Hussain and Gao *Journal of Inequalities and Applications* 2013, 2013:594
http://www.journalofinequalitiesandapplications.com/content/2013/1/594

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Multidimensional Hausdorff operators and commutators on Herz-type spaces

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Abstract

In this paper, we give necessary and sufficient conditions for the boundedness of the n-dimensional Hausdorff operators on Herz-type spaces. In addition, the sufficient condition for the boundedness of commutators generated by Lipschitz functions and the fractional Hausdorff operators on Morrey-Herz space is also provided.

MSC: 26D15; 42B35; 42B99

Keywords: Hausdorff operators; Herz-type spaces; Lipschitz functions; commutators

1 Introduction

Recall that for a locally integrable function Φ on $(0,\infty)$, the one-dimensional Hausdorff operator is defined by

$$h_\Phi f(x) = \int_0^\infty \frac{\Phi(t)}{t} f\left(\frac{x}{t}\right) dt.$$

The boundedness of this operator on the real Hardy space $H^1(R)$ was proved in [1]. Subsequently, the problem of boundedness of h_Φ in H^p, $0 < p < 1$ was considered in [2, 3] and [4]. In [5], the same operator was studied on product of Hardy spaces. Due to its close relation with the summability of the classical Fourier series, it was natural to study h_Φ in high-dimensional space R^n. With such an objective, Chen *et al.* [6] considered three extensions of the one-dimensional Hausdorff operator in R^n. One of them is the operator

$$H_\Phi f(x) = \int_{R^n} \frac{\Phi(y)}{|y|^n} f\left(\frac{x}{|y|}\right) dy.$$

The second multidimensional extension of the Hausdorff operator provided in [6] is the following operator:

$$\widetilde{H}_{\Phi,\Omega} f(x) = \int_{R^n} \frac{\Phi(x/|y|)}{|y|^n} \Omega(y') f(y)\, dy,$$

where Φ is a radial function defined on R^+, and $\Omega(y')$ is an integrable function defined on the unit sphere S^{n-1}. Here and in what follows, we denote $\widetilde{H}_{\Phi,1} = \widetilde{H}_\Phi$. In [6], the authors discussed the boundedness of these operators on various function spaces and found that they have better performance on Herz-type Hardy spaces $H\dot{K}_q^{\alpha,p}$ than their performance on the Hardy spaces H^p when $0 < p < 1$.

Recently, Lin and Sun [4] defined the n-dimensional fractional Hausdorff operator initially on the Schwartz class S by

$$H_{\Phi,\gamma} = \int_{R^n} \frac{\Phi(|x|/|y|)}{|y|^{n-\gamma}} f(y)\,dy, \quad 0 \le \gamma < n,$$

and obtained $H^p(R^n) \to L^q(R^n)$ and $L^p(|x|^\alpha\,dx) \to L^q(|x|^\alpha\,dx)$ boundedness for $H_{\Phi,\gamma}$. Furthermore, it is easy to show that the n-dimensional fractional Hardy operator

$$H_\gamma f(x) = \frac{1}{|x|^{n-\gamma}} \int_{|y|<|x|} f(y)\,dy$$

and its adjoint operator

$$H_\gamma^* f(x) = \int_{|y|\ge|x|} \frac{f(y)}{|y|^{n-\gamma}}\,dy$$

are special cases of $H_{\Phi,\gamma}$ if one chooses $\Phi(t) = \Phi_1(t) = t^{-n+\gamma}\chi_{(1,\infty)}(t)$ and $\Phi(t) = \Phi_2(t) = \chi_{(0,1]}(t)$, respectively.

In recent years, the interest in obtaining sharp bounds for integral operators has grown rapidly, mainly because of their appearance in various branches of pure and applied sciences. In [7], Xaio obtained the sharp bounds for the Hardy Littlewood averaging operator on Lebesgue and BMO spaces. Later on the problem was extended to p-adic fields in [8] and [9]. In [10] and [11], Fu with different co-author have considered the same problem for m-linear p-adic Hardy and classical Hardy operators, respectively.

As the development of linear as well as multilinear integral operators, their commutators have been well studied. A well-known theorem by Coifman *et al.* [12] states that the commutator $[b, T]$ defined by

$$[b,T](f)(x) = b(x)T(f)(x) - T(bf)(x),$$

where T is a Calderón-Zygmund singular integral operator, is bounded on $L^p(R^n)$, $1 < p < \infty$, if and only if $b \in BMO(R^n)$. One can find a vast literature devoted to the study of the boundedness properties for such commutators. More recently, Gao and Jia [13] defined the commutator of the high-dimensional Hausdorff operator as

$$\widetilde{H}_{\Phi,b} f(x) = \int_{R^n} \frac{\Phi(x/|y|)}{|y|^n}\big(b(x) - b(y)\big) f(y)\,dy$$

and studied it on Lebesgue and Herz-type spaces.

Motivated by the work cited above, in this paper, we obtain some sharp bounds for H_Φ on Herz-type spaces. Furthermore, we give a sufficient condition for the boundedness of commutators generated by the Lipschitz functions b and the n-dimensional fractional Hausdorff operators $H_{\Phi,\gamma}$, defined by

$$H^b_{\Phi,\gamma} f(x) = \int_{R^n} \frac{\Phi(|x|/|y|)}{|y|^{n-\gamma}}\big(b(x) - b(y)\big) f(y)\,dy,$$

on Morrey-Herz space. Following [14], our method is direct and straightforward. In addition, the problem of boundedness of commutators of n-dimensional fractional Hardy operators [15] is also achieved as a special case of our results. Before going into the detailed proof of these results, let us first recall some definitions. For any $k \in Z$, we set $B_k = \{x \in R^n : |x| \leq 2^k\}$, $C_k = B_k \backslash B_{k-1}$.

