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Some inequalities for the minimum eigenvalue of the Hadamard product of an M-matrix and its inverse

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Abstract

In this paper, some new inequalities for the minimum eigenvalue of the Hadamard product of an M-matrix and its inverse are given. These inequalities are sharper than the well-known results. A simple example is shown.

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Keywords: Hadamard product; M-matrix; inverse M-matrix; strictly diagonally dominant matrix; eigenvalue

1 Introduction

A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called a nonnegative matrix if $a_{ij} \geq 0$. A matrix $A \in \mathbb{R}^{n\times n}$ is called a nonsingular M-matrix [1] if there exist $B \geq 0$ and $s > 0$ such that

$$A = sI_n - B \quad \text{and} \quad s > \rho(B),$$

where $\rho(B)$ is a spectral radius of the nonnegative matrix B, I_n is the $n \times n$ identity matrix. Denote by $\mathcal{M}_n$ the set of all $n \times n$ nonsingular M-matrices. The matrices in $\mathcal{M}_n^{-1} := \{A^{-1} : A \in \mathcal{M}_n\}$ are called inverse M-matrices. Let us denote

$$\tau(A) = \min\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\},$$

and $\sigma(A)$ denotes the spectrum of A. It is known that [2]

$$\tau(A) = \frac{1}{\rho(A^{-1})}$$

is a positive real eigenvalue of $A \in \mathcal{M}_n$ and the corresponding eigenvector is nonnegative. Indeed

$$\tau(A) = s - \rho(B),$$

if $A = sI_n - B$, where $s > \rho(B)$, $B \geq 0$.

For any two $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, the Hadamard product of A and B is $A \circ B = (a_{ij}b_{ij})$. If $A, B \in \mathcal{M}_n$, then $A \circ B^{-1}$ is also an M-matrix [3].

A matrix A is irreducible if there does not exist a permutation matrix P such that

$$PAP^T = \begin{bmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{bmatrix},$$

where $A_{1,1}$ and $A_{2,2}$ are square matrices.

For convenience, the set $\{1, 2, \dots, n\}$ is denoted by N, where n (≥ 3) is any positive integer. Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a strictly diagonally dominant by row, denote

$$R_i = \sum_{k\neq i} |a_{ik}|, \qquad d_i = \frac{R_i}{|a_{ii}|}, \quad \forall i \in N;$$

$$s_{ji} = \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| d_k}{|a_{jj}|}, \quad j \neq i, \forall j \in N; \qquad s_i = \max_{j\neq i}\{s_{ij}\}, \quad \forall i \in N;$$

$$r_{ji} = \frac{|a_{ji}|}{|a_{jj}| - \sum_{k\neq j,i} |a_{jk}|}, \quad j \neq i, \forall j \in N; \qquad r_i = \max_{j\neq i}\{r_{ji}\}, \quad \forall i \in N;$$

$$m_{ji} = \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| r_i}{|a_{jj}|}, \quad j \neq i, \forall j \in N; \qquad m_i = \max_{j\neq i}\{m_{ij}\}, \quad \forall i \in N;$$

$$u_{ji} = \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| m_{ki}}{|a_{jj}|}, \quad j \neq i, \forall j \in N; \qquad u_i = \max_{j\neq i}\{u_{ij}\}, \quad \forall i \in N.$$

Recently, some lower bounds for the minimum eigenvalue of the Hadamard product of an M-matrix and an inverse M-matrix have been proposed. Let $A \in \mathcal{M}_n$, for example, $\tau(A \circ A^{-1}) \leq 1$ has been proven by Fiedler *et al.* in [4]. Subsequently, $\tau(A \circ A^{-1}) > \frac{1}{n}$ was given by Fiedler and Markham in [3], and they conjectured that $\tau(A \circ A^{-1}) > \frac{2}{n}$. Song [5], Yong [6] and Chen [7] have independently proven this conjecture. In [8], Li *et al.* improved the conjecture $\tau(A \circ A^{-1}) \geq \frac{2}{n}$ when A^{-1} is a doubly stochastic matrix and gave the following result:

$$\tau\left(A \circ A^{-1}\right) \geq \min_i \left\{ \frac{a_{ii} - s_i R_i}{1 + \sum_{j\neq i} s_{ji}} \right\}.$$

In [9], Li *et al.* gave the following result:

$$\tau\left(A \circ A^{-1}\right) \geq \min_i \left\{ \frac{a_{ii} - m_i R_i}{1 + \sum_{j\neq i} m_{ji}} \right\}.$$

Furthermore, if $a_{11} = a_{22} = \cdots = a_{nn}$, they have obtained

$$\min_i \left\{ \frac{a_{ii} - m_i R_i}{1 + \sum_{j\neq i} m_{ji}} \right\} \geq \min_i \left\{ \frac{a_{ii} - s_i R_i}{1 + \sum_{j\neq i} s_{ji}} \right\},$$

i.e., under this condition, the bound of [9] is better than the one of [8].

