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Euler-Hausdorff matrix summability operator and trigonometric approximation of the conjugate of a function belonging to the generalized Lipschitz class

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Abstract

In this paper, three new estimates for the degree of approximation of a function $\tilde{f}$, the conjugate of a function f belonging to classes Lip α and Lip(ξ, r), $r \geq 1$, by $E^{(q)} \cdot \Delta_H$ summability operator of conjugate series of the Fourier series have been determined.
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1 Introduction

The degree of approximation of the function $f \in \text{Lip}\,\alpha$ and $\text{Lip}(\alpha, r)$, $r \geq 1$, classes have been determined by several investigators like Alexits [1], Sahney and Goel [2], Chandra [3], Qureshi [4, 5] and Qureshi and Nema [6]. After quite a good amount of work on the degree of approximation of functions by different summability means of its Fourier series, Lal and Singh [7] established the degree of approximation of conjugates of $\text{Lip}(\alpha, r)$ functions by $(C, 1)(E, 1)$ product means of conjugate series of a Fourier Series in the following form.

Theorem 1.1 *If $f : \mathbb{R} \to \mathbb{R}$ is a 2π-periodic and $\text{Lip}(\alpha, r)$ function, then the degree of approximation of its conjugate function $\tilde{f}$ by the $(C, 1)(E, 1)$ product means of conjugate series of the Fourier series of f satisfies*

$$M_n(\tilde{f}) = \min \left\| (CE)_n^1 - \tilde{f} \right\|_r = O\left(\frac{1}{(n+1)^{\alpha - 1/p}} \right),$$

where $(CE)_n^1 = \frac{1}{n+1} \sum_{k=0}^{n} \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} s_j$ is $(C, 1)(E, 1)$ means of conjugate series of the Fourier series.

Recently Nigam and Sharma have obtained the degree of approximation by the Karmata summability method [8] and also by $(C, 1)(E, q)$ means of its Fourier series [9] as follows.

Theorem 1.2 *If a function f is 2π-periodic, Lebesgue integrable on $[0, 2\pi]$ and belongs to $\text{Lip}(\xi(t), r)$ class, then its degree of approximation by $(C, 1)(E, q)$ summability means of its*

Fourier series is given by

$$\left\| C_n^1 E_n^q - f \right\|_r = O\left[(n+1)^{\frac{1}{r}} \xi\left(\frac{1}{n+1} \right) \right]$$

provided $\xi(t)$ satisfies the following conditions:

$$\left\{ \int_0^{\frac{1}{n+1}} \left(\frac{t|\phi(t)|}{\xi(t)} \right)^r dt \right\}^{\frac{1}{r}} = o\left(\frac{1}{n+1} \right) \tag{1.1}$$

and

$$\left\{ \int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)} \right)^r dt \right\}^{\frac{1}{r}} = O\left((n+1)^{\delta} \right), \tag{1.2}$$

where δ is an arbitrary positive number such that $s(1-\delta) - 1 > 0$, $\frac{1}{r} + \frac{1}{s} = 1$, $1 \le r \le \infty$, conditions (1.1) and (1.2) hold uniformly in x and $C_n^1 E_n^q$ is $(C,1)(E,q)$ means of the Fourier series.

Working in a slightly different direction, in this paper, the degree of approximation of a function $\tilde{f}$, the conjugate of a function f belonging to classes Lip α and Lip(ξ, r), $r \ge 1$, by the product summability operator $E^{(q)} \cdot \Delta_H$ of conjugate series of the Fourier series has been established. The results are new, sharper and better than previously known results. Furthermore, some interesting particular estimates have been also derived from the main theorems.

2 Preliminaries

Let $f(x)$ be a 2π-periodic function, Lebesgue integrable on $[0, 2\pi]$ and belong to Lip(ξ, r) class. The Fourier series of $f(x)$ is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(x) \tag{2.1}$$

with nth partial sum $s_n(f; x)$.

The series

$$\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) = -\sum_{n=1}^{\infty} B_n(x) \tag{2.2}$$

is called 'conjugate series' of the Fourier series (2.1) with nth partial sum $\tilde{s}_n(f; x)$.

A Housdorff matrix is a lower triangular matrix with entries

$$h_{n,k} = \binom{n}{k} \Delta^{n-k} \mu_k,$$

where Δ is the forward difference operator defined by $\Delta\mu_k = \mu_k - \mu_{k+1}$ and $\Delta(\Delta^n \mu_k) = \Delta^{n+1}\mu_k$.

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series with nth partial sum $s_n = \sum_{k=0}^{n} u_k$.

If $t_n^H = \sum_{k=0}^{n} h_{n,k} s_k \to s$ as $n \to \infty$, $\sum_{n=0}^{\infty} u_n$ is said to be summable to the sum s by the Hausdorff matrix summability method (Δ_H means) (Boos and Cass [10]).

The Hausdorff matrix H is regular, *i.e.*, H preserves the limit of each convergent sequence if and only if

$$\int_0^1 \left| d\big(\alpha(z)\big) \right| < \infty,$$

where the mass function $\alpha \in BV[0,1]$, $\alpha(0+) = \alpha(0) = 0$, and $\alpha(1) = 1$. In this case, the μ_n have the representation

$$\mu_n = \int_0^1 z^n \, d\alpha(z).$$

Let $E_n^{(q)} = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k$, $q > 0$.

If $E_n^{(q)} \to s$ as $n \to \infty$, $\sum_{n=0}^{\infty} u_n$ is said to be summable to s by the Euler method (E, q) (Hardy [11], p.180).

