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An iterative method for variational inequality problems

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Abstract
In this paper, we present some properties of generalized proximity operators and propose an iterative method of approximating solutions for a class of generalized variational inequalities and show its convergence in uniformly convex and smooth Banach spaces.

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Keywords: Banach space; proximity operator; variational inequality; iterative method

1 Introduction

Let f be a lower semi-continuous proper convex function from a Hilbert space H to $(-\infty,+\infty]$. The Moreau envelope of the function f is defined as

$$e^f(x) = \inf_{y\in H}\left\{f(y) + \frac{1}{2}\|x-y\|^2\right\}. \tag{1.1}$$

It is well known that $e^f(x)$ is a continuous convex function, and for every $x \in H$, the infimum in (1.1) is achieved at a unique point $\operatorname{prox}_f(x)$. The operator prox_f from H to H, *i.e.*,

$$\operatorname{prox}_f x = \arg\min_{y\in H}\left\{f(y) + \frac{1}{2}\|x-y\|^2\right\} \tag{1.2}$$

thus defined, is called the proximity operator of f. When $f = \iota_K$ is the indicator function of a closed convex set K in H, then $\operatorname{prox}_f(x) = P_K(x)$ becomes the metric projection operator on K.

In 1994, Alber extended the metric projection operator to uniformly convex and uniformly smooth Banach spaces. Let K be a closed convex subset of a uniformly convex and uniformly smooth Banach space X, Alber [1] introduced the generalized projections $\pi_K : X^* \to K$ and $\Pi_K : X \to K$,

$$\pi_K(x^*) = \arg\min_{x\in K}\left\{\|x^*\|^2 - 2\langle x^*, x\rangle + \|x\|^2\right\}$$

and

$$\Pi_K(x) = \arg\min_{y\in K}\left\{\|Jx\|^2 - 2\langle Jx, y\rangle + \|y\|^2\right\},$$

where J is the duality mapping from X to X^*, and studied their properties in detail. In [2], Alber presented some applications of the generalized projections to approximately solving variational inequalities in Banach spaces. Recently, Li [3] extended the generalized projection operator π_K from uniformly convex and uniformly smooth Banach spaces to reflexive Banach spaces and studied some properties of the generalized projection operator with applications to solving the variational inequality in Banach spaces. By employing the generalized projection operators, Zeng and Yao [4] established some existence results for the variational inequality problem in uniformly convex and uniformly smooth Banach spaces and convergence results for the variational inequality. In [5], Wu and Huang further introduced and studied a class of generalized f-projection operators in Banach spaces. As applications, they proposed an iterative method of approximating solutions for the variational inequality problem: find $\bar{x} \in K$ such that

$$\langle A\bar{x}, y - \bar{x}\rangle + f(y) - f(\bar{x}) \geq 0, \quad \forall y \in K, \tag{1.3}$$

where K is a nonempty closed convex subset of X, $A : K \to X^*$ is a mapping and $f : X \to (-\infty, +\infty]$ is a proper convex, lower semicontinuous and positively homogeneous function, via

$$x_{n+1} = \pi_K^f\bigl(Jx_n - \alpha_n J\bigl(x_n - \pi_K^f(Jx_n - \rho Ax_n)\bigr)\bigr), \tag{1.4}$$

where

$$\pi_K^f\bigl(x^*\bigr) = \arg\min_{x\in K}\bigl\{2\rho f(x) + \bigl\|x^*\bigr\|^2 - 2\bigl\langle x^*, x\bigr\rangle + \|x\|^2\bigr\},$$

and the parameter sequence $\{\alpha_n\}$ satisfies

$$0 \leq \alpha_n \leq 1, \quad \sum_{n=0}^{+\infty} \alpha_n(1 - \alpha_n) = +\infty, \quad \rho > 0,$$

and they proved that $\{x_n\}$ has a subsequence converging to a solution of (1.3) when K is a nonempty compact convex subset of a uniformly convex and uniformly smooth Banach space.

