\right]^2. \end{aligned} \tag{2.23}$$

Thus, we finally have

$$\left(V^*A^2V\right)^{1/2} - V^*AV \leq \frac{(m-M)^2}{4(M+m)}I - D^2(A,V),$$

where $D(A,V) = (\frac{1}{(m+M)}V^*A^2V)^{1/2} - \frac{(M+m)^{1/2}}{2}I$. □

Remark It is obvious that $D^2(A,V) \geq 0$. Thus, Theorem 2.5 indeed presents an improvement of the Kantorovich-type inequality (1.12) in [12].

For an application to the Hadamard product, we have the following corollary.

Corollary 2.6 *Let A_1 and A_2 be $n \times n$ positive definite matrices with eigenvalues of $A_1 \otimes A_2$ contained in the interval $[m, M]$. Then*

$$\left(A_1^2 \circ A_2^2\right)^{1/2} - A_1 \circ A_2 \leq \frac{(M-m)^2}{4(m+M)}I - D^2(A_1 \otimes A_2, V),$$

where V is the selection matrix of order $n^2 \times n$ with the property $V^*(A_1 \otimes A_2)V = A_1 \circ A_2$ ($\otimes$ and $\circ$ indicate the tensor and the Hadamard product, respectively).

3 Conclusion

In this paper, we introduce some new bounds for several Kantorovich-type inequalities for commutative positive definite Hermitian matrix pairs. As a particular situation, in Corollary 2.4, when A and B are both positive definite, the result provides a sharpened upper bound for the matrix version of the well-known Greub-Rheinboldt inequality. Moreover, it holds for negative definite Hermite matrices. Also, a refinement of Kantorovich-type inequalities concerning positive definite matrices is presented together with an application to the Hadamard product.

Competing interests
The authors did not provide this information.

Authors' contributions
The authors did not provide this information.

Author details
[1]LMIB, School of Mathematics and System Science, Beihang University, Beijing, China. [2]Faculty of Science and Technology, University of Macau, Macau, C. Postal 3001, China.

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