The (E, q) transform of the t_n^H transform defines the $E^{(q)} \cdot \Delta_H$ transform of (s_n). It is denoted by t_n^{EH}. Thus,

$$t_n^{EH} = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} t_k^{\Delta_H} = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sum_{v=0}^{k} h_{k,v} s_v.$$

If $t_n^{EH} \to s$ as $n \to \infty$, $\sum u_n$ is said to be summable to s by the Euler-Hausdorff matrix summability means, *i.e.*, $E^{(q)} \cdot \Delta_H$ means.

Thus, if the method of summability (E, q) is superimposed on the t_n^H method, another new method of summability $E^{(q)} \cdot \Delta_H$ is obtained.

$$s_n \to s \quad \Rightarrow \quad t_n^H \to s, \quad \text{as } n \to \infty, \Delta_H \text{ method is regular,}$$

$$\Rightarrow \quad E_n^q\big(t_n^H\big) = t_n^{EH} \to s, \quad \text{as } n \to \infty, (E, q) \text{ method is regular,}$$

$$\Rightarrow \quad \big(E^q \cdot \Delta_H\big) \text{ method is regular.}$$

The important particular cases of $E^q \cdot \Delta_H$ means are as follows:
1. $(E, q)(C, \delta)$ if $\alpha(z) = \prod_{k=1}^{\delta} z^k$, $\delta \geq 1$,
2. $(E, 1)(C, \delta)$ if $q = 1$ and $\alpha(z) = \prod_{k=1}^{\delta} z^k$, $\delta \geq 1$,
3. $(E, q)(C, 1)$ if $\alpha(z) = z$,
4. $(E, 1)(C, 1)$ if $q = 1$ and $\alpha(z) = z$.

Let $\| \, \|_\infty$ of a function $f : [0, 2\pi] \to \mathbb{R}$ be defined by

$$\|f\|_\infty = \operatorname{ess}\,\sup_{0 \leq x \leq 2\pi} \left| f(x) \right|$$

and the degree of approximation of a function $f : [0, 2\pi] \to \mathbb{R}$ by a trigonometric polynomial of order n, $t_n = \frac{1}{2} a_0 + \sum_{v=1}^{n} (a_v \cos vx + b_v \sin vx)$, under ess sup norm $\| \, \|_\infty$ be defined by (Zygmund [12], p.114)

$$\|t_n - f\|_\infty = \operatorname{ess}\,\sup_{0 \leq x \leq 2\pi} \left| t_n(x) - f(x) \right|.$$

We define the norm $L^r = \| \ \|_r$ by $\|f\|_r = \{\frac{1}{2\pi}\int_0^{2\pi}|f(x)|^r \, dr\}^{1/r}$, $r \geq 1$, and let the degree of trigonometric approximation $E_n(f)$ be given by

$$E_n(f) = \min_{t_n}\|f - t_n\|_r.$$

Define the norm $\| \ \|_r^{(\xi)}$ on class $L^r_{(\xi)}$ of functions by

$$\|f\|_r^{(\xi)} = \sup_{t\neq 0}\frac{\|f(\cdot + t) - f(\cdot)\|_r}{\xi(|t|)}.$$

The degree of approximation $E_n(f; L_r^{(\xi)})$ of a function f belonging to Lip(ξ, r) class under the norm $\| \ \|_r^{(u)}$ is given by

$$E_n\big(f; L^r_{(\xi)}\big) = \|t_n - f\|_r^{(u)}.$$

A function $f \in \text{Lip}\,\alpha$ if $|f(x + t) - f(x)| = O(|t|^\alpha)$ for $0 < \alpha \leq 1$ (Titchmarsh [13], p.426).

$f \in \text{Lip}(\alpha, r)$ if $\{\int_0^{2\pi}|f(x + t) - f(x)|^r \, dx\}^{\frac{1}{r}} = O(|t|^\alpha)$ for $0 < \alpha \leq 1$, $r \geq 1$ (def. 5.38 of McFadden [14]).

$f \in \text{Lip}(\xi, r)$ if $\{\int_0^{2\pi}|f(x+t) - f(x)|^r \, dx\}^{1/r} = O(\xi(t))$ $(r \geq 1)$, where ξ is a modulus of continuity, that is, ξ is a non-negative, non-decreasing, continuous function with the properties $\xi(0) = 0$, $\xi(t_1 + t_2) \leq \xi(t_1) + \xi(t_2)$ (Khan [15]).

If $\xi(t) = t^\alpha$, Lip(ξ, r) class coincides with the class Lip(α, r), and if $r \to \infty$, then Lip(α, r) reduces to Lip$\,\alpha$ class. Thus, it is obvious that $\text{Lip}\,\alpha \subseteq \text{Lip}(\alpha, r) \subseteq \text{Lip}(\xi, r)$.

Let u be a modulus of continuity such that $\frac{\xi(t)}{u(t)}$ is positive, non-decreasing.

Then $\|f\|_r^{(u)} \leq \max(1, \frac{\xi(2\pi)}{u(\pi)})\|f\|_r^{(\xi)}$. Thus, $L^r_{(\xi)} \subseteq L^r_{(u)} \subseteq L^r$.

