Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

An analogue of the Bernstein-Walsh lemma in Jordan regions of the complex plane

Fahreddin G Abdullayev* and Naciye Pelin Özkartepe

*Correspondence:
fabdul@mersin.edu.tr;
fahreddinabdullayev@gmail.com
Faculty of Arts and Science,
Department of Mathematics, Mersin
University, Mersin, 33343, Turkey

Abstract
In this paper we continue to study two-dimensional analogues of Bernstein-Walsh estimates for arbitrary Jordan domains.
MSC: Primary 30A10; 30C10; secondary 41A17

Keywords: algebraic polynomials; conformal mapping; Bernstein lemma

1 Introduction and main results

Let $G \subset \mathbb{C}$ be a finite region, with $0 \in G$, bounded by a Jordan curve $L := \partial G$, $\Delta := \{w : |w| > 1\}$, $\Omega := \operatorname{ext} \overline{G}$ (with respect to $\overline{\mathbb{C}}$). Let $w = \Phi(z)$ be the univalent conformal mapping of Ω onto the Δ normalized by $\Phi(\infty) = \infty$, $\Phi'(\infty) > 0$, and $\Psi := \Phi^{-1}$.

Let $\wp_n$ denote the class of arbitrary algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N}$.

Let $A_p(G)$, $p > 0$, denote the class of functions f which are analytic in G and satisfy the condition

$$\|f\|_{A_p(G)} := \left(\iint_G \left|f(z)\right|^p d\sigma_z \right)^{1/p} < \infty,$$

where σ denotes a two-dimensional Lebesgue measure.

When L is rectifiable, let $\mathcal{L}_p(L)$, $p > 0$, denote the class of functions f which are integrable on L and satisfy the condition

$$\|f\|_{\mathcal{L}_p(L)} := \left(\int_L \left|f(z)\right|^p |dz| \right)^{1/p} < \infty.$$

From the well-known Bernstein-Walsh lemma [1, p.101], we see that

$$\left|P_n(z)\right| \le \left|\Phi(z)\right|^n \|P_n\|_{C(\overline{G})}, \quad z \in \Omega. \tag{1.1}$$

For $R > 1$, let us set $L_R := \{z : |\Phi(z)| = R\}$, $G_R := \operatorname{int} L_R$, $\Omega_R := \operatorname{ext} L_R$. Then (1.1) can be written as follows:

$$\|P_n\|_{C(\overline{G}_R)} \le R^n \|P_n\|_{C(\overline{G})}. \tag{1.2}$$

Hence, setting $R = 1 + \frac{1}{n}$, according to (1.2), we see that the C-norm of a polynomial $P_n(z)$ in $\overline{G}_R$ and $\overline{G}$ is equivalent, *i.e.*, the norm $\|P_n\|_{C(\overline{G}_R)}$ increases with no more than a constant with respect to $\|P_n\|_{C(\overline{G})}$.

In the case when L is rectifiable, a similar estimate of (1.2) type in space $\mathcal{L}_p(L)$ was obtained in [2] as follows:

$$\|P_n\|_{\mathcal{L}_p(L_R)} \le R^{n+\frac{1}{p}} \|P_n\|_{\mathcal{L}_p(L)}, \quad p > 0. \tag{1.3}$$

The Berstein-Walsh type estimation for regions with quasiconformal boundary [3, p.97] in the space $A_p(G)$, $p > 0$, is contained in [4]:

$$\|P_n\|_{A_p(G_R)} \le c_2 R^{*^{n+\frac{1}{p}}} \|P_n\|_{A_p(G)}, \quad p > 0, \tag{1.4}$$

where $R^* := 1 + c_1(R-1)$ and $c_1 > 0$, $c_2 = c_2(c_1, p, G) > 0$ are constants. Therefore, if we choose $R = 1 + \frac{c_3}{n}$, then (1.4) we can see that the A_p-norm of polynomials $P_n(z)$ in G_R and G is equivalent.

In this work, we study a problem similar to (1.4) in $A_p(G)$, $p > 0$, for regions with arbitrary Jordan boundary.

Now we can state our new result.

Theorem 1.1 *Let $p > 0$; G be a Jordan region. Then, for any $P_n \in \wp_n$, $R_1 = 1 + \frac{1}{n}$ and arbitrary R, $R > R_1$, we have*

$$\|P_n\|_{A_p(G_R)} \le c_4 R^{n+\frac{2}{p}} \|P_n\|_{A_p(G_{R_1})}, \tag{1.5}$$

where $c_4 = (\frac{2}{e^p-1})^{\frac{1}{p}}[1 + O(\frac{1}{n})]$, $n \to \infty$.