Motivated and inspired by the above works, we continue to study some properties of generalized proximity operators and propose an iterative method of approximating solutions for the following generalized variational inequality problem: find $\bar{x} \in \operatorname{dom} f$ such that

$$\langle A\bar{x}, y - \bar{x}\rangle + f(y) - f(\bar{x}) \geq 0, \quad \forall y \in \operatorname{dom} f, \tag{1.5}$$

where $f : X \to (-\infty, +\infty]$ is a proper convex and lower semicontinuous function, $A : X \to X^*$ is a norm-to-weak continuous operator. Our iterative method is different from that given in [5]. We also prove a convergence result for this iterative method in smooth and uniformly convex Banach spaces. Let K be a nonempty closed convex set of X. If we replace f by $f + I_K$ in (1.5), where I_K is the indicator function of K, then (1.5) reduces to (1.3).

2 Preliminaries

Let X be a reflexive, smooth and strictly convex Banach space with the dual space X^*. We denote by $x_n \to x$ and $x_n \rightharpoonup x$ the strong and the weak convergence to x of a sequence $\{x_n\}$ in a Banach space X, respectively. Let $\Gamma_0(X)$ denote the class of all lower semi-continuous proper convex functions from X to $(-\infty, +\infty]$. Let $B(x,\delta)$ denote the closed ball of $x \in X$ and radius $\delta > 0$. Let $S(X) = \{x \in X : \|x\| = 1\}$ be the unit sphere.

A Banach space X is said to be strictly convex if $\frac{1}{2}\|x+y\| < 1$ for all $x, y \in S(X)$ and $x \neq y$. The Banach space X is said to be smooth provided

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in S(X)$. We recall that uniform convexity of X means that for any given $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$ with $\|x\| \le 1$, $\|y\| \le 1$, and $\|x - y\| = \epsilon$, the inequality

$$\|x+y\| \le 2(1-\delta)$$

holds.

A subset C of X is called boundedly compact if for any $\delta > 0$ the intersection $C \cap B(0,\delta)$ is empty or compact.

The duality mapping $J : X \rightrightarrows X^*$ is defined by

$$J(x) = \{x^* \in X^* | \langle x^*, x\rangle = \|x^*\|^2 = \|x\|^2\}, \quad \forall x \in X.$$

The following basic results concerning the duality mapping are well known [2, 6, 7]:

(1) X is reflexive if and only if J is surjective;
(2) X is strictly convex if and only if J is injective;
(3) X is smooth if and only if J is single-valued;
(4) if X is smooth, then J is norm-to-weak star continuous;
(5) J is monotone, *i.e.*, $\langle Jx - Jy, x - y\rangle \ge 0$, $\forall x, y \in X$;
(6) if X is strictly convex and smooth, then $\langle Jx - Jy, x - y\rangle = 0 \Rightarrow x = y$, $\forall x, y \in X$;
(7) if a Banach space X is reflexive strictly convex and smooth, then the duality mapping J^* from X^* into X is the inverse of J, that is, $J^{-1} = J^*$.

Consider the following envelope function:

$$e_V^f(x^*) = \inf_{x\in X} \left\{ f(x) + \frac{1}{2} V(x^*, x) \right\}, \tag{2.1}$$

where $V(x^*, x) = \|x^*\|^2 - 2\langle x^*, x\rangle + \|x\|^2$. Since the function $(x, x^*) \to f(x) + \frac{1}{2}V(x^*, x)$ is lower semicontinuous convex, one sees that $e_V^f(x^*)$ is lower semicontinuous and convex by Proposition 4.4 in [8].

