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Unilateral global bifurcation for p-Laplacian with singular weight

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Abstract
In this paper, we establish a Dancer-type unilateral global bifurcation theorem for the one-dimensional p-Laplacian with a singular weight which may not be in L^1. As the applications of this theorem, we prove the existence of nodal solutions for p-Laplacian with $f_0 \in [0,+\infty]$ or $f_\infty \in [0,+\infty]$, where $f(s)/(|s|^{p-2}s)$ approaches f_0 and f_∞ as s approaches 0 and ∞, respectively.
MSC: 34B16; 34C10; 34C23

Keywords: unilateral bifurcation; nodal solutions; singular weight

1 Introduction

In this paper, we shall establish a unilateral global bifurcation theorem for the following one-dimensional p-Laplacian problem

$$\begin{cases} -(\varphi_p(u'))' = \lambda m(x) f(u), & \text{a.e. } x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where $\varphi_p(s) = |s|^{p-2}s$, $1 < p < +\infty$, λ is a positive parameter, $m(x)$ and $f \in C(\mathbb{R},\mathbb{R})$ satisfy the following assumptions:

(A1) $m(x) \in \mathcal{A}$, $m(x) \geq 0$ and $m(x) \not\equiv 0$ on any subinterval of $(0,1)$, where

$$\mathcal{A} = \left\{ m(x) \in L^1_{\mathrm{loc}}(0,1) \,\middle|\, \int_0^1 x^{p-1}(1-x)^{p-1} m(x)\,dx < +\infty \right\};$$

(A2) $f(s)s > 0$ for $s \neq 0$.

Let $\mathscr{S}$ denote the closure of the set of nontrivial solutions to problem (1.1), and let λ_k denote the kth eigenvalue which is obtained in [1, Theorem 2.1] of the following problem

$$\begin{cases} -(\varphi_p(u'))' = \lambda m(x) \varphi_p(u), & \text{a.e. } x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{1.2}$$

Let $f_0 := \lim_{s\to 0} f(s)/\varphi_p(s)$. By an argument similar to Rabinowitz's unilateral global bifurcation theory [2, Theorem 1.27], Kajikiya *et al.* [3] established the following result.

Theorem 1.1 *Assume that* (A1)-(A2) *hold and* $f_0 \in (0,+\infty)$. *Then, for each* $k \in \mathbb{N}$, *there exist two unbounded sub-continua* $\mathcal{C}_k^\pm$ *in* $\mathscr{S}$ *bifurcating from* $(\lambda_k/f_0, 0)$. *Furthermore,* $\mathcal{C}_k^\pm \cap$

$(\mathbb{R} \times \{0\}) = \{(\lambda_k/f_0, 0)\}$ *and if* $(\lambda, u) \in \mathcal{C}_k^+ \setminus \{(\lambda_k/f_0, 0)\}$ $(\mathcal{C}_k^- \setminus \{(\lambda_k/f_0, 0)\})$, *then* u *is a* $(k-1)$-*nodal solution in* $(0,1)$ *satisfying* $u'(0) > 0$ $(u'(0) < 0)$, *respectively.*

However, as pointed out by Dancer [4, 5], López-Gómez [6] and Shi and Wang [7], the original statement of Theorem 1.27 of [2] is stronger than what one can actually prove so far. In [4], Dancer gave a corrected version of the unilateral global bifurcation theorem for a linear operator which has been extended to the one-dimensional p-Laplacian problem by Dai and Ma [8]. The first purpose of the present work is to repair the proof of Theorem 1.1 by the methods which we used in [8].