Definition 1.1 ([16]) Let $\alpha \in R$, $0 < p \leq \infty$, $0 < q < \infty$. The homogeneous Herz space $\dot{K}_q^{\alpha,p}(R^n)$ is defined by

$$\dot{K}_q^{\alpha,p}(R^n) = \{f \in L_{\text{loc}}^q(R^n \backslash \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}(R^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(R^n)} = \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_{C_k}\|_{L^q(R^n)}^p\right)^{1/p},$$

with the usual modification made when $p = \infty$.

Remark 1.2 $\dot{K}_q^{\alpha,p}(R^n)$ is the generalization of $L^q(R^n, |x|^\alpha)$, the Lebesgue space with power weights. Also, it is easy to see that $\dot{K}_q^{0,q}(R^n) = L^q(R^n)$ and $\dot{K}_q^{\alpha/q,q}(R^n) = L^q(R^n, |x|^\alpha)$.

Definition 1.3 Let $\alpha \in R$, $0 < p \leq \infty$, $0 < q < \infty$ and $\lambda \geq 0$. The homogeneous Morrey-Herz space $M\dot{K}_{p,q}^{\alpha,\lambda}(R^n)$ is defined by

$$M\dot{K}_{p,q}^{\alpha,\lambda}(R^n) = \{f \in L_{\text{loc}}^q(R^n \backslash \{0\}) : \|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(R^n)} < \infty\},$$

where

$$\|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(R^n)} = \sup_{k_0 \in Z} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|f\chi_{C_k}\|_{L^q(R^n)}^p\right)^{1/p},$$

with the usual modification made when $p = \infty$.

In [17] the Morrey space $M_q^\lambda(R^n)$ is defined by

$$M_q^\lambda(R^n) = \left\{f \in L_{\text{loc}}^q(R^n) : \sup_{\lambda > 0, x \in R^n} \frac{1}{r^\lambda} \int_{|x-y|<r} |f(y)|^q \, dy < \infty\right\}.$$

Obviously, $M\dot{K}_{p,q}^{\alpha,0}(R^n) = \dot{K}_q^{\alpha,p}(R^n)$ and $M_q^\lambda(R^n) \subset M\dot{K}_{q,q}^{0,\lambda}(R^n)$.

Definition 1.4 ([18]) Let $0 < \beta < 1$. The Lipschitz space $\dot{\Lambda}_\beta(R^n)$ is defined by

$$\|f\|_{\dot{\Lambda}_\beta(R^n)} = \sup_{x,h \in R^n} \frac{|f(x+h) - f(x)|}{|h|^\beta} < \infty.$$

In the next section we will obtain some sharp bounds for H_Φ. Finally, the Lipschitz estimates for the commutators $H_{\Phi,\gamma}^b$ will be studied in the last section.

2 Sharp bounds for H_Φ

The main result of this section is as follows:

Theorem 2.1 *Let $\alpha \in R$, $\lambda \geq 0$, $1 < p, q < \infty$. If Φ is a non-negative valued function and*

$$A_1 = \int_{R^n} \frac{\Phi(y)}{|y|^n} |y|^{\alpha + \frac{n}{q} - \lambda}\, dy < \infty,$$

then H_Φ is a bounded operator on $M\dot{K}_{p,q}^{\alpha,\lambda}(R^n)$.

Conversely, suppose that H_Φ is a bounded operator on $M\dot{K}_{p,q}^{\alpha,\lambda}(R^n)$. If $\lambda = 0$, or if $\lambda > \max\{0, \alpha\}$, then $A_1 < \infty$. In addition, the operator H_Φ satisfies the following operator norm:

$$\|H_\Phi\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(R^n) \to M\dot{K}_{p,q}^{\alpha,\lambda}(R^n)} = A_1.$$

Proof By definition and using Minkowski's inequality

$$\begin{aligned}
\|H_\Phi f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(R^n)} &= \sup_{k_0 \in Z} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \big\| (H_\Phi f)\chi_{C_k} \big\|_{L^q(R^n)}^p \right\}^{\frac{1}{p}} \\
&= \sup_{k_0 \in Z} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \left(\int_{C_k} \left| \int_{R^n} \frac{\Phi(y)}{|y|^n} f\left(\frac{x}{|y|}\right) dy \right|^q dx \right)^{\frac{p}{q}} \right\}^{\frac{1}{p}} \\
&\leq \sup_{k_0 \in Z} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \left(\sum_{j=-\infty}^{\infty} \int_{C_j} \frac{\Phi(y)}{|y|^n} \left\| f\left(\frac{\cdot}{|y|}\right) \right\|_{L^q(C_k)} dy \right)^p \right\}^{\frac{1}{p}}.
\end{aligned}$$

Now, it is easy to see that for $y \in C_j$ [6]

$$\left\| f\left(\frac{\cdot}{|y|}\right) \right\|_{L^q(C_k)} = |y|^{\frac{n}{q}} \|f\chi_{C_{k-j}}\|_{L^q(R^n)}.$$

Therefore, by Minkowski's inequality, we get

$$\begin{aligned}
\|H_\Phi f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(R^n)} &\leq \sum_{j=-\infty}^{\infty} \int_{C_j} \frac{\Phi(y)}{|y|^n} |y|^{\frac{n}{q}} \sup_{k_0 \in Z} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|f\chi_{C_{k-j}}\|_{L^q(R^n)}^p \right\}^{\frac{1}{p}} dy \\
&\leq \|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(R^n)} \sum_{j=-\infty}^{\infty} \int_{C_j} \frac{\Phi(y)}{|y|^n} |y|^{\frac{n}{q}} 2^{-j(\lambda-\alpha)}\, dy \\
&\leq \|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(R^n)} \int_{R^n} \frac{\Phi(y)}{|y|^n} |y|^{\alpha + \frac{n}{q} - \lambda}\, dy.
\end{aligned}$$

Hence, we conclude that

$$\|H_\Phi\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(R^n) \to M\dot{K}_{p,q}^{\alpha,\lambda}(R^n)} \leq A_1. \tag{2.1}$$

Conversely, suppose that H_Φ is bounded on $M\dot{K}_{p,q}^{\alpha,\lambda}(R^n)$. Then we consider the following two cases.