For every $x^* \in X^*$, the infimum in (2.1) is achieved at a unique point $\pi_f(x^*)$, *i.e.*,

$$\pi_f(x^*) := \arg\min_{x\in K} \left\{ f(x) + \frac{1}{2} V(x^*, x) \right\}.$$

The operator π_f is called the generalized proximity operator. It can be characterized by the inclusion

$$x^* - J\pi_f(x^*) \in \partial f(\pi_f(x^*)), \tag{2.2}$$

equivalently,

$$\pi_f = (J + \partial f)^{-1}. \tag{2.3}$$

From (2.3), we easily know that π_f is maximal monotone by Theorem 2.6.2 in [6]. Observe that when $\rho = 1$,

$$\pi_K^f(x^*) = \pi_{f+I_K}(x^*).$$

If, in addition, $\operatorname{dom} f = K$, then $\pi_K^f(x^*) = \pi_f(x^*)$.

Lemma 2.1 ([9]) *Let X be a smooth, strictly convex and reflexive Banach space, let $\{x_n\}$ be a sequence in X, and $x \in X$. If $\langle x_n - x, Jx_n - Jx\rangle \to 0$, then $x_n \rightharpoonup x$, $Jx_n \rightharpoonup Jx$ and $\|x_n\| \to \|x\|$.*

Lemma 2.2 ([10]) *Let $r > 0$ be a fixed real number. Then a Banach space X is uniformly convex if and only if there is a continuous, strictly increasing and convex function $g : R^+ \to R^+$ with $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|), \quad \forall x, y \in B_r, 0 \le \lambda \le 1,$$

where $B_r = \{x \in X : \|x\| \le r\}$.

3 Main results

Proposition 3.1 *Let $f \in \Gamma_0(X)$. Then the following hold*:

(i) $\langle \pi_f(x^*) - \pi_f(y^*), (x^* - J\pi_f(x^*)) - (y^* - J\pi_f(y^*))\rangle \ge 0$, $\forall x^*, y^* \in X^*$;

(ii) *π_f is bounded on each nonempty bounded subset of $C \subset X^*$*;

(iii) *if $\{x_n^*\}$ is a sequence in X^* such that $x_n^* \to x^*$, then $\pi_f(x_n^*) \rightharpoonup \pi_f(x^*)$, $J\pi_f(x_n^*) \rightharpoonup J\pi_f(x^*)$ and $\|\pi_f(x_n^*)\| \to \|\pi_f(x^*)\|$*;

(iv) *if $\operatorname{dom} f$ is a nonempty boundedly compact convex subset, then π_f is weak-to-norm continuous, that is, if $x_n^* \rightharpoonup x^*$, then $\pi_f(x_n^*) \to \pi_f(x^*)$.*

Proof (i) Take $x^*, y^* \in X^*$. Then (2.2) yields

$$\langle x^* - J\pi_f(x^*), \pi_f(y^*) - \pi_f(x^*)\rangle \le f(\pi_f(y^*)) - f(\pi_f(x^*))$$

and

$$\langle y^* - J\pi_f(y^*), \pi_f(x^*) - \pi_f(y^*)\rangle \le f(\pi_f(x^*)) - f(\pi_f(y^*)).$$

Adding these two inequalities, we obtain

$$\langle \pi_f(x^*) - \pi_f(y^*), (x^* - J\pi_f(x^*)) - (y^* - J\pi_f(y^*))\rangle \ge 0.$$

(ii) Suppose that π_f is not bounded on some nonempty bounded subset of C. Then there exists a bounded sequence $\{x_n^*\} \subset C$ such that $\|\pi_f(x_n^*)\| \to \infty$. Fix $x^* \in X^*$. From (i), we obtain the following:

$$\begin{aligned}\|\pi_f(x_n^*) - \pi_f(x^*)\| \|x_n^* - x^*\| &\geq \langle \pi_f(x_n^*) - \pi_f(x^*), x_n^* - x^* \rangle \\ &\geq \langle \pi_f(x_n^*) - \pi_f(x^*), J\pi_f(x_n^*) - J\pi_f(x^*) \rangle \\ &= \frac{1}{2}\big(V(J\pi_f(x_n^*), \pi_f(x^*)) + V(J\pi_f(x^*), \pi_f(x_n^*))\big) \\ &\geq \big(\|\pi_f(x_n^*)\| - \|\pi_f(x^*)\|\big)^2.\end{aligned}$$

So, we have $\|x_n^*\| \to \infty$. This is a contradiction.