Let $f_\infty := \lim_{s\to+\infty} f(s)/\varphi_p(s)$. Based on Theorem 1.1, Kajikiya *et al.* [3] studied the existence of positive solutions as well as sign-changing solutions of problem (1.1) with $f_0 \in (0, +\infty)$ and $f_\infty = 0$. Later, they [9] again considered the case of $f_0 \in (0, +\infty)$ and $f_\infty = +\infty$. Another aim of this paper is to investigate the existence of nodal solutions for problem (1.1) with all of the following six cases:

(1) $f_0 \in (0, +\infty)$ and $f_\infty \in (0, +\infty)$;
(2) $f_0 = 0$ and $f_\infty \in (0, +\infty)$;
(3) $f_0 = +\infty$ and $f_\infty \in (0, +\infty)$;
(4) $f_0 = 0$ and $f_\infty = +\infty$;
(5) $f_0 = \infty$ and $f_\infty = +\infty$;
(6) $f_0 = 0$ and $f_\infty = 0$.

When $p = 2$, $m(x) \in C[0,1]$, Ma and Thompson [10] considered the interval of λ, in which there exist nodal solutions of problem (1.1) under some suitable assumptions on f. In [11], Ma extended the above results to the case of $m \in C(0,1)$ satisfying $0 < \int_0^1 x(1-x)m(x)\,dx < +\infty$. For $p \neq 2$, Del Pino *et al.* [12] investigated the existence of solutions for problem (1.1) with $m \equiv 1$ using the Leray-Schauder degree by the deformation along p. By the upper and lower solutions method, fixed point index theory on cones and the shooting method, the authors of [13–16] studied the existence of positive solutions or sign-changing solutions for problem (1.1) under some suitable assumptions on m and f. In [17, 18], Lee and Sim studied the existence of positive solutions as well as sign-changing solutions for problem (1.1) when $m \in L^1(0,1)$. Recently, Dai [19] studied the existence of nodal solutions for problem (1.1) when $m \in C[0,1]$ and $f_0 \notin (0, +\infty)$ or $f_\infty \notin (0, +\infty)$. In this paper, we extend the corresponding results of [19] to the case of m satisfying (A1). Clearly, the above six cases for problem (1.1) have not been studied by now.

The main results of the present paper are the following two theorems.

Theorem 1.2 *Let* (A1)-(A2) *hold and* $f_0 \in (0, +\infty)$. *Then from each* $(\lambda_k/f_0, 0)$ *it bifurcates an unbounded continuum* $\mathcal{C}_k$ *of solutions to problem* (1.1), *with exactly* $k-1$ *simple zeros.*

Theorem 1.3 *Let* (A1)-(A2) *hold and* $f_0, f_\infty \in (0, +\infty)$. *If* $\lambda \in (\lambda_k/f_\infty, \lambda_k/f_0) \cup (\lambda_k/f_0, \lambda_k/f_\infty)$, *then problem* (1.1) *has at least two solutions* u_k^+ *and* u_k^- *such that* u_k^+ *has exactly* $k-1$ *simple zeros in* $(0,1)$ *and is positive near* 0, *and* u_k^- *has exactly* $k-1$ *simple zeros in* $(0,1)$ *and is negative near* 0.

The rest of this paper is arranged as follows. In Section 2, we establish the unilateral global bifurcation theory for problem (1.1). In Section 3, we prove the existence of nodal solutions for problem (1.1) with any one of the above six cases.

2 Unilateral global bifurcation

Let E be the Banach space $C_0^1[0,1]$ with the norm $\|u\| = \|u\|_\infty + \|u'\|_\infty$, where $\|u\|_\infty = \max_{x\in[0,1]} |u|$. Consider the following auxiliary problem

$$\begin{cases} (\varphi_p(u'))' = h, & \text{a.e. } x \in (0,1), \\ u(0) = u(1) = 0 \end{cases} \tag{2.1}$$

for a given $h \in L^1(0,1)$. By a solution of problem (2.1), we understand a function $u \in E$ with $\varphi_p(u')$ absolutely continuous which satisfies problem (2.1). Problem (2.1) is equivalently written as

$$u(x) = G_p(h)(x) := \int_0^x \varphi_p^{-1}\left(a(h) + \int_0^s h(\tau)\,d\tau\right) ds,$$

where $a : L^1(0,1) \to \mathbb{R}$ is a continuous function satisfying

$$\int_0^1 \varphi_p^{-1}\left(a(h) + \int_0^s h(\tau)\,d\tau\right) ds = 0.$$

It is well known that $G_p : L^1(0,1) \to E$ is continuous and maps equi-integrable sets of $L^1(0,1)$ into relatively compacts of E. One may refer to Lee and Sim [17] and Manásevich and Mawhin [20] for details.