We write $\psi(x, t) = f(x + t) - f(x - t)$, $\tilde{f}(x) = -\frac{1}{2\pi}\int_0^\pi \psi(x, t)\cot\frac{t}{2}\,dt$,

$$(\widetilde{EH})_n(t) = \frac{1}{2\pi(1 + q)^n}\sum_{k=0}^n\binom{n}{k}q^{n-k}\sum_{\nu=0}^k\int_0^1\binom{k}{\nu}z^\nu(1 - z)^{k-\nu}\,d\alpha(z)\frac{\cos(\nu + \frac{1}{2})t}{\sin\frac{t}{2}}.$$

3 Results

In this paper, three new estimates for the degree of approximation of a function $\tilde{f}$, the conjugate of a function f belonging to the classes Lip$\,\alpha$ and Lip(ξ, r) $(r \geq 1)$ by $E^{(q)} \cdot \Delta_H$ summability operator have been determined in the following form.

Theorem 3.1 *If* $f : [0, 2\pi] \to \mathbb{R}$ *is* 2π-*periodic, Lebesgue integrable on* $[0, 2\pi]$ *and belongs to the class* Lip$\,\alpha$, *then the degree of approximation of* $\tilde{f}$, *the conjugate of* f, *by* $E^{(q)} \cdot \Delta_H$ *means*

$$t_n^{EH} = \frac{1}{(1 + q)^n}\sum_{k=0}^n\binom{n}{k}q^{n-k}\sum_{\nu=0}^k h_{k,\nu}s_\nu$$

of conjugate series (2.2) *of its Fourier series* (2.1) *satisfies, for* $n = 0, 1, 2, 3, \ldots,$

$$\big\|\tilde{t}_n^{EH} - \tilde{f}\big\|_\infty = \text{ess}\sup_{0\leq x\leq 2\pi}\big\{\tilde{t}_n^{EH}(x) - \tilde{f}(x)\big\} = \begin{cases} O((n + 1)^{-\alpha}), & 0 < \alpha < 1, \\ O(\frac{\log(n+1)\pi}{(n+1)}), & \alpha = 1. \end{cases} \tag{3.1}$$

Theorem 3.2 *Let $\xi(t)$ be the modulus of continuity such that*

$$\int_0^v \frac{\xi(t)}{t}\, dt = O\big(\xi(v)\big), \quad 0 < v < \pi. \tag{3.2}$$

If $f : [0, 2\pi] \to \mathbb{R}$ is 2π-periodic, Lebesgue integrable on $[0, 2\pi]$ and belongs to the class $\mathrm{Lip}(\xi, r)$ ($r \geq 1$), then the degree of approximation of $\tilde{f}$, the conjugate of f, by $E^{(q)} \cdot \Delta_H$ means $\tilde{t}_n^{EH}$ of conjugate series (2.2) of its Fourier series (2.1) is given by

$$\big\| \tilde{t}_n^{EH} - \tilde{f} \big\|_r = O\left(\frac{1}{n+1} \int_{\frac{1}{n+1}}^{\pi} \frac{\xi(t)}{t^2}\, dt \right) \quad \text{for } n = 0, 1, 2, 3, \ldots. \tag{3.3}$$

Theorem 3.3 *The degree of approximation $E_n(f; L_r^{(\xi)})$ of a function $\tilde{f}$, the conjugate of a function f belonging to $\mathrm{Lip}(\xi, r)$ class, by $\tilde{t}_n^{EH}$ means of conjugate series (2.2) is given by*

$$\big\| \tilde{t}_n^{EH} - \tilde{f} \big\|_r^{(u)} = O\left(\frac{1}{n+1} \int_{\frac{1}{n+1}}^{\pi} \frac{\xi(t)}{t^2 u(t)}\, dt \right), \tag{3.4}$$

where $\xi(t)$ and $u(t)$ are the modulus of continuity such that

$$\int_0^v \frac{\xi(t)}{t u(t)}\, dt = O\left(\frac{\xi(v)}{u(v)} \right), \quad 0 < v < \pi. \tag{3.5}$$

4 Lemmas

For the proof of our theorems, the following lemmas are required.

Lemma 4.1 $(\widetilde{EH})_n(t) = O(\frac{1}{t})$ *for* $0 < t \leq \frac{1}{n+1}$.

Proof For $0 < t \leq \frac{1}{n+1}$, $|\cos t| \leq 1$, $\sin(t/2) \geq (t/\pi)$ and $\sup_{0 \leq z \leq 1} |\alpha'(z)| = N$, we have

$$
\begin{aligned}
\big| (\widetilde{EH})_n(t) \big| &= \left| \frac{1}{2\pi(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sum_{v=0}^{k} \int_0^1 \binom{k}{v} z^v (1-z)^{k-v}\, d\alpha(z) \frac{\cos(v+\frac{1}{2})t}{\sin \frac{t}{2}} \right| \\
&\leq \frac{1}{2\pi(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sum_{v=0}^{k} \int_0^1 \binom{k}{v} z^v (1-z)^{k-v}\, d\alpha(z) \frac{|\cos(v+\frac{1}{2})t|}{|\sin \frac{t}{2}|} \\
&\leq \frac{1}{2\pi(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sum_{v=0}^{k} \int_0^1 \binom{k}{v} z^v (1-z)^{k-v}\, d\alpha(z) \frac{1}{t/\pi} \\
&\leq \frac{N}{2t(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \int_0^1 \sum_{v=0}^{k} \binom{k}{v} z^v (1-z)^{k-v}\, dz \\
&= \frac{N}{2t(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \int_0^1 (z+1-z)^k\, dz \\
&= \frac{N}{2t(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} = O\left(\frac{1}{t} \right).
\end{aligned}
$$

$\square$

Lemma 4.2 $(\widetilde{EH})_n(t) = O(\frac{1}{(n+1)t^2})$ *for* $\frac{1}{n+1} < t \leq \pi$.

