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Inverse result in simultaneous approximation by Baskakov-Durrmeyer-Stancu operators

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Abstract
In the present paper, we study an inverse result in simultaneous approximation for Baskakov-Durrmeyer-Stancu type operators.
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1 Introduction

Verma *et al.* [1] considered Baskakov-Durrmeyer-Stancu (abbr. BDS) operators for $0 \leq \alpha \leq \beta$ as

$$D_{n,\alpha,\beta}(f,x) = \sum_{k=1}^{\infty} p_{n,k}(x) \int_0^{\infty} b_{n,k}(t) f\left(\frac{nt+\alpha}{n+\beta}\right) dt + p_{n,0}(x) f\left(\frac{\alpha}{n+\beta}\right), \tag{1.1}$$

where $b_{n,k}(t) = \frac{1}{B(k+1,n)} \frac{t^{k-1}}{(1+t)^{n+k+1}}$ and $p_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}$.

For $\alpha = \beta = 0$, these operators reduce to Baskakov-Durrmeyer operators $D_n(f,x) = D_{n,0,0}(f,x)$. Note that this case was investigated in [2]. Several other researchers have studied in this direction and obtained different approximation properties of many operators, and we mention some of them as [3–8] *etc.* Verma *et al.* [1] also studied some approximation properties, asymptotic formula and better estimates for these operators. Recently, Gupta *et al.* [9] and Mishra and Khatri [10] established point-wise convergence, a Voronovskaja-type asymptotic formula and an error estimate in terms of modulus of continuity of the function and investigated moments of these operators using hypergeometric series, errors estimation in simultaneous approximation, respectively.

Let $C_\nu[0,\infty)$, where $\nu > 0$, be the class of all continuous functions defined on $[0,\infty)$ satisfying the growth condition $|f(t)| = O(1+t)^\nu$. The norm $\|\cdot\|_\nu$ on $C_\nu[0,\infty)$ is defined as $\|f\|_\nu = \sup_{0<t<\infty} |f(t)|(1+t)^{-\nu}$.

Let

$$N_{n,\alpha,\beta}(x,t) = \sum_{k=1}^{\infty} p_{n,k}(x) b_{n,k}(t) + p_{n,0}(x)\delta\left(\frac{nt+\alpha}{n+\beta}\right),$$

here $\delta(\frac{nt+\alpha}{n+\beta})$ being a type of the Dirac delta function. Then operators (1.1) can be written in the following form:

$$D_{n,\alpha,\beta}(f,x) = \int_0^\infty N_{n,\alpha,\beta}(x,t) f\left(\frac{nt+\alpha}{n+\beta}\right) dt.$$

The operators $D_{n,\alpha,\beta}(f,x)$ are well defined for $f \in C_\nu[0,\infty)$. It is easily checked that the operators $D_{n,\alpha,\beta}$ defined above are linear positive operators and $D_{n,\alpha,\beta}(f,x) = 1$. It turns out that the order of approximation by these operators is at best $O(n^{-1})$ as $n \to \infty$, howsoever smooth the function f cab is. Throughout this paper, we denote by $C[a,b]$ the space of all continuous functions on the interval $[a,b]$, the norm $\|\cdot\|_{C[a,b]}$ denotes the sup norm on the space $C[a,b]$. For $f \in C[a,b]$ and a positive integer $k \geq 1$, the kth order modulus of continuity is defined as

$$\omega_k(f,\delta;a,b) = \sup\left\{\left|\Delta_h^k f(x)\right| : |h| \leq \delta \text{ and } x, x+kh \in [a,b]\right\},$$

where $\Delta_h^k f(x)$ is kth forward difference with step length h.