Lemma 2.3 of [3] shows that $m(x)f(v) \in L^1(0,1)$ for any $v \in E$ and f satisfying (A2). Hence, for $(\lambda, u) \in \mathbb{R} \times E$, we can define

$$T_\lambda(u) = G_p\big(-\lambda f_0 m(x)\varphi_p(u)\big) \quad \text{and} \quad F(\lambda,u) = G\big(-\lambda m(x)f(u)\big).$$

Lemma 2.4 of [3] has shown that T_λ and F are completely continuous from $\mathbb{R} \times E$ to E. So $I - T_\lambda$ is a completely continuous vector field in $C^1[0,1]$. Thus the Leray-Schauder degree $\mathrm{d}_{\mathrm{LS}}(I - T_\lambda, B_r(0), 0)$ is well defined for an arbitrary r-ball $B_r(0)$ and $\lambda \neq \lambda_k$, $k \in \mathbb{N}$.

Lemma 2.1 ([3, Theorem 3.2]) *Assume that* (A1) *holds and let* $\{\lambda_k\}_{k\in\mathbb{N}}$ *be the sequence of eigenvalues of problem* (1.2). *Let* λ *be a constant with* $\lambda \neq \lambda_k$ *for all* $k \in \mathbb{N}$. *Then, for arbitrary* $r > 0$,

$$\deg\big(I - T_\lambda, B_r(0), 0\big) = (-1)^\beta,$$

where β *is the number of eigenvalues* λ_k *of problem* (1.2) *less than* λ.

Using Lemma 2.1 and the famous global interval bifurcation theorem due to Schmitt and Thompson [21], the authors of [3] established the following result.

Lemma 2.2 ([3, Lemma 4.4]) *Assume that* (A1)-(A2) *hold and* $f_0 \in (0,+\infty)$. *Then* $(\lambda_k/f_0, 0)$ *is a bifurcation point of* $\mathscr{S}$ *and the associated bifurcation branch* $\mathcal{C}_k$ *in* $\mathbb{R} \times E$ *whose closure contains* $(\lambda_k/f_0, 0)$ *is either unbounded or contains a pair* $(\lambda_j/f_0, 0)$ *with* $j \neq k$.

Next, we shall prove that the first choice of the alternative of Lemma 2.2 is the only possibility. Let S_k^+ denote the set of functions in E which have exactly $k-1$ interior nodal zeros in $(0,1)$ and are positive near $x = 0$, and set $S_k^- = -S_k^+$, and $S_k = S_k^+ \cup S_k^-$. It is clear that S_k^+ and S_k^- are disjoint and open in E. Finally, let $\Phi_k^{\pm} = \mathbb{R} \times S_k^{\pm}$ and $\Phi_k = \mathbb{R} \times S_k$ under the product topology.

Lemma 2.3 *Under the assumptions of Lemma* 2.2, *the last alternative of Lemma* 2.2 *is impossible if* $\mathcal{C}_k \subset (\Phi_k \cup \{(\lambda_k/f_0, 0)\})$.