In this paper, our motives are to improve the lower bounds for the minimum eigenvalue $\tau(A \circ A^{-1})$. The main ideas are based on the ones of [8] and [9].

2 Some preliminaries and notations

In this section, we give some notations and lemmas which mainly focus on some inequalities for the entries of the inverse M-matrix and the strictly diagonally dominant matrix.

Lemma 2.1 [6] *Let $A \in \mathbb{R}^{n\times n}$ be a strictly diagonally dominant matrix by row, i.e.,*

$$|a_{ii}| > \sum_{j\neq i} |a_{ij}|, \quad \forall i \in N.$$

If $A^{-1} = (b_{ij})$, then

$$|b_{ji}| \le \frac{\sum_{k\neq j} |a_{jk}|}{|a_{jj}|} |b_{ii}|, \quad j \neq i, \forall j \in N.$$

Lemma 2.2 *Let $A \in \mathbb{R}^{n\times n}$ be a strictly diagonally dominant M-matrix by row. If $A^{-1} = (b_{ij})$, then*

$$b_{ji} \le \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| m_{ki}}{a_{jj}} b_{ii} \le u_j b_{ii}, \quad j \neq i, \forall i \in N.$$

Proof Firstly, we consider $A \in \mathbb{R}^{n\times n}$ is a strictly diagonally dominant M-matrix by row. For $i \in N$, let

$$r_i(\varepsilon) = \max_{j\neq i} \left\{ \frac{|a_{ji}| + \varepsilon}{a_{jj} - \sum_{k\neq j,i} |a_{jk}|} \right\}$$

and

$$m_{ji}(\varepsilon) = \frac{r_i(\varepsilon)(\sum_{k\neq j,i} |a_{jk}| + \varepsilon) + |a_{ji}|}{a_{jj}}, \quad j \neq i.$$

Since A is strictly diagonally dominant, then $r_{ji} < 1$ and $m_{ji} < 1$. Therefore, there exists $\varepsilon > 0$ such that $0 < r_i(\varepsilon) < 1$ and $0 < m_{ji}(\varepsilon) < 1$. Let us define one positive diagonal matrix

$$M_i(\varepsilon) = \operatorname{diag}\bigl(m_{1i}(\varepsilon), \dots, m_{i-1,i}(\varepsilon), 1, m_{i+1,i}(\varepsilon), \dots, m_{ni}(\varepsilon)\bigr).$$

Similarly to the proofs of Theorem 2.1 and Theorem 2.4 in [8], we can prove that the matrix $AM_i(\varepsilon)$ is also a strictly diagonally dominant M-matrix by row for any $i \in N$. Furthermore, by Lemma 2.1, we can obtain the following result:

$$m_{ji}^{-1}(\varepsilon) b_{ji} \le \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| m_{ki}(\varepsilon)}{m_{ji}(\varepsilon) a_{jj}} b_{ii}, \quad j \neq i, j \in N,$$

i.e.,

$$b_{ji} \le \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| m_{ki}(\varepsilon)}{a_{jj}} b_{ii}, \quad j \neq i, j \in N.$$

Let $\varepsilon \longrightarrow 0^+$ to get

$$b_{ji} \le \frac{|a_{ji}| + \sum_{k\ne j,i}|a_{jk}|m_{ki}}{a_{jj}} b_{ii} \le u_j b_{ii}, \quad j \ne i, j \in N.$$

This proof is completed. □

Lemma 2.3 *Let $A = (a_{ij}) \in \mathcal{M}_n$ be a strictly diagonally dominant matrix by row and $A^{-1} = (b_{ij})$, then we have*

$$\frac{1}{a_{ii}} \le b_{ii} \le \frac{1}{a_{ii} - \sum_{j\ne i}|a_{ij}|u_{ji}}, \quad \forall i \in N.$$

Proof Let $B = A^{-1}$. Since A is an M-matrix, then $B \ge 0$. By $AB = BA = I_n$, we have

$$1 = \sum_{j=1}^{n} a_{ij}b_{ji} = a_{ii}b_{ii} - \sum_{j\ne i}|a_{ij}|b_{ji}, \quad \forall i \in N.$$