A function f is said to belong to the generalized Zygmund class $\mathrm{Liz}(\alpha,k;a,b)$ if for $\delta > 0$ there exists a constant C such that $\omega_{2k}(f,\delta;a,b) \leq C\delta^{\alpha k}$. In particular for $k = 1$, we simply write $\mathrm{Lip}^{\cdot}(\alpha,a,b)$ instead of $\mathrm{Liz}(\alpha,1;a,b)$. By C_0 we mean the class of continuous functions defined on $(0,\infty)$ having a compact support and C_0^s the subclass of C_0, consisting of s-times continuously differentiable functions with $\mathrm{supp}[a',b'] \subset (a,b)$ and $[a,b] \subset (0,\infty)$. Also let

$$G^{(s)} = \left\{g \in C_0^{s+2} : \mathrm{supp}\, g \subset \left[a',b'\right]\right\}.$$

For $f \in C_0^s$ with $\mathrm{supp} f \subset [a',b']$, Peetre's K-functionals are defined as

$$K_s(\xi,f;a,b) = \inf_{g \in G^s}\left\{\left\|f^{(s)} - g^{(s)}\right\|_{C[a',b']} + \xi\left(\left\|g^{(s)}\right\|_{C[a',b']} + \left\|g^{(s+2)}\right\|_{C[a',b']}\right)\right\}, \quad 0 < \xi \leq 1.$$

For $0 < \alpha < 2$ and $f \in C_0^s$ with $\mathrm{supp} f \subset [a',b']$, we say that $f \in C_0^s(\alpha,k+1;a',b')$ if

$$\left\|f^{(s)}\right\|_{\alpha,s} \equiv \sup_{0<\xi\leq 1} \xi^{-\alpha/2} K_s(\xi,f) < \infty.$$

2 Auxiliary results

In the sequel we shall need several lemmas.

Lemma 1 [10] *For $n > 0$, $m > 0$ and $s \geq 0$, we have*

$$D_n\left(t^s,x\right) = \frac{\Gamma(n-s+1)\Gamma(s+1)}{\Gamma(n+1)}\left[(1+x)^s {}_2F_1\left(1-n,-s;1;\frac{x}{1+x}\right) - (1+x)^{-n}\right]. \tag{2.1}$$

Moreover,

$$D_n\left(t^s,x\right) = \frac{(n+s-1)!(n-s)!}{n!(n-1)!}x^s + \frac{s(s-1)(n+s-2)!(n-s)!}{n!(n-1)!}x^{s-1} + O\left(n^{-m}\right). \tag{2.2}$$

Lemma 2 [10] *For $0 \leq \alpha \leq \beta$ and $m > 0$, we have*

$$\begin{aligned}
&D_{n,\alpha,\beta}\left(t^s, x\right) \\
&\quad = x^s \frac{n^s}{(n+\beta)^s} \frac{(n+s-1)!(n-s)!}{n!(n-1)!} \\
&\qquad + x^{s-1}\left\{s(s-1)\frac{n^s}{(n+\beta)^s}\frac{(n+s-2)!(n-s)!}{n!(n-1)!} + s\alpha\frac{n^{s-1}}{(n+\beta)^s}\frac{(n+s-2)!(n-s+1)!}{n!(n-1)!}\right\} \\
&\qquad + x^{s-2}\left\{s(s-1)^2\alpha\frac{n^{s-1}}{(n+\beta)^s}\frac{(n+s-3)!(n-s+1)!}{n!(n-1)!}\right. \\
&\qquad \left. + \frac{s(s-1)}{2}\alpha^2\frac{n^{s-2}}{(n+\beta)^s}\frac{(n+s-3)!(n-s+2)!}{n!(n-1)!}\right\} + O\left(n^{-m}\right).
\end{aligned}$$

Lemma 3 [11] *For $m \in \mathbb{N} \cup \{0\}$, if*

$$U_{n,m}(x) = \sum_{k=0}^{\infty} p_{n,k}(x)\left(\frac{k}{n} - x\right)^m,$$

then $U_{n,0}(x) = 1$, $U_{n,1}(x) = 0$, and we have the recurrence relation:

$$nU_{n,m+1}(x) = x(1+x)\left[U'_{n,m}(x) + mU_{n,m-1}(x)\right].$$

Consequently, $U_{n,m}(x) = O(n^{-[(m+1)/2]})$, where $[m]$ is an integral part of m.