(iii) Let $\{x_n^*\}$ be a sequence in X^* such that $x_n^* \to x^*$. It follows from (ii) that $\{\pi_f(x_n^*)\}$ is bounded. From (i), we have

$$0 \leq \langle \pi_f(x_n^*) - \pi_f(x^*), J\pi_f(x_n^*) - J\pi_f(x^*) \rangle \leq \|\pi_f(x_n^*) - \pi_f(x^*)\| \|x_n^* - x^*\| \to 0.$$

Thus, Lemma 2.1 implies that $\pi_f(x_n^*) \rightharpoonup \pi_f(x^*)$, $J\pi_f(x_n^*) \rightharpoonup J\pi_f(x^*)$ and $\|\pi_f(x_n^*)\| \to \|\pi_f(x^*)\|$.

(iv) From (2.2), we know that

$$\langle x_n^* - J\pi_f(x_n^*), y - \pi_f(x_n^*) \rangle \leq f(y) - f(\pi_f(x_n^*)), \quad \forall y \in \operatorname{dom} f. \tag{3.1}$$

Since $x_n^* \rightharpoonup x^*$, $\{x_n^*\}$ is bounded. It follows from (ii) that $\{\pi_f(x_n^*)\}$ is bounded. Since $\operatorname{dom} f$ is boundedly compact, there exists a subsequence $\{x_{n_i}^*\}$ of $\{x_n^*\}$ such that

$$\pi_f(x_{n_i}^*) \to \bar{x} \in \operatorname{dom} f \quad \text{as } i \to +\infty.$$

Since J is norm-to-weak star continuous and f is lower semicontinuous, we obtain that

$$\langle x^* - J\bar{x}, y - \bar{x} \rangle \leq f(y) - f(\bar{x}), \quad \forall y \in \operatorname{dom} f. \tag{3.2}$$

Then, by (2.2), we have $\bar{x} = \pi_f(x^*)$. Similar to the above arguments, we know that $\pi_f(x^*)$ is the unique limit point of $\{\pi_f(x_n^*)\}$. Hence, $\pi_f(x_n^*) \to \pi_f(x^*)$. □

With the help of the operator π_f, we can show that the envelope function e_V^f is Gâteaux differentiable.

Proposition 3.2 *Let $f \in \Gamma_0(X)$. Then e_V^f is Gâteaux differentiable and* $\nabla e_V^f(x^*) = J^* x^* - \pi_f(x^*)$.

Proof For any $h \in X^*$, by definitions of e_V^f and π_f, we have

$$\begin{aligned}&\frac{e_V^f(x^* + th) - e_V^f(x^*)}{t} \\ &\quad = \frac{f(\pi_f(x^* + th)) + \frac{1}{2}V(\pi_f(x^* + th), x^* + th) - f(\pi_f(x^*)) - \frac{1}{2}V(\pi_f(x^*), x^*)}{t}\end{aligned}$$

$$\begin{aligned}&\le \frac{f(\pi_f(x^*)) + \frac{1}{2}V(\pi_f(x^*), x^* + th) - f(\pi_f(x^*)) - \frac{1}{2}V(\pi_f(x^*), x^*)}{t}\\&= \frac{\frac{1}{2}\|x^* + th\|^2 - \frac{1}{2}\|x^*\|^2 - \langle \pi_f(x^*), th\rangle}{t}.\end{aligned}$$