Proof Suppose on the contrary that if there exists $(\lambda_m, u_m) \to (\lambda_j/f_0, 0)$ when $m \to +\infty$ with $(\lambda_m, u_m) \in \mathcal{C}_k$, $u_m \not\equiv 0$ and $j \neq k$. Let $f(s) = f_0\varphi_p(s) + \xi(s)$ with $\xi(s)/\varphi_p(s) \to 0$ as $s \to 0$. Set $v_m := u_m/\|u_m\|$, then v_m should be a solution of the problem

$$v = G_p\left(-\lambda_m m(x) f_0 \varphi_p\big(v(x)\big) - \frac{\xi(u_m(x))}{\|u_m(x)\|^{p-1}}\right).$$

Let

$$\widetilde{\xi}(u) = \max_{0\le|s|\le u} \big|\xi(s)\big|,$$

then $\widetilde{\xi}$ is nondecreasing with respect to u and

$$\lim_{u\to 0^+} \frac{\widetilde{\xi}(u)}{u^{p-1}} = 0. \tag{2.2}$$

It follows from (2.2) that

$$\frac{\xi(u)}{\|u\|^{p-1}} \le \frac{\widetilde{\xi}(|u|)}{\|u\|^{p-1}} \le \frac{\widetilde{\xi}(\|u\|_\infty)}{\|u\|^{p-1}} \le \frac{\widetilde{\xi}(\|u\|)}{\|u\|^{p-1}} \to 0 \quad \text{as } \|u\| \to 0. \tag{2.3}$$

By (2.3) and the continuity and compactness of G_p, we obtain that for some convenient subsequence $v_m \to v_0$ as $m \to +\infty$. Now v_0 verifies the equation

$$-\big(\varphi_p\big(v_0'\big)\big)' = \lambda_j m(x) \varphi_p(v_0)$$

and $\|v_0\| = 1$. Hence $v_0 \in S_j$ which is an open set in E, and as a consequence for some m large enough, $v_m \in S_j$, and this is a contradiction. □

Proof of Theorem 1.2 Taking into account Lemma 2.2 and Lemma 2.3, we only need to prove that $\mathcal{C}_k \subset (\Phi_k \cup \{(\lambda_k/f_0, 0)\})$. By an argument similar to that of Lemma 2.3, we can show that there exists a neighborhood $\mathscr{O}$ of $(\lambda_k/f_0, 0)$ such that $\mathscr{O} \cap \mathcal{C}_k \subset (\Phi_k \cup \{(\lambda_k/f_0, 0)\})$. Suppose $\mathcal{C}_k \not\subset (\Phi_k \cup \{(\lambda_k/f_0, 0)\})$. Then there exists $(\lambda, u) \in \mathcal{C}_k \cap (\mathbb{R} \times \partial S_k)$ such that $(\lambda, u) \neq (\lambda_k/f_0, 0)$, $u \notin S_k$, and $(\lambda_n, u_n) \to (\lambda, u)$ with $(\lambda_n, u_n) \in \mathcal{C}_k \cap (\mathbb{R} \times S_k)$. Since $u \in \partial S_k$, by Lemma 4.1 of [3], $u \equiv 0$. Let $w_n := u_n/\|u_n\|$, then w_n should be a solution of the problem

$$w = G_p\left(-\lambda_n m(x) f_0 \varphi_p\big(w(x)\big) - \frac{\xi(u_n(x))}{\|u_n(x)\|^{p-1}}\right). \tag{2.4}$$

By (2.3), (2.4) and the continuity and compactness of G_p, we obtain that for some convenient subsequence $w_n \to w_0 \neq 0$ as $n \to +\infty$. Now w_0 verifies the equation

$$-\big(\varphi_p\big(w_0'\big)\big)' = \lambda f_0 m(x) \varphi_p(w_0)$$

and $\|w_0\| = 1$. Hence $\lambda f_0 = \lambda_j$ for some $j \neq k$. Therefore, $(\lambda_n, u_n) \to (\lambda_j/f_0, 0)$ with $(\lambda_n, u_n) \in \mathcal{C}_k \cap (\mathbb{R} \times S_k)$. This contradicts Lemma 2.3. □

Proof of Theorem 1.1 Applying a similar method to prove [8, Theorem 3.2] with obvious changes (we only need to replace $g(t, u; \mu)$ with $\xi(u)$ in the proof of Lemmas 3.1, 3.2 and 3.4 of [8]), we can obtain the result of Theorem 1.1. □

3 Nodal solutions

In this section, we use Theorem 1.1 to prove the existence of nodal solutions for problem (1.1) with all of the six cases introduced at the start.