Lemma 4 [1] *For $m \in \mathbb{N} \cup \{0\}$, if*

$$\begin{aligned}
\mu_{n,m}(x) &= D_{n,\alpha,\beta}\left((t-x)^m, x\right) \\
&= \sum_{k=1}^{\infty} p_{n,k}(x)\int_0^{\infty} b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta} - x\right)^m dt + p_{n,0}(x)\left(\frac{\alpha}{n+\beta} - x\right)^m,
\end{aligned}$$

then

$$\mu_{n,0}(x) = 1, \qquad \mu_{n,1}(x) = \frac{\alpha - \beta x}{n+\beta},$$

and for $n > m$ we have the recurrence relation:

$$\begin{aligned}
(n-m)\left(\frac{n+\beta}{n}\right)\mu_{n,m+1}(x) &= x(1+x)\left[\mu'_{n,m}(x) + m\mu_{n,m-1}(x)\right] \\
&\quad + \left[(m+nx) + \left(\frac{n+\beta}{n}\right)\left(\frac{\alpha}{n+\beta} - x\right)(n-2m)\right]\mu_{n,m}(x) \\
&\quad - \left(\frac{\alpha}{n+\beta} - x\right)\left[\left(\frac{\alpha}{n+\beta} - x\right)\left(\frac{n+\beta}{n}\right) - 1\right]m\mu_{n,m-1}(x).
\end{aligned}$$

From the recurrence relation, it is easily verified that for all $x \in [0, \infty)$, we have

$$\mu_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right).$$

Lemma 5 [11] *There exist polynomials $q_{i,j,s}(x)$ on $[0,\infty)$, independent of n and k, such that*

$$x^s(1+x)^s \frac{d^s}{dx^s} p_{n,k}(x) = \sum_{\substack{2i+j\leq s\\ i,j\geq 0}} n^i(k-nx)^j q_{i,j,s}(x) p_{n,k}(x).$$

Lemma 6 *Let $0<a<a'<a''<b''<b'<b<\infty$ and $f^{(s)}\in C_0$ with* $\operatorname{supp} f\subset[a'',b'']$. *If*

$$\left\| D^{(s)}_{n,\alpha,\beta}(f,\star) - f^{(s)}\right\|_{C[a,b]} = O\left(n^{-\alpha/2}\right),$$

then

$$K_s(\xi,f) = C_1\left\{n^{-\alpha/2} + n\xi K_s\left(n^{-1},f\right)\right\}. \tag{2.3}$$

Consequently, $K_s(\xi,f)\leq C_2\xi^{\alpha/2}$, i.e., $f\in C_0^s(\alpha,1;a',b')$, where C_1 and C_2 are some positive constants.

Proof To prove (2.3), it is sufficient to show that

$$K_s(\xi,f)\leq C_1\left\{n^{-\alpha/2} + n\xi K_s\left(n^{-1},f\right)\right\} \quad \text{for all } n \text{ sufficiently large.}$$

Since $\operatorname{supp} f\subset[a'',b'']$, therefore by Theorem 2 there exists a function $e^{(i)}\in G^{(s)}$ such that for $i=s$ and $i=s+2$,

$$\left\| D^{(i)}_{n,\alpha,\beta}(f,\star) - e^{(i)}\right\|_{C[a,b]} \leq C_3 n^{-1},$$

which implies that

$$\begin{aligned} K_s(\xi,f) &\leq 3C_3n^{-1} + \left\| D^{(s)}_{n,\alpha,\beta}(f,\star) - f^{(s)}\right\|_{C[a',b']} \\ &\quad + \xi\left(\left\| D^{(s)}_{n,\alpha,\beta}(f,\star)\right\|_{C[a',b']} + \left\| D^{(s+2)}_{n,\alpha,\beta}(f,\star)\right\|_{C[a',b']}\right). \end{aligned}$$

Thus, it is sufficient to show that there exits a constant C_4 such that for each $g\in G^{(s)}$,

$$\left\| D^{(s+2)}_{n,\alpha,\beta}(f,\star)\right\|_{C[a',b']} \leq C_4 n\left(\left\| f^{(s)} - g^{(s)}\right\|_{C[a',b']} + n^{-1}\left\| g^{(s+2)}\right\|_{C[a',b']}\right). \tag{2.4}$$

In fact, by the linearity property, we have

$$\left\| D^{(s+2)}_{n,\alpha,\beta}(f,\star)\right\|_{C[a',b']} \leq \left\| D^{(s+2)}_{n,\alpha,\beta}(f-g,\star)\right\|_{C[a',b']} + \left\| D^{(s+2)}_{n,\alpha,\beta}(g,\star)\right\|_{C[a',b']}. \tag{2.5}$$