Since $\nabla(\|z\|^2) = 2J^*(z)$, for any $z \in X^*$, we get that

$$\begin{aligned}\limsup_{t\to 0} \frac{e_V^f(x^* + th) - e_V^f(x^*)}{t} &\le \lim_{t\to 0} \frac{\frac{1}{2}\|x^* + th\|^2 - \frac{1}{2}\|x^*\|^2}{t} - \langle \pi_f(x^*), h\rangle\\&= \langle J^*x^* - \pi_f(x^*), h\rangle.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\frac{e_V^f(x^* + th) - e_V^f(x^*)}{t}\\&= \frac{f(\pi_f(x^* + th)) + \frac{1}{2}V(\pi_f(x^* + th), x^* + th) - f(\pi_f(x^*)) - \frac{1}{2}V(\pi_f(x^*), x^*)}{t}\\&\ge \frac{f(\pi_f(x^* + th)) + \frac{1}{2}V(\pi_f(x^* + th), x^* + th) - f(\pi_f(x^* + th)) - \frac{1}{2}V(\pi_f(x^* + th), x^*)}{t}\\&= \frac{\frac{1}{2}\|x^* + th\|^2 - \frac{1}{2}\|x^*\|^2}{t} - \langle \pi_f(x^* + th), h\rangle.\end{aligned}$$

By Proposition 3.1(iii), we have $\pi_f(x^* + th) \rightharpoonup \pi_f(x^*)$ as $t \to 0$. Hence, we get that

$$\begin{aligned}\liminf_{t\to 0} \frac{e_V^f(x^* + th) - e_V^f(x^*)}{t} &\ge \lim_{t\to 0} \frac{\frac{1}{2}\|x^* + th\|^2 - \frac{1}{2}\|x^*\|^2}{t} - \langle \pi_f(x^* + th), h\rangle\\&= \langle J^*x^* - \pi_f(x^*), h\rangle.\end{aligned}$$

This implies that

$$\lim_{t\to 0} \frac{e_V^f(x^* + th) - e_V^f(x^*)}{t} = \langle J^*x^* - \pi_f(x^*), h\rangle.$$

Hence e_V^f is Gâteaux differentiable and $\nabla e_V^f(x^*) = J^*x^* - \pi_f(x^*)$. □

In the following, we propose a modification of the iterative method given in [5] and prove that the iterative sequence has a subsequence converging to a solution of (1.5) when X is a smooth and uniformly convex Banach space and f is not necessarily positively homogeneous.

By (2.2), we can easily prove the following result.

Proposition 3.3 *Let $f \in \Gamma_0(X)$. Then the point $\bar{x} \in \operatorname{dom} f$ is a solution of the variational inequality*

$$\langle Ax, y - x\rangle + f(y) - f(x) \ge 0, \quad \forall y \in \operatorname{dom} f$$

if and only if $\bar{x} \in \operatorname{dom} f$ is a solution of the following inclusion:

$$x = \pi_f(Jx - Ax).$$

The following lemma will be used in proving the convergence of the iterative method for variational inequality problem (1.5).

Lemma 3.1 *Let $f \in \Gamma_0(X)$. If $f(x) \geq 0$ for all $x \in \operatorname{dom} f$ and $f(0) = 0$, then*

$$\left\|\pi_f(x^*)\right\| \leq \left\|x^*\right\|. \tag{3.3}$$

Proof From (2.2), we know that

$$\langle x^* - J\pi_f(x^*), y - \pi_f(x^*)\rangle \leq f(y) - f(\pi_f(x^*)), \quad \forall y \in \operatorname{dom} f.$$

Noticing that $f(x) \geq 0$ for all $x \in \operatorname{dom} f$ and $f(0) = 0$, it follows that

$$\langle x^* - J\pi_f(x^*), -\pi_f(x^*)\rangle \leq -f(\pi_f(x^*)) \leq 0.$$

Hence,

$$\left\|\pi_f(x^*)\right\|^2 \leq \langle x^*, \pi_f(x^*)\rangle,$$

and hence

$$\left\|\pi_f(x^*)\right\| \leq \left\|x^*\right\|. \qquad \square$$

Proposition 3.4 *Let X be a smooth and uniformly convex Banach space. Let $A : X \to X^*$ be a norm-to-weak continuous operator. Suppose that $f \in \Gamma_0(X)$ and $\operatorname{dom} f$ is nonempty boundedly compact convex. Suppose that*