Case I: $\lambda > 0$.

In this case, we choose $f_0 \in L^q_{\mathrm{loc}}(R^n\backslash\{0\})$, such that

$$f_0(x) = |x|^{-\alpha-\frac{n}{q}+\lambda}.$$

An easy computation shows that

$$\|f_0\chi_{C_k}\|_{L^q(R^n)} = 2^{k(\lambda-\alpha)}\left[\frac{(1-2^{q(\alpha-\lambda)})|S^{n-1}|}{\lambda-\alpha}\right]^{\frac{1}{q}},$$

where $|S^{n-1}|$ denotes the volume of unit sphere S^{n-1}. Now, by definition

$$\begin{aligned}\|f_0\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(R^n)} &= \sup_{k_0\in Z} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\|f_0\chi_{C_k}\|^p_{L^q(R^n)}\right\}^{\frac{1}{p}} \\ &= \left[\frac{(1-2^{q(\alpha-\lambda)})|S^{n-1}|}{\lambda-\alpha}\right]^{\frac{1}{q}} \sup_{k_0\in Z} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\right\}^{\frac{1}{p}} \\ &= \left[\frac{(1-2^{q(\alpha-\lambda)})|S^{n-1}|}{\lambda-\alpha}\right]^{\frac{1}{q}} \frac{2^\lambda}{(2^{\lambda p}-1)^{\frac{1}{p}}} < \infty.\end{aligned}$$

On the other hand, it is easy to check that

$$H_\Phi f_0(x) = f_0(x)\int_{R^n}\frac{\Phi(y)}{|y|^n}|y|^{\alpha+\frac{n}{q}-\lambda}\,dy.$$

Under the assumption that H_Φ is bounded on $M\dot{K}^{\alpha,\lambda}_{p,q}(R^n)$, we get

$$\int_{R^n}\frac{\Phi(y)}{|y|^n}|y|^{\alpha+\frac{n}{q}-\lambda}\,dy \le \|H_\Phi\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(R^n)\to M\dot{K}^{\alpha,\lambda}_{p,q}(R^n)} < \infty. \tag{2.2}$$

Furthermore, combing (2.2) with (2.1), we immediately obtain

$$\|H_\Phi\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(R^n)\to M\dot{K}^{\alpha,\lambda}_{p,q}(R^n)} = \int_{R^n}\frac{\Phi(y)}{|y|^n}|y|^{\alpha+\frac{n}{q}-\lambda}\,dy.$$

Case II: $\lambda = 0$.

In this case, we have $M\dot{K}^{\alpha,\lambda}_{p,q}(R^n) = \dot{K}^{\alpha,p}_q(R^n)$. To prove the converse relation we take the sequence of function $\{f_m\}$ ($m \ge 0$) as follows:

$$f_m(x) = \begin{cases} 0 & \text{if } |x| < 1, \\ |x|^{-\alpha-\frac{n}{q}-2^{-m}} & \text{if } |x| \ge 1. \end{cases}$$

Obviously for $k < 0$, we have $f_m\chi_{C_k} = 0$. Hence, for $k \ge 0$, we obtain

$$\begin{aligned}\|f_m\chi_{C_k}\|^q_{L^q(R^n)} &= \int_{C_k}|x|^{-\alpha-\frac{n}{q}-2^{-m}}\,dx \\ &= \frac{(2^{q(\alpha+2^{-m})}-1)}{q(\alpha+2^{-m})}|S|^{n-1}2^{-kq(\alpha+2^{-m})}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|f_m\|_{\dot K_q^{\alpha,p}(R^n)} &= \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f_m\chi_{C_k}\|_{L^q(R^n)}^p\right\}^{\frac{1}{p}}\\ &= \left[\frac{(2^{q(\alpha+2^{-m})}-1)}{q(\alpha+2^{-m})}|S|^{n-1}\right]^{\frac{1}{q}}\left\{\sum_{k=0}^{\infty}2^{k\alpha p}2^{-kp(\alpha+2^{-m})}\right\}^{\frac{1}{p}}\\ &= \left[\frac{(2^{q(\alpha+2^{-m})}-1)}{q(\alpha+2^{-m})}|S|^{n-1}\right]^{\frac{1}{q}}\frac{2^{\frac{1}{2^m}}}{(2^{\frac{p}{2^m}}-1)^{\frac{1}{p}}}<\infty.\end{aligned}$$

On the other hand, we write

$$H_\Phi f_m(x)=\begin{cases}0 & \text{if } |x|<1,\\ |x|^{-\alpha-\frac{n}{q}-2^{-m}}\int_{|y|\le|x|}\frac{\Phi(y)}{|y|^n}|y|^{\alpha+\frac{n}{q}+2^{-m}}\,dy & \text{if } |x|\ge 1.\end{cases}$$

This implies that $(H_\Phi f_m)\chi_{C_k}=0$ for $k<0$. Thus for $k\ge 0$, we get

$$\left\|(H_\Phi f_m)\chi_{C_k}\right\|_{L^q(R^n)}^q=\int_{C_k}\left(|x|^{-\alpha-\frac{n}{q}-2^{-m}}\int_{|y|\le|x|}\frac{\Phi(y)}{|y|^n}|y|^{\alpha+\frac{n}{q}+2^{-m}}\,dy\right)^q dx.$$