Applying Lemma 5, we have

$$\begin{aligned} \int_0^\infty \left| \frac{\partial^{s+2}}{\partial x^{s+2}} N_{n,\alpha,\beta}(x,t)\right| dt &\leq \sum_{\substack{2i+j\leq s+2\\ i,j\geq 0}} \sum_{k=1}^{\infty} \frac{n^i |q_{i,j,s+2}|(x)}{[x(1+x)]^{s+2}} p_{n,k}(x)|k-nx|^j \int_0^\infty b_{n,k}(t)\,dt \\ &\quad + \frac{d^{s+2}}{dx^{s+2}}\left[(1+x)^{-n}\right]. \end{aligned}$$

Therefore, by the Cauchy-Schwarz inequality and Lemma 3, we get

$$\left\|D^{(s+2)}_{n,\alpha,\beta}(f-g,\star)\right\|_{C[a',b']} \le C_5 n \left\|f^{(s)}-g^{(s)}\right\|_{C[a',b']}, \tag{2.6}$$

where the constant N_4 is independent of f and g. Next, by Taylor's expansion, we have

$$g(t) = \sum_{i=0}^{s+1} \frac{g^{(i)}(x)}{i!}(t-x)^i + \frac{g^{(s+2)}(\xi)}{(s+2)!}(t-x)^{s+2},$$

where ξ lies between t and x. Using the above expansion and the fact that

$$\int_0^\infty \frac{\partial^s}{\partial x^s} N_{n,\alpha,\beta}(x,t)(t-x)^i\,dt = 0 \quad \text{for } s > i, \tag{2.7}$$

we get

$$\left\|D^{(s+2)}_{n,\alpha,\beta}(g,\star)\right\|_{C[a',b']} \le C_6 \left\|g^{(s+2)}\right\|_{C[a',b']} \left\| \int_0^\infty \frac{\partial^{s+2}}{\partial x^{s+2}} N_{n,\alpha,\beta}(x,t)(t-x)^{s+2}\,dt \right\|_{C[a',b']}. \tag{2.8}$$

Also, by Lemmas 3, 4 and 5 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
J &\equiv \int_0^\infty \left| \frac{\partial^{s+2}}{\partial x^{s+2}} N_{n,\alpha,\beta}(x,t) \right| (t-x)^{s+2}\,dt \\
&\le \sum_{\substack{2i+j\le s+2\\ i,j\ge 0}} \sum_{k=1}^\infty \frac{n^i |q_{i,j,s+2}|(x)}{[x(1+x)]^{s+2}} p_{n,k}(x)|k-nx|^j \int_0^\infty b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{s+2} dt \\
&\quad + \frac{d^{s+2}}{dx^{s+2}}\left[\left(\frac{\alpha}{n+\beta}-x\right)^{s+2}(1+x)^{-n}\right] \\
&\le \sum_{\substack{2i+j\le s+2\\ i,j\ge 0}} \frac{n^i |q_{i,j,s+2}|(x)}{[x(1+x)]^{s+2}} \left(\sum_{k=1}^\infty p_{n,k}(x)(k-nx)^{2j}\right)^{\frac{1}{2}} \\
&\quad \times \left(\sum_{k=1}^\infty p_{n,k}(x)\int_0^\infty b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{2s+4} dt\right)^{\frac{1}{2}} \left(\int_0^\infty b_{n,k}(t)\,dt\right)^{\frac{1}{2}} \\
&\quad + \frac{d^{s+2}}{dx^{s+2}}\left[\left(\frac{\alpha}{n+\beta}-x\right)^{s+2}(1+x)^{-n}\right] \\
&= C_7 \sum_{\substack{2i+j\le s+2\\ i,j\ge 0}} n^i O\left(n^{j/2}\right) O\left(n^{-(s+2)/2}\right) = C_8 O(1).
\end{aligned}$$

Hence

$$\left\|D^{(s+2)}_{n,\alpha,\beta}(g,\star)\right\|_{C[a',b']} \le C_9 \left\|g^{(s+2)}\right\|_{C[a',b']}. \tag{2.9}$$

Combining the estimates (2.5)-(2.9), we get (2.4). The other consequence follows from [12]. This completes the proof of the lemma. □

Lemma 7 [5] *Let* $0 < a < a' < a'' < b'' < b' < b < \infty$ *and* $f^{(s)} \in C_0$ *with* $\operatorname{supp} f \subset [a'', b'']$. *If* $f \in C_0^s(\alpha, 1; a', b')$, *then* $f^{(s)} \in \mathrm{Lip}^\star(\alpha, a', b')$.