(i) $f(x) \geq 0$ *for all* $x \in \operatorname{dom} f$ *and* $f(0) = 0$;
(ii) *for any* $x \in \operatorname{dom} f$,

$$\|Jx - Ax\| \leq \|x\|.$$

Let $x_0 \in \operatorname{dom} f$ and the sequence $\{x_n\}$ be generated by the following iteration scheme:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n\pi_f(Jx_n - Ax_n),$$

where $\{\alpha_n\}$ satisfies the conditions:

(a) $0 \leq \alpha_n \leq 1$ *for all* $n = 0, 1, 2, \ldots$;
(b) $\sum_{n=0}^{+\infty} \alpha_n(1 - \alpha_n) = +\infty$.

Then generalized variational inequality (1.5) *has a solution $\bar{x} \in \operatorname{dom} f$, and there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to \bar{x}$ as $i \to \infty$.*

Proof By (3.3), we have

$$\left\|\pi_f(Jx_n - Ax_n)\right\| \leq \|Jx_n - Ax_n\|. \tag{3.4}$$

By (3.4) and condition (ii), we obtain

$$\|x_{n+1}\| \leq (1 - \alpha_n)\|x_n\| + \alpha_n\left\|\pi_f(Jx_n - Ax_n)\right\| \leq \|x_n\|.$$

Then $\{x_n\}$ and $\{\pi_f(Jx_n - Ax_n)\}$ are bounded. Hence, by Lemma 2.2, there exists a continuous, strictly increasing and convex function $g : R^+ \to R^+$ with $g(0) = 0$ such that

$$\begin{aligned} \|x_{n+1}\|^2 &= \big\|(1-\alpha_n)x_n + \alpha_n \pi_f(Jx_n - Ax_n)\big\|^2 \\ &\le (1-\alpha_n)\|x_n\|^2 + \alpha_n \big\|\pi_f(Jx_n - Ax_n)\big\|^2 \\ &\quad - \alpha_n(1-\alpha_n) g\big(\big\|x_n - \pi_f(Jx_n - Ax_n)\big\|\big). \end{aligned} \tag{3.5}$$

It follows from (3.5), (3.4) and condition (ii) that

$$\|x_{n+1}\|^2 \le (1-\alpha_n)\|x_n\|^2 + \alpha_n\|x_n\|^2 - \alpha_n(1-\alpha_n) g\big(\big\|x_n - \pi_f(Jx_n - Ax_n)\big\|\big).$$

That is,

$$\|x_{n+1}\|^2 \le \|x_n\|^2 - \alpha_n(1-\alpha_n) g\big(\big\|x_n - \pi_f(Jx_n - Ax_n)\big\|\big). \tag{3.6}$$

Taking the sum for $n = 0, 1, 2, \ldots, m$ in (3.6), we get

$$\begin{aligned} &\sum_{n=0}^{m} \alpha_n(1-\alpha_n) g\big(\big\|x_n - \pi_f(Jx_n - Ax_n)\big\|\big) \\ &\quad \le \|x_0\|^2 - \|x_{m+1}\|^2 \\ &\quad \le \|x_0\|^2. \end{aligned}$$

Hence,

$$\sum_{n=0}^{+\infty} \alpha_n(1-\alpha_n) g\big(\big\|x_n - \pi_f(Jx_n - Ax_n)\big\|\big) < +\infty. \tag{3.7}$$