The sharpness of (1.5) can be seen from the following remark:

Remark 1.1 For any $n = 1, 2, \dots$, there exist a polynomial $P_n^* \in \wp_n$, region $G^* \subset \mathbb{C}$ and number $R > R_1 = 1 + \frac{1}{n}$ such that

$$\|P_n^*\|_{A_p(G_R^*)} \ge \left(\frac{2}{e^p - 1}\right)^{\frac{1}{p}} R^{n+\frac{2}{p}} \|P_n^*\|_{A_p(G_{R_1}^*)}. \tag{1.6}$$

2 Some auxiliary results

Let $G \subset \mathbb{C}$ be a finite region bounded by the Jordan curve L. Let $L_R := \{z : |\Phi(z)| = R, R > 1\}$, $G_t := \operatorname{int} L_t$, $\Omega_t := \operatorname{ext} L_t$.

We note that, throughout this paper, $c_1, c_2, \dots$ (in general, different in different relations) are positive constants.

Lemma 2.1 *Let $p > 0$; f be an analytic function in $|z| > 1$ and have a pole of degree at most n, $n \ge 1$ at $z = \infty$. Then, for any R_1 and $R > R_1$, we have*

$$\|f\|_{A_p(R_1<|z|<R)} \le \left(\frac{R^{np+2} - R_1^{np+2}}{R_1^{np+2} - 1}\right)^{\frac{1}{p}} \|f\|_{A_p(1<|z|<R_1)}. \tag{2.1}$$

Proof The function $g(z) := \frac{f(z)}{z^n}$ is analytic in $|z| > 1$ and continuous in $|z| \ge 1$. Applying Hardy's convexity theorem [5, p.9: Th.1.5], for any arbitrary R_1 and R ($R > R_1$), and ρ, s

such that $R_1 \le \rho < R$, $1 < s \le R_1$, we can write

$$\int_{|z|=\rho} \left|\frac{f(z)}{z^{n+\frac{1}{p}}}\right|^p |dz| \le \int_{|z|=R_1} \left|\frac{f(z)}{z^{n+\frac{1}{p}}}\right|^p |dz|, \tag{2.2}$$

$$\int_{|z|=R_1} \left|\frac{f(z)}{z^{n+\frac{1}{p}}}\right|^p |dz| \le \int_{|z|=s} \left|\frac{f(z)}{z^{n+\frac{1}{p}}}\right|^p |dz|, \tag{2.3}$$

respectively. Thus,

$$\int_{|z|=\rho} |f(z)|^p |dz| \le \rho^{np+1} \int_{|z|=R_1} \left|\frac{f(z)}{z^{n+\frac{1}{p}}}\right|^p |dz|, \tag{2.4}$$

$$s^{np+1} \int_{|z|=R_1} \left|\frac{f(z)}{z^{n+\frac{1}{p}}}\right|^p |dz| \le \int_{|z|=s} |f(z)|^p |dz|. \tag{2.5}$$

Integrating (2.4) over ρ from R_1 to R, and (2.5) over s from 1 to R_1, we get

$$\int_{R_1}^{R} \int_{|z|=\rho} |f(z)|^p |dz|\, d\rho \le \frac{1}{np+2}\left(R^{np+2} - R_1^{np+2}\right) \int_{|z|=R_1} \left|\frac{f(z)}{z^{n+\frac{1}{p}}}\right|^p |dz|,$$

$$\frac{1}{np+2}\left(R_1^{np+2} - 1\right) \int_{|z|=R_1} \left|\frac{f(z)}{z^{n+\frac{1}{p}}}\right|^p |dz| \le \int_1^{R_1} \int_{|z|=s} |f(z)|^p |dz|\, ds.$$

After calculation we have

$$\iint_{R_1<|z|<R} |f(z)|^p \, d\sigma_z \le \frac{R^{np+2} - R_1^{np+2}}{R_1^{np+2} - 1} \iint_{1<|z|<R_1} |f(z)|^p \, d\sigma_z, \tag{2.6}$$

and we see that (2.1) is true. □

Corollary 2.2 *Under the assumptions of Lemma* 2.1 *for* $R_1 = 1 + \frac{1}{n}$, *we have*

$$\|f\|_{A_p(R_1<|z|<R)} \le c_1 R^{n+\frac{2}{p}} \|f\|_{A_p(1<|z|<R_1)}, \tag{2.7}$$

where $c_1 := c_1(p,n) = (\frac{1}{e^p-1})^{\frac{1}{p}}[1 + O(\frac{1}{n})]$, $n \to \infty$.