3 Known and inverse results

In this section, first we give some known results and then we estimate an inverse theorem in simultaneous approximation for Baskakov-Durrmeyer-Stancu operators. Now, this section is devoted to the following inverse theorem in simultaneous approximation.

Theorem 1 [9] *If* $s \in \mathbb{N}, f \in C_\nu[0,\infty)$ *for some* $\nu > 0$, *and* $f^{(s)}$ *exists at a point* $x \in (0,\infty)$, *then*

$$\lim_{n\to\infty} D^{(s)}_{n,\alpha,\beta}(f,x) = f^{(s)}(x). \tag{3.1}$$

Theorem 2 [9] *Let* $f \in C_\nu[0,\infty)$ *for some* $\nu > 0$, *and* $f^{(s+2)}$ *exists at a point* $x \in (0,\infty)$. *Then*

$$\begin{aligned}&\lim_{n\to\infty} n\bigl(D^{(s)}_{n,\alpha,\beta}(f,x) - f^{(s)}(x)\bigr)\\ &\quad = s(s-1-\beta)f^{(s)}(x) + \bigl[(2s-\beta)x + (s+\alpha)\bigr]f^{(s+1)}(x) + x(1+x)f^{(s+2)}(x).\end{aligned} \tag{3.2}$$

Theorem 3 [9] *Let* $f \in C_\nu[0,\infty)$ *for some* $\nu > 0$, *and* $0 < a < a_1 < b_1 < b < \infty$. *Then, for sufficiently large* n, *we have*

$$\bigl\|D^{(s)}_{n,\alpha,\beta}(f,\star) - f^{(s)}\bigr\|_{C[a_1,b_1]} \le C_1\omega\bigl(f^{(s)}, n^{-1/2}, a, b\bigr) + C_2 n^{-k}\|f\|_\nu, \tag{3.3}$$

where $C_1 = C_1(s)$, $C_2 = C_2(s,f)$.

Theorem 4 *Let* $0 < \alpha < 2$, $0 < a_1 < a_2 < b_2 < b_1 < \infty$, *and suppose* $f \in C_\nu[o,\infty)$. *Then in the following statements* (i) $\Longrightarrow$ (ii):

(i) $\|D^{(s)}_{n,\alpha,\beta}(f,\star)\|_{C[a_1,b_1]} = O(n^{-\alpha/2})$,

(ii) $f^{(s)} \in \mathrm{Lip}^\star(\alpha, a_2, b_2)$,

where $\mathrm{Lip}^\star(\alpha, a_2, b_2)$ *denotes the Zygmund class satisfying* $\omega_2(f,\delta,a_2,b_2) \le C\delta^\alpha$.

Proof Let us choose a', a'', b', b'' in such a way that $a_1 < a' < a'' < a_2 < b_2 < b < b'' < b_1$. Also suppose $g \in C_0^\infty$ with $\operatorname{supp} g \in [a'', b'']$ and $g(x) = 1$ on the interval $[a_2, b_2]$. For $x \in [a', b']$ with $D \equiv \frac{d}{dx}$, we have

$$\begin{aligned}&D^{(s)}_{n,\alpha,\beta}(fg,x) - (fg)^{(s)}(x)\\ &\quad = D^s\bigl(D_{n,\alpha,\beta}\bigl((fg)(t) - (fg)(x)\bigr), x\bigr)\\ &\quad = D^s\bigl(D_{n,\alpha,\beta}\bigl(f(t)\bigl[g(t) - g(x)\bigr], x\bigr)\bigr) + D^s\bigl(D_{n,\alpha,\beta}\bigl(g(x)\bigl[f(t) - f(x)\bigr], x\bigr)\bigr)\\ &\quad =: E_1 + E_2.\end{aligned}$$