Hence

$$1 \le a_{ii}b_{ii}, \quad \forall i \in N,$$

or equivalently,

$$\frac{1}{a_{ii}} \le b_{ii}, \quad \forall i \in N.$$

Furthermore, by Lemma 2.2, we get

$$1 = a_{ii}b_{ii} - \sum_{j\ne i}|a_{ij}|b_{ji} \ge \left(a_{ii} - \sum_{j\ne i}|a_{ij}|u_{ji}\right)b_{ii}, \quad \forall i \in N,$$

i.e.,

$$b_{ii} \le \frac{1}{a_{ii} - \sum_{j\ne i}|a_{ij}|u_{ji}}, \quad \forall i \in N.$$

Thus the proof is completed. □

Lemma 2.4 [10] *Let $A \in \mathbb{C}^{n\times n}$ and $x_1, x_2, \ldots, x_n$ be positive real numbers. Then all the eigenvalues of A lie in the region*

$$\bigcup_{1}^{n}\left\{z \in C : |z - a_{ii}| \le x_i \sum_{j\ne i}\frac{1}{x_j}|a_{ji}|\right\}.$$

Lemma 2.5 [11] *If A^{-1} is a doubly stochastic matrix, then $Ae = e$, $A^T e = e$, where $e = (1, 1, \ldots, 1)^T$.*

3 Main results

In this section, we give two new lower bounds for $\tau(A \circ A^{-1})$ which improve the ones in [8] and [9].

Lemma 3.1 *If $A \in \mathcal{M}_n$ and $A^{-1} = (b_{ij})$ is a doubly stochastic matrix, then*

$$b_{ii} \geq \frac{1}{1 + \sum_{j \neq i} u_{ji}}, \quad \forall i \in N.$$

Proof This proof is similar to the ones of Lemma 3.2 in [8] and Theorem 3.2 in [9]. □

Theorem 3.1 *Let $A \in \mathcal{M}_n$ and $A^{-1} = (b_{ij})$ be a doubly stochastic matrix. Then*

$$\tau\left(A \circ A^{-1}\right) \geq \min_i \left\{ \frac{a_{ii} - u_i R_i}{1 + \sum_{j \neq i} u_{ji}} \right\}.$$

Proof Firstly, we assume that A is irreducible. By Lemma 2.5, we have

$$a_{ii} = \sum_{j \neq i} |a_{ij}| + 1 = \sum_{j \neq i} |a_{ji}| + 1 \quad \text{and} \quad a_{ii} > 1, \quad i \in N.$$

Denote

$$u_j = \max_{i \neq j} \{u_{ji}\} = \max \left\{ \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| m_{ki}}{a_{jj}} \right\}, \quad j \in N.$$

Since A is an irreducible matrix, we know that $0 < u_j \leq 1$. So, by Lemma 2.4, there exists $i_0 \in N$ such that

$$|\lambda - a_{i_0 i_0} b_{i_0 i_0}| \leq u_{i_0} \sum_{j \neq i_0} \frac{1}{u_j} |a_{j i_0} b_{j i_0}|,$$

or equivalently,

$$\begin{aligned}
|\lambda| &\geq a_{i_0 i_0} b_{i_0 i_0} - u_{i_0} \sum_{j \neq i_0} \frac{1}{u_j} |a_{j i_0} b_{j i_0}| \\
&\geq a_{i_0 i_0} b_{i_0 i_0} - u_{i_0} \sum_{j \neq i_0} \frac{1}{u_j} |a_{j i_0}| u_j b_{i_0 i_0} \quad \text{(by Lemma 2.2)} \\
&\geq \left(a_{i_0 i_0} - u_{i_0} \sum_{j \neq i_0} |a_{j i_0}| \right) b_{i_0 i_0} \\
&= (a_{i_0 i_0} - u_{i_0} R_{i_0}) b_{i_0 i_0} \\
&\geq \frac{a_{i_0 i_0} - u_{i_0} R_{i_0}}{1 + \sum_{j \neq i_0} u_{j i_0}} \quad \text{(by Lemma 3.1)} \\
&\geq \min_i \left\{ \frac{a_{ii} - u_i R_i}{1 + \sum_{j \neq i} u_{ji}} \right\}.
\end{aligned}$$