Proof of Theorem 1.3 Applying Theorem 1.1 to problem (1.1), we have that there are two distinct unbounded continua $\mathcal{C}_k^+$ and $\mathcal{C}_k^-$, consisting of the bifurcation branch $\mathcal{C}_k$ emanating from $(\lambda_k/f_0, 0)$, such that

$$\mathcal{C}_k^\nu \subset \left(\left\{(\lambda_k, 0)\right\} \cup \left(\mathbb{R} \times S_k^\nu\right)\right).$$

To complete the proof of this theorem, it will be enough to show that $\mathcal{C}_k^\nu$ joins $(\lambda_k/f_0, 0)$ to $(\lambda_k/f_\infty, +\infty)$. Let $(\lambda_n, u_n) \in \mathcal{C}_k^\nu$ satisfy $\lambda_n + \|u_n\| \to +\infty$. We note that $\lambda_n > 0$ for all $n \in \mathbb{N}$ since $(0, 0)$ is the only solution of problem (1.1) for $\lambda = 0$ and $\mathcal{C}_k^\nu \cap (\{0\} \times E) = \emptyset$.

We divide the rest of the proof into two steps.

Step 1. We show that there exists a constant M such that $\lambda_n \in (0, M]$ for $n \in \mathbb{N}$ large enough.

On the contrary, we suppose that $\lim_{n\to+\infty} \lambda_n = +\infty$. On the other hand, we note that

$$\begin{cases} -(\varphi_p(u_n'))' = \lambda_n m(x)\widetilde{f}_n(x)\varphi_p(u_n), & \text{a.e. } x \in (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

where

$$\widetilde{f}_n(x) = \begin{cases} \frac{f(u_n)}{\varphi_p(u_n)} & \text{if } u_n \neq 0, \\ f_0 & \text{if } u_n = 0. \end{cases}$$

The signum condition implies that there exists a positive constant ϱ such that $\widetilde{f}_n(x) \geq \varrho$ for any $x \in [0, 1]$. By Theorem 2.1 of [1], we get u_n must change its sign more than $k - 1$ times in $(0, 1)$ for n large enough, and this contradicts the fact that $u_n \in \mathcal{C}_k^\nu$.

Step 2. We show that $\mathcal{C}_k^\nu$ joins $(\lambda_k/f_0, 0)$ to $(\lambda_k/f_\infty, +\infty)$.

It follows from Step 1 that $\|u_n\| \to +\infty$. Let $\eta \in C(\mathbb{R})$ such that $f(s) = f_\infty \varphi_p(s) + \eta(s)$. Then $\lim_{|s|\to+\infty} \eta(s)/\varphi_p(s) = 0$. Let $\widetilde{\eta}(u) = \max_{0\leq|s|\leq u} |\eta(s)|$. Then $\widetilde{\eta}$ is nondecreasing and

$$\lim_{u\to+\infty} \frac{\widetilde{\eta}(u)}{\varphi_p(u)} = 0. \tag{3.1}$$

We divide the equation

$$\begin{cases} -(\varphi_p(u_n'))' = \lambda_n m(x) f_\infty \varphi_p(u_n) + \lambda_n m(x)\eta(u_n), & \text{a.e. } x \in (0,1), \\ u(0) = u(1) = 0 \end{cases}$$