Therefore, for any $m\le k$, we have

$$\begin{aligned}\left\|(H_\Phi f_m)\chi_{C_k}\right\|_{L^q(R^n)} &\ge \left(\int_{C_k}|x|^{-\alpha q-n-2^{-m}q}\,dx\right)^{\frac{1}{q}}\int_{|y|\le 2^{m-1}}\frac{\Phi(y)}{|y|^n}|y|^{\alpha+\frac{n}{q}+2^{-m}}\,dy\\ &= 2^{-k(\alpha+2^{-m})}\left[\frac{(2^{q(\alpha+2^{-m})}-1)}{q(\alpha+2^{-m})}|S|^{n-1}\right]^{\frac{1}{q}}\\ &\quad\times\int_{|y|\le 2^{m-1}}\frac{\Phi(y)}{|y|^n}|y|^{\alpha+\frac{n}{q}+2^{-m}}\,dy.\end{aligned}$$

Now, it is easy to show that

$$\begin{aligned}\|H_\Phi f_m\|_{\dot K_q^{\alpha,p}(R^n)} &\ge \left[\frac{(2^{q(\alpha+2^{-m})}-1)}{q(\alpha+2^{-m})}|S|^{n-1}\right]^{\frac{1}{q}}\left\{\sum_{k=m}^{\infty}2^{-\frac{kp}{2^m}}\right\}^{\frac{1}{p}}\\ &\quad\times\int_{|y|\le 2^{m-1}}\frac{\Phi(y)}{|y|^n}|y|^{\alpha+\frac{n}{q}+2^{-m}}\,dy\\ &= \left[\frac{(2^{q(\alpha+2^{-m})}-1)}{q(\alpha+2^{-m})}|S|^{n-1}\right]^{\frac{1}{q}}\left\{\sum_{k=0}^{\infty}2^{-\frac{kp}{2^m}}\right\}^{\frac{1}{p}}\\ &\quad\times 2^{-\frac{m}{2^m}}\int_{|y|\le 2^{m-1}}\frac{\Phi(y)}{|y|^n}|y|^{\alpha+\frac{n}{q}+2^{-m}}\,dy\\ &= \|f_m\|_{\dot K_q^{\alpha,p}(R^n)}2^{-\frac{m}{2^m}}\int_{|y|\le 2^{m-1}}\frac{\Phi(y)}{|y|^n}|y|^{\alpha+\frac{n}{q}+2^{-m}}\,dy.\end{aligned}$$

Consequently,

$$\|H_\Phi\|_{\dot{K}_q^{\alpha,p}(R^n)\to\dot{K}_q^{\alpha,p}(R^n)} \geq 2^{-\frac{m}{2^m}} \int_{|y|\leq 2^{m-1}} \frac{\Phi(y)}{|y|^n}|y|^{\alpha+\frac{n}{q}+2^{-m}}\,dy.$$

Finally, we let $m \to +\infty$ to obtain

$$\|H_\Phi\|_{\dot{K}_q^{\alpha,p}(R^n)\to\dot{K}_q^{\alpha,p}(R^n)} \geq \int_{R^n} \frac{\Phi(y)}{|y|^n}|y|^{\alpha+\frac{n}{q}}\,dy. \tag{2.3}$$

In view of (2.3) with (2.1), we get

$$\|H_\Phi\|_{\dot{K}_q^{\alpha,p}(R^n)\to\dot{K}_q^{\alpha,p}(R^n)} = \int_{R^n} \frac{\Phi(y)}{|y|^n}|y|^{\alpha+\frac{n}{q}}\,dy.$$

Thus, we finish the proof of Theorem 2.1. □

3 Lipschitz estimates for *n*-dimensional fractional Hausdorff operator

In this section, we will prove that the commutator generated by Lipschitz function b and the fractional Hausdorff operator $H_{\Phi,\gamma}$ is bounded on the Morrey-Herz space. Similar estimates for high-dimensional fractional Hardy operators are also obtained as a special case of the following theorem.

Theorem 3.1 *Let* $b \in \dot{\Lambda}_\beta(R^n)$, $0 < \beta < 1 < q_2 < q_1 < \infty$, $0 < p < \infty$, $\lambda > 0$, $\mu = \alpha + \beta + \gamma + \frac{n}{q_2} - \frac{n}{q_1}$. *If*

$$A_2 = \int_0^\infty \frac{|\Phi(t)|}{t} t^{\alpha+\frac{n}{q_2}-\lambda} \max\{1, t^\beta\}\,dt < \infty,$$

then $H^b_{\Phi,\gamma}$ *is bounded from* $M\dot{K}^{\mu,\lambda}_{p,q_1}(R^n)$ *to* $M\dot{K}^{\alpha,\lambda}_{p,q_2}(R^n)$ *and satisfies the following inequality*:

$$\|H^b_{\Phi,\gamma}f\|_{M\dot{K}^{\alpha,\lambda}_{p,q_2}(R^n)} \leq CA_2\|b\|_{\dot{\Lambda}_\beta(R^n)}\|f\|_{M\dot{K}^{\mu,\lambda}_{p,q_1}(R^n)}.$$

In proving Theorem 3.1, we need the following lemmas.