Proof For $\frac{1}{n+1} < t \le \pi$, $\sin(n+1)t \le 1$, $\sin(t/2) \ge (t/\pi)$ and $\sup_{0 \le z \le 1} |\alpha'(z)| = N$

$$\sum_{k=0}^{n} \int_0^1 \binom{n}{k} z^k (1-z)^{n-k} e^{i(k+\frac{1}{2})t} \, d\alpha(z)$$

$$= \int_0^1 e^{it/2} \sum_{k=0}^{n} \binom{n}{k} z^k (1-z)^{n-k} e^{ikt} \, d\alpha(z)$$

$$= \int_0^1 e^{it/2} (1-z)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{z}{1-z}\right)^k e^{ikt} \, d\alpha(z)$$

$$= \int_0^1 e^{it/2} \left(1 - z + z e^{it}\right)^n \, d\alpha(z)$$

$$\le N \int_0^1 e^{it/2} \left(1 - z + z e^{it}\right)^n \, dz$$

$$= \frac{N e^{it/2}}{(n+1)} \left(\frac{e^{i(n+1)t} - 1}{e^{it} - 1}\right)$$

$$= \frac{N}{(n+1)} \left(\frac{e^{i(n+1)t} - 1}{2i \sin \frac{t}{2}}\right)$$

$$= \frac{N}{(n+1)} \left(\frac{(\cos(n+1)t - 1) + i \sin(n+1)t}{2i \sin \frac{t}{2}}\right).$$

Equating real parts from both sides, we get

$$\sum_{k=0}^{n} \int_0^1 \binom{n}{k} z^k (1-z)^{n-k} \cos\left(k + \frac{1}{2}\right) d\alpha(z) = \frac{N}{(n+1)} \left(\frac{\sin(n+1)t}{2 \sin \frac{t}{2}}\right)$$

$$= O\left(\frac{1}{(n+1)t}\right).$$

Therefore, using $\sum_{\nu=0}^{n} \binom{n}{\nu} \frac{q^{n-\nu}}{\nu+1} = O(\frac{(1+q)^n}{n+1})$, we get

$$(\widetilde{EH})_n(t) = \frac{1}{2\pi(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sum_{\nu=0}^{k} \int_0^1 \binom{k}{\nu} z^\nu (1-z)^{k-\nu} \, d\alpha(z) \frac{\cos(\nu + \frac{1}{2})t}{\sin \frac{t}{2}}$$

$$\le \frac{1}{2t(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sum_{\nu=0}^{k} \int_0^1 \binom{k}{\nu} z^\nu (1-z)^{k-\nu} \, d\alpha(z) \cos\left(\nu + \frac{1}{2}\right)t$$

$$= \frac{1}{2t(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{O\left(\frac{1}{(k+1)t}\right)\right\}$$

$$= O\left(\frac{1}{2t^2(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} \frac{q^{n-k}}{k+1}\right) = O\left(\frac{1}{(n+1)t^2}\right). \qquad \square$$

Lemma 4.3 *If $f \in \operatorname{Lip}\alpha$, then*

$$|\psi(x,t)| = O(t^\alpha).$$

Proof Clearly,

$$\big|\psi(x,t)\big| = \big|f(x+t) - f(x-t)\big|$$
$$= \big|f(x+t) - f(x) + f(x) - f(x-t)\big|$$
$$\leq \big|f(x+t) - f(x)\big| + \big|f(x-t) - f(x)\big|,$$
$$\big|\psi(x,t)\big| = O\big(t^{\alpha}\big) + O\big(t^{\alpha}\big) = O\big(t^{\alpha}\big). \qquad \square$$

Lemma 4.4 *If* $f \in \mathrm{Lip}(\xi, r)$, $r \geq 1$, *then*

$$\left[\int_0^{2\pi} \big|\psi(x,t)\big|^r \, dx\right]^{1/r} = O\big(\xi(t)\big).$$

Proof Using Lemma 4.3 and Minkowski's inequality, we have

$$\big|\psi(x,t)\big| \leq \big|f(x+t) - f(x)\big| + \big|f(x-t) - f(x)\big|,$$
$$\left[\int_0^{2\pi} \big|\psi(x,t)\big|^r \, dx\right]^{1/r} \leq \left[\int_0^{2\pi} \big|f(x+t) - f(x)\big|^r \, dx\right]^{1/r} + \left[\int_0^{2\pi} \big|f(x-t) - f(x)\big|^r \, dx\right]^{1/r}$$
$$= O\big(\xi(t)\big) + O\big(\xi(t)\big) = O\big(\xi(t)\big). \qquad \square$$

Lemma 4.5

$$\big\|\psi(\cdot + y, t) - \psi(\cdot, t)\big\|_r = O\left(u\big(|y|\big)\frac{\xi(t)}{u(t)}\right).$$

Proof

$$\big|\psi(x+y,t) - \psi(x,t)\big| = \big|\{f(x+y+t) - f(x+y-t)\} - \{f(x+t) - f(x-t)\}\big|$$
$$\leq \big|f(x+y+t) - f(x+t)\big| + \big|f(x+y-t) - f(x-t)\big|.$$

Applying Minkowski's inequality, we have

$$\big\|\psi(\cdot + y, t) - \psi(\cdot, t)\big\|_r \leq \big\|f(\cdot + y + t) - f(\cdot + t)\big\|_r + \big\|f(\cdot + y - t) - f(\cdot + t)\big\|_r$$
$$= O\big(\xi(|y|)\big) + O\big(\xi(|y|)\big) = O\big(\xi(|y|)\big). \tag{4.1}$$