By the Leibniz formula, we have

$$\begin{aligned}E_1 &= \frac{\partial^s}{\partial x^s}\int_0^\infty N_{n,\alpha,\beta}(x,t)f(t)\bigl[g(t) - g(x)\bigr]\,dt\\ &= \sum_{i=0}^{s}\binom{s}{i}\int_0^\infty N^{(i)}_{n,\alpha,\beta}(x,t)\frac{\partial^{s-i}}{\partial x^{s-i}}\bigl[f(t)\bigl(g(t) - g(x)\bigr)\bigr]\,dt\end{aligned}$$

$$= -\sum_{i=0}^{s-1}\binom{s}{i}g^{(s-i)}(x)D_{n,\alpha,\beta}^{(i)}(f,x) + \int_0^\infty N_{n,\alpha,\beta}^{(s)}(x,t)f(t)\big(g(t)-g(x)\big)\,dt$$
$$=: E_3 + E_4.$$

Applying Theorem 3, we have

$$E_3 = -\sum_{i=0}^{s-1}\binom{s}{i}g^{(s-i)}(x)f^{(i)}(x) + O\big(n^{-\alpha/2}\big),$$

uniformly in $x \in [a', b']$. By Taylor's expansion of $f(t)$ and $g(t)$, we have

$$f(t) = \sum_{i=0}^{s}\frac{f^{(i)}(x)}{i!}(t-x)^i + O(t-x)^s$$

and

$$g(t) = \sum_{i=0}^{s+1}\frac{g^{(i)}(x)}{i!}(t-x)^i + O(t-x)^{s+1}.$$

Substituting the above expansions in E_4 and using Theorem 2, the Schwarz inequality and Lemma 4, we obtain

$$E_4 = \sum_{i=0}^{s}\frac{g^{(i)}(x)f^{(s-i)}(x)}{i!(s-i)!}s! + O\big(n^{-1/2}\big)$$
$$= \sum_{i=0}^{s}\binom{s}{i}g^{(i)}(x)f^{(s-i)}(x) + O\big(n^{-\alpha/2}\big),$$

uniformly in $x \in [a', b']$. Again using the Leibniz formula, we have

$$E_2 = \sum_{i=0}^{s}\binom{s}{i}\int_0^\infty N_{n,\alpha,\beta}^{(i)}(x,t)\frac{\partial^{s-i}}{\partial x^{s-i}}\big[g(t)\big(f(t)-f(x)\big)\big]\,dt$$
$$= \sum_{i=0}^{s}\binom{s}{i}g^{(s-i)}(x)D_{n,\alpha,\beta}^{(i)}(f,x) - (fg)^s(x)$$
$$= \sum_{i=0}^{s}\binom{s}{i}g^{(s-i)}(x)f^{(i)}(x) - (fg)^s(x) + O\big(n^{-\alpha/2}\big)$$
$$= O\big(n^{-\alpha/2}\big),$$

uniformly in $x \in [a', b']$. Combining the above estimates, we get

$$\big\|D_{n,\alpha,\beta}^{(s)}(fg,\star) - (fg)^s\big\|_{C[a',b']} = O\big(n^{-\alpha/2}\big).$$

Thus by Lemmas 5 and 7, we have $(fg)^{(s)} \in \mathrm{Lip}^\star(\alpha, a', b')$ also $g(x) = 1$ on the interval $[a_2, b_2]$, and it proves that $f^{(s)} \in \mathrm{Lip}^\star(\alpha, a_2, b_2)$. This completes the validity of the implication (i) $\Longrightarrow$ (ii) for the case $0 < \alpha \le 1$.

To prove the result for $1 < \alpha < 2$ for any interval $[a^*, b^*] \subset (a_1, b_1)$, let a_2^*, b_2^* be such that $(a_2, b_2) \subset (a_2^*, b_2^*)$ and $(a_2^*, b_2^*) \subset (a_1^*, b_1^*)$. Letting $\delta > 0$ we shall prove the assertion