Lemma 3.2 *For* $1 < p < \infty$, *we have*

$$\|(H_{\Phi,\gamma}f)\chi_{C_k}\|_{L^p(R^n)} \leq 2^{k\gamma}|S^{n-1}| \int_0^\infty \frac{|\Phi(t)|}{t^{1+\gamma}} t^{\frac{n}{p}} \|f\chi_{t^{-1}C_k}\|_{L^p(R^n)}\,dt.$$

Proof The lemma can be proved in a way similar to Theorem 3.1 in [6]. □

Lemma 3.3 ([18]) *For any* $x, y \in R^n$, *if* $f \in \dot{\Lambda}_\beta(R^n)$, $0 < \beta < 1$, *then* $|f(x) - f(y)| \leq |x - y|^\beta \|f\|_{\dot{\Lambda}_\beta(R^n)}$. *Furthermore, for any cube* $Q \subset R^n$, $\sup_{x\in Q}|f(x) - f_Q| \leq C|Q|^{\frac{\beta}{n}}\|f\|_{\dot{\Lambda}_\beta(R^n)}$, *where* $f_Q = \frac{1}{|Q|}\int_Q f$.

Lemma 3.4 ([18]) *Let* $f \in \dot{\Lambda}_\beta(R^n)$, $0 < \beta < 1$, Q *and* Q^* *are cubes in* R^n. *If* $Q^* \subset Q$, *then*

$$|f_{Q^*} - f_Q| \leq C|Q|^{\frac{\beta}{n}}\|f\|_{\dot{\Lambda}_\beta(R^n)}.$$

Proof of Theorem 3.1 Notice that

$$\begin{aligned}\|(H^b_{\Phi,\gamma}f)\chi_{C_k}\|_{L^{q_2}(R^n)} &= \left\|\left(\int_{R^n}\frac{\Phi(|x|/|y|)}{|y|^{n-\gamma}}\big(b(x)-b(y)\big)f(y)\,dy\right)\chi_{C_k}\right\|_{L^{q_2}(R^n)}\\ &\le \left\|\left(\int_{R^n}\frac{\Phi(|x|/|y|)}{|y|^{n-\gamma}}\big(b(x)-b_{B_k}\big)f(y)\,dy\right)\chi_{C_k}\right\|_{L^{q_2}(R^n)}\\ &\quad+\left\|\left(\int_{R^n}\frac{\Phi(|x|/|y|)}{|y|^{n-\gamma}}\big(b(y)-b_{B_k}\big)f(y)\,dy\right)\chi_{C_k}\right\|_{L^{q_2}(R^n)}\\ &= I+J.\end{aligned}$$

Let $\frac{1}{r}=\frac{1}{q_2}-\frac{1}{q_1}$. Then by Hölder's inequality, Lemma 3.2, and Lemma 3.3, we have

$$\begin{aligned}I &\le \left(\int_{C_k}\left|b(x)-b_{B_k}\right|^r dx\right)^{\frac{1}{r}}\left(\int_{C_k}\left|\int_{R^n}\frac{\Phi(|x|/|y|)}{|y|^{n-\gamma}}f(y)\,dy\right|^{q_1}dx\right)^{\frac{1}{q_1}}\\ &\le C|B_k|^{\frac{\beta}{n}+\frac{1}{r}}\|b\|_{\dot\Lambda_\beta(R^n)}\left\|(H_{\Phi,\gamma}f)\chi_{C_k}\right\|_{L^{q_1}(R^n)}\\ &\le C2^{k(\beta+\gamma+\frac{n}{r})}\|b\|_{\dot\Lambda_\beta(R^n)}\int_0^\infty\frac{|\Phi(t)|}{t^{1+\gamma}}t^{\frac{n}{q_1}}\|f\chi_{t^{-1}C_k}\|_{L^{q_1}(R^n)}\,dt.\end{aligned}$$

Now, using polar coordinates, Minkowski's inequality and Hölder's inequality, we approximate J as

$$\begin{aligned}J &= \left\|\left(\int_0^\infty\int_{S^{n-1}}\frac{\Phi(|x|/r)}{r^{1-\gamma}}\big(b(ry')-b_{B_k}\big)f(ry')\,d\sigma(y')\,dr\right)\chi_{C_k}\right\|_{L^{q_2}(R^n)}\\ &= \left\|\left(\int_0^\infty\int_{S^{n-1}}\frac{\Phi(t)}{t}\big(|x|t^{-1}\big)^\gamma\big(b(|x|t^{-1}y')-b_{B_k}\big)f(|x|t^{-1}y')\,d\sigma(y')\,dt\right)\chi_{C_k}\right\|_{L^{q_2}(R^n)}\\ &\le 2^{k\gamma}\int_0^\infty\frac{|\Phi(t)|}{t^{1+\gamma}}\left\|\left(\int_{S^{n-1}}\big(b(|x|t^{-1}y')-b_{B_k}\big)f(|x|t^{-1}y')\,d\sigma(y')\right)\chi_{C_k}\right\|_{L^{q_2}(R^n)}dt\\ &\le 2^{k\gamma}\left|S^{n-1}\right|^{\frac{1}{q_2'}}\int_0^\infty\frac{|\Phi(t)|}{t^{1+\gamma}}\left(\int_{C_k}\int_{S^{n-1}}\left|\big(b(|x|t^{-1}y')-b_{B_k}\big)f(|x|t^{-1}y')\right|^{q_2}d\sigma(y')\,dx\right)^{\frac{1}{q_2}}dt.\end{aligned}$$