Also, we can write

$$\big|\psi(x+y,t) - \psi(x,t)\big| = \big|\{f(x+y+t) - f(x+y-t)\} - \{f(x+t) - f(x-t)\}\big|$$
$$\leq \big|f(x+y+t) - f(x+y)\big| + \big|f(x+y-t) - f(x+y)\big|$$
$$\quad + \big|f(x+t) - f(x)\big| + \big|f(x-t) - f(x)\big|,$$
$$\big\|\psi(\cdot + y, t) - \psi(\cdot, t)\big\|_r \leq \big\|f(\cdot + y + t) - f(\cdot + y)\big\|_r + \big\|f(\cdot + y - t) - f(\cdot + y)\big\|_r$$
$$\quad + \big\|f(\cdot + t) - f(\cdot)\big\|_r + \big\|f(\cdot - t) - f(\cdot)\big\|_r$$
$$= O\big(\xi(t)\big) + O\big(\xi(t)\big) + O\big(\xi(t)\big) + O\big(\xi(t)\big)$$
$$= O\big(\xi(t)\big). \tag{4.2}$$

For $t \leq |y|$, we obtain

$$\left\| \psi(\cdot + y, t) - \psi(\cdot, t) \right\|_r = O\big(\xi(t)\big)$$
$$= O\left(u(|y|) \frac{\xi(t)}{u(t)} \right).$$

Since $\frac{\xi(t)}{u(t)}$ is positive, non-decreasing, if $t \geq |y|$, then $\frac{\xi(t)}{u(t)} \geq \frac{\xi(|y|)}{u(|y|)}$, so that

$$\left\| \psi(\cdot + y, t) - \psi(\cdot, t) \right\|_r = O\big(\xi(|y|)\big)$$
$$= O\left(u(|y|) \frac{\xi(t)}{u(t)} \right). \tag{4.3}$$

$\square$

Lemma 4.6 *The inequality*

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \int_a^b g(x, t)\, dt \right|^r dx \right\}^{1/r} \leq \int_a^b \left\{ \frac{1}{2\pi} \int_0^{2\pi} |g(t, x)|^r\, dx \right\}^{1/r} dt$$

is known as generalized Minkowski's inequality where the generalization is simply replacing a finite sum by a definite (Lebesgue) integral (see Chui [16]).

5 Proof of Theorem 3.1

The nth partial sum of conjugate series (2.2) is given by

$$\tilde{s}_n(f; x) - \tilde{f}(x) = \frac{1}{2\pi} \int_0^\pi \psi(x, t) \frac{\cos(n + \frac{1}{2})t}{\sin \frac{t}{2}}\, dt.$$

Denoting the Hausdorff matrix summability transform of $\tilde{s}_n(f; x)$ by $\tilde{t}_n^H(x)$, we get

$$\tilde{t}_n^H(x) - \tilde{f}(x) = \sum_{k=0}^n h_{n,k} \big\{ \tilde{s}_k(f; x) - \tilde{f}(x) \big\}$$
$$= \sum_{k=0}^n \binom{n}{k} \Delta^{n-k} \mu_k \left\{ \frac{1}{2\pi} \int_0^\pi \psi(x, t) \frac{\cos(k + \frac{1}{2})t}{\sin \frac{t}{2}}\, dt \right\}$$
$$= \frac{1}{2\pi} \int_0^\pi \psi(x, t) \sum_{k=0}^n \binom{n}{k} \Delta^{n-k} \left(\int_0^1 z^k\, d\alpha(z) \right) \frac{\cos(k + \frac{1}{2})t}{\sin \frac{t}{2}}\, dt$$
$$= \frac{1}{2\pi} \int_0^\pi \psi(x, t) \sum_{k=0}^n \int_0^1 \binom{n}{k} z^k (1 - z)^{n-k}\, d\alpha(z) \frac{\cos(k + \frac{1}{2})t}{\sin \frac{t}{2}}\, dt.$$

The (E, q) transform of $\tilde{t}_n^H(x)$, *i.e.*, the $E^{(q)} \cdot \Delta_H$ transform of $\tilde{s}_n(f; x)$ denoted by $\tilde{t}_n^{EH}(x)$, is given by

$$\tilde{t}_n^{EH}(x) - \tilde{f}(x)$$
$$= \frac{1}{(1 + q)^n} \sum_{\nu=0}^n \binom{n}{\nu} q^{n-\nu} \big\{ \tilde{t}_n^H(x) - \tilde{f}(x) \big\}$$

$$= \frac{1}{(1+q)^n} \sum_{v=0}^{n} \binom{n}{v} q^{n-v}$$

$$\times \left\{ \frac{1}{2\pi} \int_0^\pi \psi(x,t) \sum_{v=0}^{k} \int_0^1 \binom{k}{v} z^v (1-z)^{k-v} \, d\alpha(z) \frac{\cos(v+\frac{1}{2})t}{\sin \frac{t}{2}} \, dt \right\}$$

$$= \int_0^\pi \psi(x,t) \big((\widetilde{EH})_n(t) \big) \, dt. \tag{5.1}$$

Thus,

$$\big\| \tilde{t}_n^{EC}(x) - \tilde{f}(x) \big\|_\infty \le \int_0^\pi \big| \psi(x,t) \big| \big| (\widetilde{EH})_n(t) \big| \, dt$$