Due to the condition $\sum_{n=0}^{+\infty} \alpha_n(1-\alpha_n) = +\infty$, we may assume, without loss of generality, that

$$g\big(\big\|x_n - \pi_f(Jx_n - Ax_n)\big\|\big) \to 0 \quad \text{as } n \to +\infty.$$

Applying the properties of g, we can deduce that

$$\big\|x_n - \pi_f(Jx_n - Ax_n)\big\| \to 0 \quad \text{as } n \to +\infty. \tag{3.8}$$

Since $\operatorname{dom} f$ is boundedly compact, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$x_{n_i} \to \bar{x} \in \operatorname{dom} f \quad \text{as } i \to +\infty.$$

Since A is norm-to-weak continuous and J is norm-to-weak star continuous, we get that

$$Jx_{n_i} - Ax_{n_i} \rightharpoonup J\bar{x} - A\bar{x} \quad \text{as } i \to +\infty.$$

Since π_f is weak-to-norm continuous by Proposition 3.1(iv),

$$\pi_f(Jx_{n_i} - Ax_{n_i}) \to \pi_f(J\bar{x} - A\bar{x}) \quad \text{as } i \to +\infty.$$

Hence, (3.8) yields

$$\bar{x} = \pi_f(J\bar{x} - A\bar{x}).$$

Now it follows from Proposition 3.3 that $\bar{x}$ is a solution of generalized variational inequality (1.5). □

4 Application

Let $f \in \Gamma_0(X)$ and let $g : X \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Consider the optimization problem

$$\min_{x \in X} f(x) + g(x). \tag{P}$$

We denote by Sol(P) the solution set of problem (P). Despite its simplicity, problem (P) has been shown to cover a wide range of apparently unrelated signal recovery formulations (see [11, 12]).

Notice that

$$\begin{aligned} \bar{x} \in \mathrm{Sol(P)} \quad &\Leftrightarrow \quad 0 \in \partial f(\bar{x}) + \nabla g(\bar{x}) \\ &\Leftrightarrow \quad -\nabla g(\bar{x}) \in \partial f(\bar{x}) \\ &\Leftrightarrow \quad \langle \nabla g(\bar{x}), y - \bar{x} \rangle + f(y) - f(\bar{x}) \geq 0, \quad \forall y \in X. \end{aligned}$$

Note that if g is convex and Gâteaux differentiable, then ∇g is norm-to-weak continuous from X to X^* by Corollary 3.1 in [13]. Therefore, as an application of Proposition 3.4, we have the following result.

Proposition 4.1 *Let X be a smooth and uniformly convex Banach space. Let $g : X \to \mathbb{R}$ be convex and Gâteaux differentiable. Suppose that $f \in \Gamma_0(X)$ and* dom$f$ *is a nonempty boundedly compact convex subset of X. Suppose that*

(i) $f(x) \geq 0$ *for all* $x \in \mathrm{dom} f$ *and* $f(0) = 0$;
(ii) *for any* $x \in \mathrm{dom} f$,

$$\| Jx - \nabla g(x) \| \leq \|x\|.$$

Let $x_0 \in \mathrm{dom} f$ and the sequence $\{x_n\}$ be generated by the following iteration scheme:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n \pi_f\big(Jx_n - \nabla g(x_n)\big),$$

where $\{\alpha_n\}$ satisfies the conditions:

(a) $0 \leq \alpha_n \leq 1$ *for all* $n = 0, 1, 2, \dots$;
(b) $\sum_{n=0}^{+\infty} \alpha_n(1 - \alpha_n) = +\infty$.

Then problem (P) *has a solution $\bar{x}$ and there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to \bar{x}$ as $i \to \infty$.*

5 Concluding remark

This paper has improved the iterative method of Wu and Huang [5] for solving generalized variational inequality problem (1.5), several results regarding the generalized proximity operator and its relations with the envelope function are presented. In addition, it is shown that under an appropriate assumption some optimization problem can be transformed into (1.5) and then the iterative method can be applied.

Competing interests
The author declares that they have no competing interests.

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