$\alpha < 2$. From the previous case it implies that $f^{(s)}$ exists and belongs to $\mathrm{Lip}(1-\delta, a_1^*, b_1^*)$. Let $g \in C_0^\infty$ be such that $g(x) = 1$ on the interval $[a_2, b_2]$ and $\operatorname{supp} g \subset (a_2^*, b_2^*)$. If $\chi(t)$ denotes the characteristic function of the interval $[a_1^*, b_1^*]$, we have

$$\begin{aligned}\bigl\|D^{(s)}_{n,\alpha,\beta}(fg,x) - (fg)^{(s)}(x)\bigr\|_{C[a_2^*,b_2^*]} &\le \bigl\|D^s\bigl(D_{n,\alpha,\beta}\bigl(f(t)\bigl[g(t)-g(x)\bigr],x\bigr)\bigr)\bigr\|_{C[a_2^*,b_2^*]} \\ &\quad + \bigl\|D^s\bigl(D_{n,\alpha,\beta}\bigl(g(x)\bigl[f(t)-f(x)\bigr],x\bigr)\bigr)\bigr\|_{C[a_2^*,b_2^*]} \\ &=: F_1 + F_2.\end{aligned}$$

Using the linearity property, the Leibniz formula and Theorem 3, we have

$$\begin{aligned}F_1 &= \bigl\|D^s\bigl(g(x)D_{n,\alpha,\beta}(f,x) - (fg)(x)D_{n,\alpha,\beta}(1,x)\bigr)\bigr\|_{C[a_2^*,b_2^*]} \\ &= \left\|\sum_{i=0}^{s}\binom{s}{i}g^{(s-i)}(x)D^{(i)}_{n,\alpha,\beta}(f,x) - (fg)^{(s)}\right\|_{C[a_2^*,b_2^*]} \\ &= \left\|\sum_{i=0}^{s}\binom{s}{i}g^{(s-i)}(x)f^{(i)}(x) - (fg)^{(s)}\right\|_{C[a_2^*,b_2^*]} + O\bigl(n^{-\alpha/2}\bigr) = O\bigl(n^{-\alpha/2}\bigr).\end{aligned}$$

Applying the Leibniz formula and Theorem 2, we get

$$\begin{aligned}F_2 &= \left\|-\sum_{i=0}^{s-1}\binom{s}{i}g^{(s-i)}(x)D^{(i)}_{n,\alpha,\beta}(f,x) + D^{(s)}_{n,\alpha,\beta}\bigl(f(t)\bigl[g(t)-g(x)\bigr]\chi(t),x\bigr)\right\|_{C[a_2^*,b_2^*]} + O\bigl(n^{-1}\bigr) \\ &=: \|F_3 + F_4\|_{C[a_2^*,b_2^*]} + O\bigl(n^{-1}\bigr).\end{aligned}$$

Then by Theorem 3, we have

$$F_3 = -\sum_{i=0}^{s-1}\binom{s}{i}g^{(s-i)}(x)f^{(i)}(x) + O\bigl(n^{-\alpha/2}\bigr),$$

uniformly in $x \in [a_2^*, b_2^*]$. Applying Taylor's expansion of $f(t)$, we have

$$\begin{aligned}F_4 &= \int_0^\infty N^{(s)}_{n,\alpha,\beta}(x,t)\bigl[f(t)\bigl(g(t)-g(x)\bigr)\chi(t)\bigr]\,dt \\ &= \sum_{i=0}^{s}\frac{f^{(i)}(x)}{i!}\int_0^\infty N^{(s)}_{n,\alpha,\beta}(x,t)(t-x)^i\bigl(g(t)-g(x)\bigr)\chi(t)\,dt \\ &\quad + \int_0^\infty N^{(s)}_{n,\alpha,\beta}(x,t)\left[\frac{f^{(s)}(\xi)-f^{(s)}(x)}{s!}\right](t-x)^s\bigl(g(t)-g(x)\bigr)\chi(t)\,dt \\ &=: F_5 + F_6,\end{aligned}$$

where ξ lies between t and x. Using Theorem 2, we get

$$\begin{aligned}F_5 &= \sum_{i=0}^{s}\frac{f^{(i)}(x)}{i!}\int_0^\infty N^{(s)}_{n,\alpha,\beta}(x,t)(t-x)^i\bigl(g(t)-g(x)\bigr)\,dt + O\bigl(n^{-1}\bigr), \\ &\qquad \text{uniformly in } x \in \bigl[a_2^*, b_2^*\bigr] \\ &=: F_7 + O\bigl(n^{-1}\bigr).\end{aligned}$$