Again by means of polar decomposition and change of the variables, we obtain

$$\begin{aligned}J &\le 2^{k\gamma}\left|S^{n-1}\right|\int_0^\infty\frac{|\Phi(t)|}{t^{1+\gamma}}\left(\int_{2^{k-1}}^{2^k}s^{n-1}\int_{S^{n-1}}\left|\big(b(st^{-1}y')-b_{B_k}\big)f(st^{-1}y')\right|^{q_2}d\sigma(y')\,ds\right)^{\frac{1}{q_2}}dt\\ &= C2^{k\gamma}\int_0^\infty\frac{|\Phi(t)|}{t^{1+\gamma}}t^{\frac{n}{q_2}}\left(\int_{t^{-1}C_k}\left|\big(b(y)-b_{B_k}\big)f(y)\right|^{q_2}dy\right)^{\frac{1}{q_2}}dt\\ &\le C2^{k\gamma}\int_0^\infty\frac{|\Phi(t)|}{t^{1+\gamma}}t^{\frac{n}{q_2}}\left(\int_{t^{-1}C_k}\left|\big(b(y)-b_{t^{-1}B_k}\big)f(y)\right|^{q_2}dy\right)^{\frac{1}{q_2}}dt\\ &\quad+C2^{k\gamma}\int_0^\infty\frac{|\Phi(t)|}{t^{1+\gamma}}t^{\frac{n}{q_2}}\left(\int_{t^{-1}C_k}\left|\big(b_{B_k}-b_{t^{-1}B_k}\big)f(y)\right|^{q_2}dy\right)^{\frac{1}{q_2}}dt\\ &= J_1+J_2.\end{aligned}$$

For J_1, using Hölder's inequality and Lemma 3.3, we have

$$\begin{aligned} J_1 &\le C2^{k\gamma}\int_0^\infty \frac{|\Phi(t)|}{t^{1+\gamma}} t^{\frac{n}{q_2}} \left(\int_{t^{-1}C_k} \left|b(x)-b_{t^{-1}B_k}\right|^r dx\right)^{\frac{1}{r}} \left(\int_{t^{-1}C_k} \left|f(y)\right|^{q_1} dy\right)^{\frac{1}{q_1}} dt \\ &\le C2^{k(\beta+\gamma+\frac{n}{r})}\|b\|_{\dot{\Lambda}_\beta(R^n)} \int_0^\infty \frac{|\Phi(t)|}{t^{1+\gamma}} t^{\frac{n}{q_1}} \|f\chi_{t^{-1}C_k}\|_{L^{q_1}(R^n)} t^{-\beta}\, dt. \end{aligned}$$

Observe that if $t < 1$, then $B_k \subset t^{-1}B_k$, while the reverse is true for $t > 1$. Hence, by Lemma 3.4, we obtain

$$\begin{aligned} J_2 &= C2^{k\gamma}\int_0^\infty \frac{|\Phi(t)|}{t^{1+\gamma}} t^{\frac{n}{q_2}} \left(\int_{t^{-1}C_k} \left|f(y)\right|^{q_2} dy\right)^{\frac{1}{q_2}} |b_{B_k}-b_{t^{-1}B_k}|\, dt \\ &\le C2^{k\gamma}|B_k|^{\frac{1}{r}}\int_0^\infty \frac{|\Phi(t)|}{t^{1+\gamma}} t^{\frac{n}{q_1}} \left(\int_{t^{-1}C_k} \left|f(y)\right|^{q_1} dy\right)^{\frac{1}{q_1}} |b_{B_k}-b_{t^{-1}B_k}|\, dt \\ &\le C2^{k(\beta+\gamma+\frac{n}{r})}\|b\|_{\dot{\Lambda}_\beta(R^n)} \int_0^1 \frac{|\Phi(t)|}{t^{1+\gamma}} t^{\frac{n}{q_1}} \|f\chi_{t^{-1}C_k}\|_{L^{q_1}(R^n)} t^{-\beta}\, dt \\ &\quad + C2^{k(\beta+\gamma+\frac{n}{r})}\|b\|_{\dot{\Lambda}_\beta(R^n)} \int_1^\infty \frac{|\Phi(t)|}{t^{1+\gamma}} t^{\frac{n}{q_1}} \|f\chi_{t^{-1}C_k}\|_{L^{q_1}(R^n)}\, dt \\ &\le C2^{k(\beta+\gamma+\frac{n}{r})}\|b\|_{\dot{\Lambda}_\beta(R^n)} \int_0^\infty \frac{|\Phi(t)|}{t^{1+\gamma}} t^{\frac{n}{q_1}} \|f\chi_{t^{-1}C_k}\|_{L^{q_1}(R^n)} \max\{1,t^{-\beta}\}\, dt. \end{aligned}$$

Note that for $t > 1$, $0 < \beta < 1$, we have $0 < t^{-\beta} < 1$. Therefore, by combining the estimates for I, J_1, and J_2, we get

$$\begin{aligned} \left\|\left(H^b_{\Phi,\gamma}f\right)\chi_{C_k}\right\|_{L^{q_2}(R^n)} &\le C2^{k(\beta+\gamma+\frac{n}{r})}\|b\|_{\dot{\Lambda}_\beta(R^n)} \\ &\quad \times \int_0^\infty \frac{|\Phi(t)|}{t^{1+\gamma}} t^{\frac{n}{q_1}} \|f\chi_{t^{-1}C_k}\|_{L^{q_1}(R^n)} \max\{1,t^{-\beta}\}\, dt. \end{aligned}$$

Following [19], we let $m \in Z$ such that $m-1 < -\log_2 t \le m$, then $t^{-1}C_k$ is contained in two adjacent annuli C_{k+m} and C_{k+m-1}. Therefore,

$$\begin{aligned} \left\|\left(H^b_{\Phi,\gamma}f\right)\chi_{C_k}\right\|_{L^{q_2}(R^n)} &\le C2^{k(\beta+\gamma+\frac{n}{r})}\|b\|_{\dot{\Lambda}_\beta(R^n)} \\ &\quad \times \int_0^\infty \frac{|\Phi(t)|}{t^{1+\gamma}} t^{\frac{n}{q_1}} \sum_{i=0}^{1} \|f\chi_{C_{k+m-i}}\|_{L^{q_1}(R^n)} \max\{1,t^{-\beta}\}\, dt. \end{aligned}$$