$$= O\left(\int_0^\pi t^\alpha \big((\widetilde{EH})_n(t) \big) \, dt \right), \quad \text{by Lemma 4.3}$$

$$= O\left(\int_0^{\frac{1}{(n+1)}} t^\alpha (\widetilde{EH})_n(t) \, dt \right) + O\left(\int_{\frac{1}{(n+1)}}^\pi t^\alpha (\widetilde{EH})_n(t) \, dt \right)$$

$$= I_1 + I_2 \quad \text{say.} \tag{5.2}$$

Applying Lemma 4.1 and Lemma 4.3, we have

$$I_1 = O\left(\int_0^{\frac{1}{(n+1)}} t^\alpha (\widetilde{EH})_n(t) \, dt \right)$$

$$= O\left(\int_0^{\frac{1}{(n+1)}} t^{\alpha-1} \, dt \right)$$

$$= O\big((n+1)^{-\alpha} \big). \tag{5.3}$$

Now, by Lemma 4.2 and Lemma 4.3, we get

$$I_2 = O\left(\int_{\frac{1}{(n+1)}}^\pi \frac{t^\alpha}{(n+1)t^2} \, dt \right)$$

$$= O\left[\frac{1}{(n+1)} \int_{\frac{1}{(n+1)}}^\pi t^{\alpha-2} \, dt \right]$$

$$= \begin{cases} O[(\frac{1}{(n+1)})(\frac{1}{1-\alpha})(\frac{1}{(n+1)^{\alpha-1}} - \pi^{\alpha-1})], & 0 < \alpha < 1, \\ O(\frac{\log(n+1)\pi}{(n+1)}), & \alpha = 1 \end{cases}$$

$$= \begin{cases} O((n+1)^{-\alpha}), & 0 < \alpha < 1, \\ O(\frac{\log(n+1)\pi}{(n+1)}), & \alpha = 1. \end{cases} \tag{5.4}$$

Collecting equations (5.2) to (5.4), the result is

$$\big\| \tilde{t}_n^{EH} - \tilde{f} \big\|_\infty = \operatorname*{ess\,sup}_{0 \le x \le 2\pi} \big| \tilde{t}_n^{EH}(x) - \tilde{f}(x) \big| = \begin{cases} O((n+1)^{-\alpha}), & 0 < \alpha < 1, \\ O(\frac{\log(n+1)\pi}{(n+1)}), & \alpha = 1. \end{cases}$$

This completes the proof of Theorem 3.1.

6 Proof of Theorem 3.2

Following the proof of Theorem 3.1, using equation (5.1), we have

$$
\begin{aligned}
\left\| \tilde{t}_n^{EH} - \tilde{f} \right\|_r &= \left[\int_0^{2\pi} \left| \int_0^{\pi} \psi(x,t)\big((\widetilde{EH})_n(t)\big)\,dt \right|^r dx \right]^{\frac{1}{r}} \\
&\leq \int_0^{\pi} \left\{ \int_0^{2\pi} \left| \psi(x,t) \right|^r dx \right\}^{1/r} \left| (\widetilde{EH})_n(t) \right| dt \\
&= O\left(\int_0^{\pi} \xi(t)\big((\widetilde{EH})_n(t)\big)\,dt \right), \quad \text{by Lemma 4.6} \\
&= O\left(\int_0^{\frac{1}{(n+1)}} \xi(t)(\widetilde{EH})_n(t)\,dt \right) + O\left(\int_{\frac{1}{(n+1)}}^{\pi} \xi(t)(\widetilde{EH})_n(t)\,dt \right) \\
&= I_1' + I_2' \quad \text{say.}
\end{aligned}
\tag{6.1}
$$

Applying Lemmas 4.1 and 4.4 and condition (3.2), we obtain

$$
\begin{aligned}
I_1' &= O\left(\int_0^{\frac{1}{(n+1)}} \xi(t)(\widetilde{EH})_n(t)\,dt \right) \\
&= O\left(\int_0^{\frac{1}{(n+1)}} \frac{\xi(t)}{t}\,dt \right) \\
&= O\left(\xi\left(\frac{1}{n+1} \right) \right).
\end{aligned}
\tag{6.2}
$$

By Lemmas 4.2 and 4.4, we have

$$
\begin{aligned}
I_2' &= O\left(\int_{\frac{1}{(n+1)}}^{\pi} \frac{\xi(t)}{(n+1)t^2}\,dt \right) \\
&= O\left(\frac{1}{(n+1)} \int_{\frac{1}{(n+1)}}^{\pi} \frac{\xi(t)}{t^2}\,dt \right).
\end{aligned}
\tag{6.3}
$$

Next,

$$
\begin{aligned}
\frac{1}{(n+1)} \int_{\frac{1}{(n+1)}}^{\pi} \frac{\xi(t)}{t^2}\,dt &\geq \frac{1}{(n+1)} \xi\left(\frac{1}{(n+1)} \right) \int_{\frac{1}{(n+1)}}^{\pi} \frac{1}{t^2}\,dt \\
&= \xi\left(\frac{1}{(n+1)} \right) \left\{ 1 - \frac{1}{(n+1)\pi} \right\} \geq \frac{1}{2} \xi\left(\frac{1}{(n+1)} \right).
\end{aligned}
$$

Then

$$
\xi\left(\frac{1}{(n+1)} \right) = O\left(\frac{1}{(n+1)} \int_{\frac{1}{(n+1)}}^{\pi} \frac{\xi(t)}{t^2}\,dt \right).
\tag{6.4}
$$

Combining equations (6.1) to (6.4), we get

$$
\left\| \tilde{t}_n^{EH} - \tilde{f} \right\|_r = O\left(\frac{1}{(n+1)} \int_{\frac{1}{(n+1)}}^{\pi} \frac{\xi(t)}{t^2}\,dt \right).
$$

This completes the proof of Theorem 3.2.