Again using Taylor's expansion of $g(t) \in C_0^\infty$ and using the fact that $\varepsilon(t,x) \to 0$ as $t \to x$, we have

$$\begin{aligned} F_7 &= \sum_{i=0}^{s} \frac{f^{(i)}(x)}{i!} \int_0^\infty N_{n,\alpha,\beta}^{(s)}(x,t)(t-x)^i \\ &\quad \times \left[g(x) + \sum_{j=1}^{s+2} \frac{g^{(j)}(x)}{j!}(t-x)^j + \varepsilon(t,x)(t-x)^{s+2} - g(x) \right] dt \\ &= \sum_{i=0}^{s} \frac{f^{(i)}(x)}{i!} \sum_{j=1}^{s+2} \frac{g^{(j)}(x)}{j!} \int_0^\infty N_{n,\alpha,\beta}^{(s)}(x,t)(t-x)^{i+j}\,dt \\ &\quad + \sum_{i=0}^{s} \frac{f^{(i)}(x)}{i!} \int_0^\infty N_{n,\alpha,\beta}^{(s)}(x,t)\varepsilon(t,x)(t-x)^{i+s+2}\,dt \\ &=: F_8 + F_9. \end{aligned}$$

Since $\int_0^\infty \frac{\partial^s}{\partial x^s} N_{n,\alpha,\beta}(x,t)(t-x)^k\,dt = 0$ for every $s > k$, therefore by Theorem 2 and Lemma 2, we have

$$\begin{aligned} F_8 &= \sum_{j=1}^{s} \frac{g^{(j)}(x) f^{(s-j)}(x)}{j!(s-j)!} s! + O(n^{-1}) \\ &= \sum_{j=1}^{s} \binom{s}{i} g^{(j)}(x) f^{(s-j)}(x) + O(n^{-1}), \end{aligned}$$

uniformly in $x \in [a_2^*, b_2^*]$. Also as in the proof of Theorem 1, it can be easily shown that

$$F_9 = O(n^{-\alpha/2}),$$

uniformly in $x \in [a_2^*, b_2^*]$. Next, using Lemma 5, the mean value theorem, the Schwarz inequality and Lemma 4, we have

$$\begin{aligned} &\left\| \int_0^\infty N_{n,\alpha,\beta}^{(s)}(x,t) \left[\frac{f^{(s)}(\xi) - f^{(s)}(x)}{s!} \right] (t-x)^s \big(g(t) - g(x)\big) \chi(t)\,dt \right\|_{C[a_2^*, b_2^*]} \\ &\quad \le \sum_{\substack{2i+j \le s \\ i,j \ge 0}} n^{i+j} \frac{|q_{i,j,s}(x)|}{[x(1+x)]^s} \\ &\qquad \times \left\| \int_0^\infty N_{n,\alpha,\beta}(x,t)|t-x|^{\delta+s+1} \frac{|f^{(s)}(\xi) - f^{(s)}(x)|}{s!} \big|g'(\eta)\big| \chi(t)\,dt \right\|_{C[a_2^*, b_2^*]} = O(n^{-\delta/2}), \end{aligned}$$

where η lies between t and x, and choose δ such that $0 \le \delta \le 2 - \alpha$. Combining the above estimates, we get

$$\left\| D_{n,\alpha,\beta}^{(s)}(fg, \star) - (fg)^s \right\|_{C[a_2^*, b_2^*]} = O(n^{-\alpha/2}).$$

Since supp $fg \subset (a_2^*, b_2^*)$, therefore by Lemmas 6 and 7, we have $(fg)^{(s)} \in \mathrm{Lip}^\star(\alpha, 1, a_2^*, b_2^*)$ also $g(x) = 1$ on the interval $[a_2, b_2]$, which proves that $f^{(s)} \in \mathrm{Lip}^\star(\alpha, a_2, b_2)$. This completes the validity of the implication (i) $\Longrightarrow$ (ii) for the case $1 < \alpha < 2$. This completes the proof of the theorem. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
VNM, KK, LNM and Deepmala computed the auxiliary results and inverse theorem in simultaneous approximation for Baskakov-Durrmeyer-Stancu Operators. VNM and Deepmala conceived of the study and participated in its design and coordination. VNM, KK, LNM and Deepmala contributed equally and significantly in writing this manuscript. All the authors drafted the manuscript, read and approved the final version of manuscript in JIA.

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