Hereafter, we use the notation $\widetilde{\Phi}(t) = \frac{|\Phi(t)|}{t^{1+\gamma}} t^{\frac{n}{q_1}} \max\{1,t^{-\beta}\}$ for simplicity. Then for $0 < p < 1$, we get

$$\begin{aligned} &\left\|H^b_{\Phi,\gamma}f\right\|_{M\dot{K}^{\alpha,\lambda}_{p,q_2}(R^n)} \\ &\quad \le C\|b\|_{\dot{\Lambda}_\beta(R^n)} \sup_{k_0\in Z} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\mu p}\left(\int_0^\infty \widetilde{\Phi}(t)\|f\chi_{C_{k+m}}\|_{L^{q_1}(R^n)}\, dt\right)^p\right\}^{\frac{1}{p}} \\ &\qquad + C\|b\|_{\dot{\Lambda}_\beta(R^n)} \sup_{k_0\in Z} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\mu p}\left(\int_0^\infty \widetilde{\Phi}(t)\|f\chi_{C_{k+m-1}}\|_{L^{q_1}(R^n)}\, dt\right)^p\right\}^{\frac{1}{p}} \\ &\quad = K_1 + K_2. \end{aligned}$$

Here, we approximate K_1 as

$$\begin{aligned}
K_1 &\leq \sup_{k_0\in Z} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\mu p}\right.\\
&\quad\times\left.\left(\int_0^\infty \widetilde{\Phi}(t)2^{-(k+m)\lambda}\left(\sum_{i=-\infty}^{k+m} 2^{i\mu p}\|f\chi_{C_i}\|_{L^{q_1}(R^n)}^p\right)^{\frac{1}{p}} 2^{(k+m)(\lambda-\mu)}\,dt\right)^p\right\}^{\frac{1}{p}} C\|b\|_{\dot{\Lambda}_\beta(R^n)}\\
&\leq C\|b\|_{\dot{\Lambda}_\beta(R^n)}\|f\|_{M\dot{K}_{p,q_1}^{\mu,\lambda}(R^n)} \sup_{k_0\in Z} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\left(\int_0^\infty \widetilde{\Phi}(t)2^{m(\lambda-\mu)}\,dt\right)^p\right\}^{\frac{1}{p}}\\
&\leq C\|b\|_{\dot{\Lambda}_\beta(R^n)}\|f\|_{M\dot{K}_{p,q_1}^{\mu,\lambda}(R^n)} \sup_{k_0\in Z} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\right\}^{\frac{1}{p}}\\
&\quad\times\int_0^\infty \frac{|\Phi(t)|}{t^{1+\gamma}} t^{\frac{n}{q_1}}\max\{1,t^{-\beta}\}t^{\mu-\lambda}\,dt\\
&\leq C\|b\|_{\dot{\Lambda}_\beta(R^n)}\|f\|_{M\dot{K}_{p,q_1}^{\mu,\lambda}(R^n)}\int_0^\infty \frac{|\Phi(t)|}{t} t^{\alpha+\frac{n}{q_2}-\lambda}\max\{1,t^{\beta}\}\,dt.
\end{aligned}$$

Similarly,

$$\begin{aligned}
K_2 &\leq C\|b\|_{\dot{\Lambda}_\beta(R^n)}\|f\|_{M\dot{K}_{p,q_1}^{\mu,\lambda}(R^n)} \sup_{k_0\in Z} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\left(\int_0^\infty \widetilde{\Phi}(t)2^{(m-1)(\lambda-\mu)}\,dt\right)^p\right\}^{\frac{1}{p}}\\
&\leq C\|b\|_{\dot{\Lambda}_\beta(R^n)}\|f\|_{M\dot{K}_{p,q_1}^{\mu,\lambda}(R^n)}\int_0^\infty \frac{|\Phi(t)|}{t^{1+\gamma}} t^{\frac{n}{q_1}}\max\{1,t^{-\beta}\}t^{\mu-\lambda}\,dt\\
&= C\|b\|_{\dot{\Lambda}_\beta(R^n)}\|f\|_{M\dot{K}_{p,q_1}^{\mu,\lambda}(R^n)}\int_0^\infty \frac{|\Phi(t)|}{t} t^{\alpha+\frac{n}{q_2}-\lambda}\max\{1,t^{\beta}\}\,dt.
\end{aligned}$$

Now, we consider the case $1 < p < \infty$. By Minkowski's inequality, we write

$$\begin{aligned}
\|H_{\Phi,\gamma}^b f\|_{M\dot{K}_{p,q_2}^{\alpha,\lambda}(R^n)} &\leq C\|b\|_{\dot{\Lambda}_\beta(R^n)} \sup_{k_0\in Z} 2^{-k_0\lambda}\int_0^\infty \widetilde{\Phi}(t)\left\{\sum_{k=-\infty}^{k_0} 2^{k\mu p}\|f\chi_{C_{k+m-1}}\|_{L^{q_1}(R^n)}^p\right\}^{\frac{1}{p}} dt\\
&\quad+ C\|b\|_{\dot{\Lambda}_\beta(R^n)} \sup_{k_0\in Z} 2^{-k_0\lambda}\int_0^\infty \widetilde{\Phi}(t)\left\{\sum_{k=-\infty}^{k_0} 2^{k\mu p}\|f\chi_{C_{k+m}}\|_{L^{q_1}(R^n)}^p\right\}^{\frac{1}{p}} dt\\
&= L_1 + L_2.
\end{aligned}$$