by $\|u_n\|$ and set $v_n = u_n/\|u_n\|$. Since v_n are bounded in E, after taking a subsequence if necessary, we have that $v_n \rightharpoonup \overline{v}$ for some $\overline{v} \in E$. Moreover, from (3.1) and the fact that $\widetilde{\eta}$ is nondecreasing, we have that

$$\lim_{n\to+\infty} \frac{\eta(u_n(x))}{\|u_n\|^{p-1}} = 0,$$

since

$$\frac{|\eta(u_n(x))|}{\|u_n\|^{p-1}} \leq \frac{\widetilde{\eta}(|u_n(x)|)}{\|u_n\|^{p-1}} \leq \frac{\widetilde{\eta}(\|u_n(x)\|)}{\|u_n\|^{p-1}}.$$

By the continuity and compactness of F, it follows that

$$\begin{cases} -(\varphi_p(\overline{v}'))' = \overline{\lambda} m(x) f_\infty \varphi_p(\overline{v}), & \text{a.e. } t \in (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

where $\overline{\lambda} = \lim_{n\to+\infty} \lambda_n$, again choosing a subsequence and relabeling it if necessary.

It is clear that $\|\overline{v}\| = 1$ and $\overline{v} \in \overline{\mathcal{C}_k^\nu} \subseteq \mathcal{C}_k^\nu$ since $\mathcal{C}_k^\nu$ is closed in $\mathbb{R} \times E$. Therefore, $\overline{\lambda} f_\infty = \lambda_k$, so that $\overline{\lambda} = \lambda_k/f_\infty$. Therefore, $\mathcal{C}_k^\nu$ joins $(\lambda_k/f_0, 0)$ to $(\lambda_k/f_\infty, +\infty)$. □

Theorem 3.1 *Let* (A1) *and* (A2) *hold. If* $f_0 = 0$ *and* $f_\infty \in (0, +\infty)$, *then for any* $\lambda \in (\lambda_k/f_\infty, +\infty)$, *problem* (1.1) *has at least two solutions* u_k^+ *and* u_k^- *such that* u_k^+ *has exactly* $k-1$ *simple zeros in* $(0,1)$ *and is positive near* 0, *and* u_k^- *has exactly* $k-1$ *simple zeros in* $(0,1)$ *and is negative near* 0.

Proof If (λ, u) is any solution of problem (1.1) with $\|u\|_\infty \neq 0$, dividing problem (1.1) by $\|u\|_\infty^{2(p-1)}$ and setting $w = u/\|u\|_\infty^2$ yields

$$\begin{cases} -(\varphi_p(w'))' = \lambda m(x)\left(\frac{f(u)}{\|u\|_\infty^{2(p-1)}}\right) & \text{in } (0,1), \\ w(0) = w(1) = 0. \end{cases} \tag{3.2}$$

Define $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{f}(w) = \begin{cases} \|w\|_\infty^{2(p-1)} f(w/\|w\|_\infty^2) & \text{if } w \neq 0, \\ 0 & \text{if } w = 0. \end{cases}$$

Clearly, problem (3.2) is equivalent to

$$\begin{cases} -(\varphi_p(w'))' = \lambda m(x) \widetilde{f}(w) & \text{in } (0,1), \\ w(0) = w(1) = 0. \end{cases} \tag{3.3}$$

It is obvious that $(\lambda, 0)$ is always the solution of problem (3.3). On the other hand, we have that

$$\widetilde{f}_0 = \lim_{w\to0} \frac{\widetilde{f}(w)}{\varphi_p(w)} = \lim_{w\to0} \frac{\|w\|^{2(p-1)} f(w/\|w\|^2)}{\varphi_p(w)} = \lim_{|u|\to+\infty} \frac{f(u)}{\varphi_p(u)} = f_\infty.$$

Similarly, we can also show that $\widetilde{f}_\infty = f_0$.

Now, applying Theorem 5.1 of [3] and the inversion $w \to w/\|w\|_\infty^2 = u$, we can achieve the conclusion. □

The following result is a direct corollary of Theorem 2.4 of [9].