Secondly, if A is reducible, without loss of generality, we may assume that A has the following block upper triangular form:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1K} \\ 0 & A_{22} & \cdots & A_{2K} \\ 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & A_{KK} \end{bmatrix},$$

where $A_{ii} \in \mathcal{M}_{n_i}$ is an irreducible diagonal block matrix, $i = 1, 2, \ldots, K$. Obviously, $\tau(A \circ A^{-1}) = \min_i \tau(A_{ii} \circ A_{ii}^{-1})$. Thus the reducible case is converted into the irreducible case. This proof is completed. □

Theorem 3.2 *If $A = (a_{ij}) \in \mathcal{M}_n$ is a strictly diagonally dominant by row, then*

$$\min_i \left\{ \frac{a_{ii} - u_i R_i}{1 + \sum_{j \neq i} u_{ji}} \right\} \geq \min_i \left\{ \frac{a_{ii} - s_i R_i}{1 + \sum_{j \neq i} s_{ji}} \right\}.$$

Proof Since A is strictly diagonally dominant by row, for any $j \neq i$, we have

$$\begin{aligned} d_j - m_{ji} &= \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}|}{a_{jj}} - \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| r_i}{a_{jj}} \\ &= \frac{(1 - r_i) \sum_{k \neq j,i} |a_{jk}|}{a_{jj}} \\ &\geq 0, \end{aligned}$$

or equivalently,

$$d_j \geq m_{ji}, \quad j \neq i, \forall j \in N. \tag{1}$$

So, we can obtain

$$u_{ji} = \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| m_{ki}}{a_{jj}} \leq \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| d_k}{a_{jj}} = s_{ji}, \quad j \neq i, \forall j \in N, \tag{2}$$

and

$$u_i \leq s_i, \quad \forall i \in N.$$

Therefore, it is easy to obtain that

$$\frac{a_{ii} - u_i R_i}{1 + \sum_{j \neq i} u_{ji}} \geq \frac{a_{ii} - s_i R_i}{1 + \sum_{j \neq i} s_{ji}}, \quad \forall i \in N.$$

Obviously, we have the desired result

$$\min_i \left\{ \frac{a_{ii} - u_i R_i}{1 + \sum_{j \neq i} u_{ji}} \right\} \geq \min_i \left\{ \frac{a_{ii} - s_i R_i}{1 + \sum_{j \neq i} s_{ji}} \right\}.$$

This proof is completed. □

Theorem 3.3 *If $A = (a_{ij}) \in \mathcal{M}_n$ is strictly diagonally dominant by row, then*

$$\min_i\left\{\frac{a_{ii} - u_i R_i}{1 + \sum_{j\neq i} u_{ji}}\right\} \geq \min_i\left\{\frac{a_{ii} - m_i R_i}{1 + \sum_{j\neq i} m_{ji}}\right\}.$$

Proof Since A is strictly diagonally dominant by row, for any $j \neq i$, we have

$$\begin{aligned} r_i - m_{ji} &= r_i - \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| r_i}{a_{jj}} \\ &= \frac{r_i - |a_{ji}| - \sum_{k\neq j,i} |a_{jk}|}{a_{jj}} \\ &= \frac{r_i(a_{jj} - \sum_{k\neq j,i} |a_{jk}|) - |a_{ji}|}{a_{jj}} \\ &= \frac{a_{jj} - \sum_{k\neq j,i} |a_{jk}|}{a_{jj}}\left(r_i - \frac{|a_{ji}|}{a_{jj} - \sum_{k\neq j,i} |a_{jk}|}\right) \\ &\geq 0, \end{aligned}$$

i.e.,

$$r_i \geq m_{ji}, \quad j \neq i, \forall j \in N. \tag{3}$$

So, we can obtain

$$u_{ji} = \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| m_{ki}}{a_{jj}} \leq \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| r_i}{a_{jj}} = m_{ji}, \quad j \neq i, \forall j \in N, \tag{4}$$

and

$$u_i \leq m_i, \quad \forall i \in N.$$

Therefore, it is easy to obtain that

$$\frac{a_{ii} - u_i R_i}{1 + \sum_{j\neq i} u_{ji}} \geq \frac{a_{ii} - m_i R_i}{1 + \sum_{j\neq i} m_{ji}}, \quad \forall i \in N.$$

Obviously, we have the desired result

$$\min_i\left\{\frac{a_{ii} - u_i R_i}{1 + \sum_{j\neq i} u_{ji}}\right\} \geq \min_i\left\{\frac{a_{ii} - m_i R_i}{1 + \sum_{j\neq i} m_{ji}}\right\}.$$

□

Remark 3.1 According to inequalities (1) and (3), it is easy to know that

$$b_{ji} \leq \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| m_{ki}}{a_{jj}} b_{ii} \leq \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| d_k}{a_{jj}} b_{ii}, \quad \forall i \in N.$$

and

$$b_{ji} \leq \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| m_{ki}}{a_{jj}} b_{ii} \leq \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| r_i}{a_{jj}} b_{ii}, \quad \forall i \in N.$$

That is to say, the result of Lemma 2.2 is sharper than the ones of Theorem 2.1 in [8] and Lemma 2.2 in [9]. Moreover, the results of Theorem 3.2 and Theorem 3.3 are sharper than the ones of Theorem 3.1 in [8] and Theorem 3.3 in [9], respectively.