Here, we estimate L_1 as

$$\begin{aligned}
L_1 &\leq C\|b\|_{\dot{\Lambda}_\beta(R^n)}\int_0^\infty \widetilde{\Phi}(t)\sup_{k_0\in Z} 2^{-(k_0+m-1)\lambda}\left\{\sum_{k=-\infty}^{k_0+m-1} 2^{k\mu p}\|f\chi_{C_k}\|_{L^{q_1}(R^n)}^p\right\}^{\frac{1}{p}} 2^{(m-1)(\lambda-\mu)}\,dt\\
&\leq C\|b\|_{\dot{\Lambda}_\beta(R^n)}\|f\|_{M\dot{K}_{p,q_1}^{\mu,\lambda}(R^n)}\int_0^\infty \frac{|\Phi(t)|}{t^{1+\gamma}} t^{\frac{n}{q_1}}\max\{1,t^{-\beta}\}t^{\mu-\lambda}\,dt\\
&= C\|b\|_{\dot{\Lambda}_\beta(R^n)}\|f\|_{M\dot{K}_{p,q_1}^{\mu,\lambda}(R^n)}\int_0^\infty \frac{|\Phi(t)|}{t} t^{\alpha+\frac{n}{q_2}-\lambda}\max\{1,t^{\beta}\}\,dt.
\end{aligned}$$

Similarly,

$$\begin{aligned} L_2 &\le C\|b\|_{\dot{\Lambda}_\beta(R^n)}\|f\|_{M\dot{K}^{\mu,\lambda}_{p,q_1}(R^n)}\int_0^\infty \frac{|\Phi(t)|}{t^{1+\gamma}}t^{\frac{n}{q_1}}\max\{1,t^{-\beta}\}t^{\mu-\lambda}\,dt \\ &= C\|b\|_{\dot{\Lambda}_\beta(R^n)}\|f\|_{M\dot{K}^{\mu,\lambda}_{p,q_1}(R^n)}\int_0^\infty \frac{|\Phi(t)|}{t}t^{\alpha+\frac{n}{q_2}-\lambda}\max\{1,t^{\beta}\}\,dt. \end{aligned}$$

Thus, we finish the proof of Theorem 3.1. □

Now, we deduce the Lipschitz estimates for the commutators of n-dimensional fractional Hardy operators on the Morrey-Herz space as a special case of Theorem 3.1.

Corollary 3.5 *If $\alpha+\beta+\gamma<\frac{n}{q_2'}+\lambda$, then under the same conditions as in Theorem* 3.1, *the commutator of the n-dimensional fractional Hardy operator* [15],

$$H_{\gamma,b}f(x)=\frac{1}{|x|^{n+\gamma}}\int_{|y|<|x|}\bigl(b(x)-b(y)\bigr)f(y)\,dy,$$

is bounded from $M\dot{K}^{\mu,\lambda}_{p,q_1}(R^n)$ to $M\dot{K}^{\alpha,\lambda}_{p,q_2}(R^n)$.

Proof In the operator $H^b_{\Phi,\gamma}f(x)$, we replace

$$\Phi(t)=\Phi_1(t)=t^{-n+\gamma}\chi_{(1,\infty)}(t),$$

then we obtain the commutator of the n-dimensional fractional Hardy operator,

$$H^b_{\Phi_1,\gamma}f(x)=H_{\gamma,b}f(x).$$

Hence, by Theorem 3.1

$$\begin{aligned} \|H_{\gamma,b}f\|_{M\dot{K}^{\alpha,\lambda}_{p,q_2}(R^n)} &\le C\|b\|_{\dot{\Lambda}_\beta(R^n)}\|f\|_{M\dot{K}^{\mu,\lambda}_{p,q_1}(R^n)}\int_1^\infty t^{\alpha+\beta+\gamma-\frac{n}{q_2'}-\lambda-1}\,dt \\ &\le C\|b\|_{\dot{\Lambda}_\beta(R^n)}\|f\|_{M\dot{K}^{\mu,\lambda}_{p,q_1}(R^n)}. \end{aligned}$$

Thus, the corollary is proved. □

Corollary 3.6 *If $\alpha+\frac{n}{q_2}>\lambda$, then under the same conditions as in Theorem* 3.1, *the commutator of the adjoint fractional Hardy operator* [15],

$$H^*_{\gamma,b}f(x)=\int_{|y|\ge|x|}\frac{1}{|y|^{n-\gamma}}\bigl(b(x)-b(y)\bigr)f(y)\,dy,$$

is bounded from $M\dot{K}^{\mu,\lambda}_{p,q_1}(R^n)$ to $M\dot{K}^{\alpha,\lambda}_{p,q_2}(R^n)$.

Proof In the operator $H^b_{\Phi,\gamma}f(x)$, we replace

$$\Phi(t)=\Phi_2(t)=\chi_{(0,1]}(t),$$

then we obtain the commutator of the n-dimensional adjoint Hardy operator

$$H^b_{\Phi_2,\gamma}f(x) = H^*_{\gamma,b}f(x).$$

Thus, by Theorem 3.1

$$\begin{aligned}\|H^*_{\gamma,b}f\|_{M\dot{K}^{\alpha,\lambda}_{p,q_2}(R^n)} &\le C\|b\|_{\dot{\Lambda}_\beta(R^n)}\|f\|_{M\dot{K}^{\mu,\lambda}_{p,q_1}(R^n)}\int_0^1 t^{\alpha+\frac{n}{q_2}-\lambda-1}\,dt. \\ &\le C\|b\|_{\dot{\Lambda}_\beta(R^n)}\|f\|_{M\dot{K}^{\mu,\lambda}_{p,q_1}(R^n)}.\end{aligned}$$

With this we finish the proof of Corollary 3.6. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Acknowledgements
The authors are grateful to the referees for their valuable suggestions and comments, which improved the earlier version of the manuscript.

Received: 1 August 2013 Accepted: 12 December 2013 Published: 30 Dec 2013

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10.1186/1029-242X-2013-594
Cite this article as: Hussain and Gao: **Multidimensional Hausdorff operators and commutators on Herz-type spaces.** *Journal of Inequalities and Applications* 2013, 2013:594