7 Proof of Theorem 3.3

By equation (5.2) of the proof of Theorem 3.1 and Lemma 4.6, we get

$$\tilde{l}_n(x) = t_n^{EH}(x) - f(x) = \int_0^\pi \psi(x,t)(EH)_n(t)\,dt, \tag{7.1}$$

$$\tilde{l}_n(x+y) - \tilde{l}_n(x) = \int_0^\pi \big(\psi(x+y,t) - \psi(x,t)\big)(EH)_n(t)\,dt, \tag{7.2}$$

$$\begin{aligned}
\big\|\tilde{l}_n(\cdot+y) - \tilde{l}_n(\cdot)\big\|_r &\leq \int_0^\pi \big\|\psi(\cdot+y,t) - \psi(\cdot,t)\big\|_r (EH)_n(t)\,dt \\
&= \left(\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^\pi\right) \big\|\psi(\cdot+y,t) - \psi(\cdot,t)\big\|_r (EH)_n(t)\,dt \\
&= I_1'' + I_2'', \quad \text{say.}
\end{aligned} \tag{7.3}$$

Using Lemma 4.1 and Lemma 4.5 and (3.5), we obtain

$$\begin{aligned}
I_1'' &= \int_0^{\frac{1}{n+1}} \big\|\psi(\cdot+y,t) - \psi(\cdot,t)\big\|_r (EH)_n(t)\,dt \\
&= O\left(\int_0^{\frac{1}{n+1}} u(|y|)\frac{\xi(t)}{tu(t)}\,dt\right) = O\left(u(|y|)\frac{\xi(\frac{1}{n+1})}{u(\frac{1}{n+1})}\right).
\end{aligned} \tag{7.4}$$

Also, using Lemma 4.2 and Lemma 4.5, we have

$$\begin{aligned}
I_2'' &= \int_{\frac{1}{n+1}}^\pi \big\|\psi(\cdot+y,t) - \psi(\cdot,t)\big\|_r (EH)_n(t)\,dt \\
&= O\left(\int_{\frac{1}{n+1}}^\pi u(|y|)\frac{\xi(t)}{u(t)}\frac{1}{(n+1)t^2}\,dt\right), \quad \text{by Lemma 4.2,} \\
&= O\left(\frac{1}{n+1}\int_{\frac{1}{n+1}}^\pi u(|y|)\frac{\xi(t)}{t^2 u(t)}\,dt\right).
\end{aligned} \tag{7.5}$$

By (7.3), (7.4) and (7.5), we have

$$\big\|\tilde{l}_n(\cdot+y) - \tilde{l}_n(\cdot)\big\|_r = O\left(u(|y|)\frac{\xi(\frac{1}{n+1})}{u(\frac{1}{n+1})}\right) + O\left(\frac{1}{n+1}\int_{\frac{1}{n+1}}^\pi u(|y|)\frac{\xi(t)}{t^2 u(t)}\,dt\right).$$

Thus,

$$\sup_{y\neq 0}\frac{\|\tilde{l}_n(\cdot+y) - \tilde{l}_n(\cdot)\|_r}{u(|y|)} = O\left(\frac{\xi(\frac{1}{n+1})}{u(\frac{1}{n+1})}\right) + O\left(\frac{1}{n+1}\int_{\frac{1}{n+1}}^\pi \frac{\xi(t)}{t^2 u(t)}\,dt\right). \tag{7.6}$$

Since ξ and u are the modulus of continuity such that $\frac{\xi(t)}{u(t)}$ is positive, non-decreasing, therefore

$$\frac{1}{n+1}\int_{\frac{1}{n+1}}^\pi \frac{\xi(t)}{t^2 u(t)}\,dt \geq \frac{\xi(\frac{1}{n+1})}{u(\frac{1}{n+1})}\left(\frac{1}{n+1}\right)\int_{\frac{1}{n+1}}^\pi \frac{dt}{t^2} \geq \frac{\xi(\frac{1}{n+1})}{2u(\frac{1}{n+1})}.$$

Then,

$$\frac{\xi(\frac{1}{n+1})}{u(\frac{1}{n+1})} = O\left(\frac{1}{n+1}\int_{\frac{1}{n+1}}^{\pi}\frac{\xi(t)}{t^2 u(t)}\right)dt. \tag{7.7}$$

Thus,

$$\left\|\tilde{l}_n(\cdot)\right\|_r^{(u)} = \sup_{y\neq 0}\frac{\|\tilde{l}_n(\cdot+y) - \tilde{l}_n(\cdot)\|_r}{u(|y|)} = O\left(\frac{1}{n+1}\int_{\frac{1}{n+1}}^{\pi}\frac{\xi(t)}{t^2 u(t)}\,dt\right). \tag{7.8}$$

Thus, Theorem 3.3 is completely established.

8 Applications

The following corollaries can be derived from our theorem.