Theorem 3.2 *Let* (A1) *and* (A2) *hold. If* $f_0 = 0$ *and* $f_\infty = +\infty$, *then for any* $\lambda \in (0, +\infty)$, *problem* (1.1) *has two solutions* u_k^+ *and* u_k^- *such that* u_k^+ *has exactly* $k - 1$ *simple zeros in* $(0, 1)$ *and is positive near* 0, *and* u_k^- *has exactly* $k - 1$ *simple zeros in* $(0, 1)$ *and is negative near* 0.

Theorem 3.3 *Let* (A1) *and* (A2) *hold. If* $f_0 = +\infty$ *and* $f_\infty \in (0, +\infty)$, *then for any* $\lambda \in (0, \lambda_1/f_\infty)$, *problem* (1.1) *has two solutions* u_k^+ *and* u_k^- *such that* u_k^+ *has exactly* $k - 1$ *simple zeros in* $(0, 1)$ *and is positive near* 0, *and* u_k^- *has exactly* $k - 1$ *simple zeros in* $(0, 1)$ *and is negative near* 0.

Proof By an argument similar to Theorem 3.1 and the conclusion of [9, Theorem 2.1], we can obtain the conclusion. □

Next, we shall need the following topological lemma.

Lemma 3.1 (see [22]) *Let* X *be a Banach space and let* C_n *be a family of closed connected subsets of* X. *Assume that*:

(i) *there exist* $z_n \in C_n$, $n = 1, 2, \ldots$, *and* $z^* \in X$ *such that* $z_n \to z^*$;

(ii) $r_n = \sup\{\|x\| | x \in C_n\} = +\infty$;

(iii) *for every* $R > 0$, $(\bigcup_{n=1}^{+\infty} C_n) \cap B_R$ *is a relatively compact set of* X, *where*

$$B_R = \{x \in X | \|x\| \leq R\}.$$

Using Theorem 1.1, Lemma 3.1 and a similar method to prove [19, Theorems 2.7 and 2.8] with obvious changes, we may obtain the following two theorems.

Theorem 3.4 *Let* (A1) *and* (A2) *hold. If* $f_0 = +\infty$ *and* $f_\infty = +\infty$, *then there exists* $\lambda^+ > 0$ *such that for any* $\lambda \in (0, \lambda^+)$, *problem* (1.1) *has two solutions* $u_{k,1}^+$ *and* $u_{k,2}^+$ *such that they have exactly* $k - 1$ *simple zeros in* $(0, 1)$ *and are positive near* 0. *Similarly, there exists* $\lambda^- > 0$ *such that for any* $\lambda \in (0, \lambda^-)$, *problem* (1.1) *has two solutions* $u_{k,1}^-$ *and* $u_{k,2}^-$ *such that they have exactly* $k - 1$ *simple zeros in* $(0, 1)$ *and are negative near* 0.

Theorem 3.5 *Let* (A1) *and* (A2) *hold. If* $f_0 = 0$ *and* $f_\infty = 0$, *then there exists* $\lambda_*^+ > 0$ *such that for any* $\lambda \in (\lambda_*^+, +\infty)$, *problem* (1.1) *has two solutions* $u_{k,1}^+$ *and* $u_{k,2}^+$ *such that they have exactly* $k - 1$ *simple zeros in* $(0, 1)$ *and are positive near* 0. *Similarly, there exists* $\lambda_*^- > 0$ *such that for any* $\lambda \in (\lambda_*^-, +\infty)$, *problem* (1.1) *has two solutions* $u_{k,1}^-$ *and* $u_{k,2}^-$ *such that they have exactly* $k - 1$ *simple zeros in* $(0, 1)$ *and are negative near* 0.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
GD conceived of the study, and participated in its design and coordination and helped to draft the manuscript. XL participated in the design of the study. All authors read and approved the final manuscript.

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