Theorem 3.4 *If $A \in \mathcal{M}_n$ is strictly diagonally dominant by row, then*

$$\tau\left(A \circ A^{-1}\right) \geq \min_i \left\{1 - \frac{1}{a_{ii}} \sum_{j \neq i} |a_{ji}| u_{ji}\right\}.$$

Proof This proof is similar to the one of Theorem 3.5 in [8]. □

Remark 3.2 According to inequalities (2) and (4), we get

$$1 - \frac{1}{a_{ii}} \sum_{j \neq i} |a_{ji}| u_{ji} \geq 1 - \frac{1}{a_{ii}} \sum_{j \neq i} |a_{ji}| s_{ji},$$

and

$$1 - \frac{1}{a_{ii}} \sum_{j \neq i} |a_{ji}| u_{ji} \geq 1 - \frac{1}{a_{ii}} \sum_{j \neq i} |a_{ji}| m_{ji}.$$

That is to say, the bound of Theorem 3.4 is sharper than the ones of Theorem 3.5 in [8] and Theorem 3.4 in [9], respectively.

Remark 3.3 Using the above similar ideas, we can obtain similar inequalities of the strictly diagonally M-matrix by column.

4 Example

For convenience, we consider the M-matrix A is the same as the matrix of [8]. Define the M-matrix A as follows:

$$A = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -2 & 5 & -1 & -1 \\ 0 & -2 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}.$$

1. Estimate the upper bounds for entries of $A^{-1} = (b_{ij})$. Firstly, by Lemma 2.2(2) in [9], we have

$$A^{-1} \leq \begin{bmatrix} 1 & 0.5833 & 0.5000 & 0.5000 \\ 0.6667 & 1 & 0.5000 & 0.5000 \\ 0.5000 & 0.6667 & 1 & 0.5000 \\ 0.5833 & 0.5833 & 0.5000 & 1 \end{bmatrix} \circ \begin{bmatrix} b_{11} & b_{22} & b_{33} & b_{44} \\ b_{11} & b_{22} & b_{33} & b_{44} \\ b_{11} & b_{22} & b_{33} & b_{44} \\ b_{11} & b_{22} & b_{33} & b_{44} \end{bmatrix}.$$

By Lemma 2.2, we have

$$A^{-1} \leq \begin{bmatrix} 1 & 0.5625 & 0.5000 & 0.5000 \\ 0.6167 & 1 & 0.5000 & 0.5000 \\ 0.4792 & 0.6458 & 1 & 0.5000 \\ 0.5417 & 0.5625 & 0.5000 & 1 \end{bmatrix} \circ \begin{bmatrix} b_{11} & b_{22} & b_{33} & b_{44} \\ b_{11} & b_{22} & b_{33} & b_{44} \\ b_{11} & b_{22} & b_{33} & b_{44} \\ b_{11} & b_{22} & b_{33} & b_{44} \end{bmatrix}.$$

By Lemma 2.3 and Theorem 3.1 in [9], we get

$$0.3637 \le b_{11} \le 0.4430, \qquad 0.3530 \le b_{22} \le 0.3870,$$
$$0.4000 \le b_{33} \le 0.4000, \qquad 0.4000 \le b_{44} \le 0.4000.$$

By Lemma 2.3 and Lemma 3.1, we get

$$0.3791 \le b_{11} \le 0.4233, \qquad 0.3609 \le b_{22} \le 0.3750,$$
$$0.4000 \le b_{33} \le 0.4000, \qquad 0.4000 \le b_{44} \le 0.4000.$$

2. Lower bounds for $\tau(A \circ A^{-1})$.

By Theorem 3.2 in [9], we obtain

$$0.9755 = \tau\left(A \circ A^{-1}\right) \ge 0.8000.$$

By Theorem 3.1, we obtain

$$0.9755 = \tau\left(A \circ A^{-1}\right) \ge 0.8250.$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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