Corollary 8.1 *Let* $\xi(t) = t^{\alpha-\beta}$, $0 \le \beta < \alpha \le 1$, *and* $f \in \mathrm{Lip}(\xi, r)$, $r \ge 1$. *Then Theorem* 3.2 *becomes*

$$\left\|\tilde{t}_n^{EH} - \tilde{f}\right\|_r = \begin{cases} O((n+1)^{\beta-\alpha}), & 0 \le \beta < \alpha < 1, \\ O(\frac{\log(n+1)\pi}{(n+1)}), & \beta = 0, \alpha = 1. \end{cases}$$

Proof Putting $\xi(t) = t^{\alpha-\beta}$, $0 \le \beta < \alpha \le 1$, in (3.3), we have

$$\left\|\tilde{t}_n^{EH} - \tilde{f}\right\|_r = O\left(\frac{1}{(n+1)}\int_{\frac{1}{(n+1)}}^{\pi} t^{\alpha-\beta-2}\,dt\right)$$

$$= O\begin{cases} ((n+1)^{\beta-\alpha}), & 0 \le \beta < \alpha < 1, \\ O(\frac{\log(n+1)\pi}{(n+1)}), & \beta = 0, \alpha = 1. \end{cases} \qquad \square$$

Corollary 8.2 *Let* $0 \le \beta < \alpha \le 1$ *and* $f \in L_{(\alpha)}^r$ $r \ge 1$. *Then*

$$\left\|\tilde{t}_n^{EH} - \tilde{f}\right\|_r^{(\beta)} = \begin{cases} O((n+1)^{\beta-\alpha}), & 0 \le \beta < \alpha < 1, \\ O(\frac{\log(n+1)\pi}{n+1}), & \beta = 0, \alpha = 1. \end{cases}$$

Proof Putting $\xi(t) = t^{\alpha}$, $u(t) = t^{\beta}$, $0 \le \beta < \alpha \le 1$, in (3.4), we have

$$\left\|\tilde{t}_n^{EH} - \tilde{f}\right\|_r^{(\beta)} = O\left(\frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi} t^{\alpha-\beta-2}\,dt\right)$$

$$= \begin{cases} O(\frac{1}{(n+1)}(\frac{(n+1)^{\beta-\alpha+1}-(\frac{1}{\pi})^{\beta-\alpha+1}}{\beta-\alpha+1})), & 0 \le \beta < \alpha \le 1, \\ O(\frac{\log(n+1)\pi}{n+1}), & \beta = 0, \alpha = 1 \end{cases}$$

$$= \begin{cases} O((n+1)^{\beta-\alpha}), & 0 \le \beta < \alpha \le 1, \\ O(\frac{\log(n+1)\pi}{n+1}), & \beta = 0, \alpha = 1. \end{cases} \qquad \square$$

Corollary 8.3 *If* $f : [0, 2\pi] \to \mathbb{R}$ *is* 2π-*periodic, Lebesgue integrable on* $[0, 2\pi]$ *and belongs to the class* $\mathrm{Lip}(\xi, r)$ $(r \ge 1)$ *and condition* (3.2) *holds, then the degree of approximation of*

a function $\tilde{f}$, the conjugate of f, by $(E,1)(C,1)$ means

$$\tilde{t}_n^{E^1 C_1} = \frac{1}{2^n} \sum_{\nu=0}^{n} \binom{n}{\nu} \frac{1}{\nu+1} \sum_{k=0}^{\nu} \tilde{s}_k$$

of conjugate series (2.2) *is given by*

$$\left\| \tilde{t}_n^{E^1 C_1} - \tilde{f} \right\|_r = O\left(\frac{1}{n+1} \int_{\frac{1}{n+1}}^{\pi} \frac{\xi(t)}{t^2} \, dt \right).$$

Corollary 8.4 *If $f : [0, 2\pi] \to \mathbb{R}$ is 2π-periodic, Lebesgue integrable on $[0, 2\pi]$ and belongs to the class* $\mathrm{Lip}(\alpha, r)$, $r \geq 1$, $0 < \alpha \leq 1$, *then the degree of approximation of its conjugate function $\tilde{f}$ by t_n^{EH} means of conjugate series* (2.2) *satisfies*

$$\left\| \tilde{t}_n^{EH} - \tilde{f} \right\|_r = \begin{cases} O((n+1)^{-\alpha}), & 0 < \alpha < 1, \\ O(\frac{\log(n+1)\pi}{(n+1)}), & \alpha = 1. \end{cases}$$

Corollary 8.5 *In addition to the conditions of Theorem 3.2, if also $(\frac{\xi(t)}{t})$ is non-increasing, then*

$$\left\| \tilde{t}_n^{EH} - \tilde{f} \right\|_r = O\left(\xi\left(\frac{1}{n+1} \right) \log(n+1)\pi \right).$$

Corollary 8.6 *In addition to the conditions of Theorem 3.3, if also $(\frac{\xi(t)}{tu(t)})$ is non-increasing, then*

$$\left\| \tilde{t}_n^{EH} - \tilde{f} \right\|_r^{(u)} = O\left(\frac{\xi(\frac{1}{n+1})}{u(\frac{1}{n+1})} \log(n+1)\pi \right).$$

9 Remarks

1. If we take $\xi(t) = t^\alpha$ in Theorem 3.2, then it reduces to Corollary 8.4.
2. The independent proofs of Corollaries 8.3 to 8.6 can be developed along the same lines as the theorems.
3. Some interesting estimates parallel to the main theorems and Corollary 8.4, Corollary 8.5 and Corollary 8.6 can also be obtained for $(E,q)(C,\delta)$, $(E,1)(C,\delta)$, $(E,q)(C,1)$, $(E,1)(C,1)$ summability operators.
4. The degree of approximation of functions of $\mathrm{Lip}(\xi; r)$ class has been determined without using conditions like (1.1) and (1.2) of Theorem 1.2.

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