Proof Let us put

$$S^p := S^p(R, R_1, n, p) := \frac{R^{np+2} - R_1^{np+2}}{R_1^{np+2} - 1} = R^{np+2} \cdot \frac{1 - (\frac{R_1}{R})^{np+2}}{R_1^{np+2} - 1},$$

and taking $R_1 = 1 + \frac{1}{n}$, we have

$$S^p = R^{np+2} \cdot \frac{1 - (\frac{R_1}{R})^{np+2}}{(1+\frac{1}{n})^{np+2} - 1} \le \frac{R^{np+2}}{(1+\frac{1}{n})^{np+2} - 1}. \tag{2.8}$$

According to the right-hand side of the well-known estimation (see, for example, [6, p.52 (Problem 170)])

$$\frac{e}{2n+2} < e - \left(1 + \frac{1}{n}\right)^n < \frac{e}{2n+1}, \quad n = 1, 2, \ldots, \tag{2.9}$$

we have

$$\left(1+\frac{1}{n}\right)^{np+2} \geq \left(1+\frac{1}{n}\right)^{np} \geq \left(e-\frac{e}{2n+1}\right)^{p} = e^{p}\cdot\left(1-\frac{1}{2n+1}\right)^{p} \geq (\varepsilon_n\cdot e)^{p},$$

where

$$\frac{2}{3} \leq \varepsilon_n := 1-\frac{1}{2n+1} \to 1, \quad n\to\infty. \tag{2.10}$$

Therefore

$$S^p \leq \frac{1}{(\varepsilon_n e)^p - 1}R^{np+2} = R^{np+2}\frac{1}{e^p-1}\left[1+O\left(\frac{1}{n}\right)\right], \quad n\to\infty. \tag{2.11}$$

From (2.8) and (2.11) we complete the proof. □

Remark 2.1 For the polynomial $Q_n(z) = z^n$, $R_1 = 1+\frac{1}{n}$ and any $R > R_1$,

$$\|Q_n\|_{A_p(R_1<|z|<R)} \geq c_2 R^{n+\frac{2}{p}}\|Q_n\|_{A_p(1<|z|<R_1)}, \tag{2.12}$$

where $c_2 := c_2(p,n) := (\frac{1}{e^p-1})^{\frac{1}{p}}[1-O(\frac{1}{n})]$, $n\to\infty$.

Proof Really, from (2.6) we get

$$S^p = R^{np+2}\cdot\frac{1-(\frac{R_1}{R})^{np+2}}{(1+\frac{1}{n})^{np+2}-1} = R^{np+2}\cdot\frac{1-\delta_n}{(1+\frac{1}{n})^{np+2}-1}, \tag{2.13}$$

where

$$\delta_n := \left(\frac{R_1}{R}\right)^{np+2} \to 0, \quad n\to\infty. \tag{2.14}$$

According to the left-hand side of (2.9), we obtain

$$\begin{aligned}\left(1+\frac{1}{n}\right)^{np+2} &= \left(1+\frac{1}{n}\right)^{np}\left(1+\frac{1}{n}\right)^{2} \leq \left(e-\frac{e}{2n+2}\right)^{p}\eta_n\\ &= e^p\cdot\left(1-\frac{1}{2n+2}\right)^{p}\eta_n \leq e^p\cdot\eta_n,\end{aligned}$$

where

$$\eta_n := \left(1+\frac{1}{n}\right)^{2} \to 1, \quad n\to\infty.$$

Therefore,

$$\begin{aligned}S^p &\geq R^{np+2}\cdot\frac{1-\delta_n}{\eta_n e^p-1}\\ &= R^{np+2}\cdot\left[\frac{1}{\eta_n e^p-1}-\frac{\delta_n}{\eta_n e^p-1}\right]\end{aligned}$$

$$\begin{aligned}&= R^{np+2} \cdot \left\{ \frac{1}{e^p - 1}\left[1 - O\left(\frac{1}{n}\right)\right] - O(\delta_n)\right\}\\&= R^{np+2} \cdot \frac{1}{e^p - 1}\left[1 - O\left(\frac{1}{n}\right)\right], \quad n \to \infty.\end{aligned}$$ □

Corollary 2.3 *For $f \equiv P_n$, we have*

$$\|P_n\|_{A_p(|z|<R)} \le c_3 R^{n+\frac{2}{p}} \|P_n\|_{A_p(|z|<R_1)},$$

where $c_3 := c_3(p,n) := (\frac{2}{e^p-1})^{\frac{1}{p}}[1 + O(\frac{1}{n})]$, $n \to \infty$.

Proof Really, (2.1) implies, for any $f \equiv P_n$,

$$\|P_n\|^p_{A_p(R_1<|z|<R)} \le S^p \cdot \|P_n\|^p_{A_p(1<|z|<R_1)}.$$

Adding $\|P_n\|^p_{A_p(|z|<R_1)}$ to the both sides, we obtain

$$\begin{aligned}\|P_n\|^p_{A_p(|z|<R)} &\le S^p \cdot \|P_n\|^p_{A_p(1<|z|<R_1)} + \|P_n\|^p_{A_p(|z|<R_1)}\\&\le 2\max\{S^p, 1\} \cdot \|P_n\|^p_{A_p(|z|<R_1)}.\end{aligned}$$

Passing to the limit as $R_1 = 1 + \frac{1}{n} \to 1$, from (2.11) we obtain

$$\|P_n\|^p_{A_p(|z|<R)} \le \frac{2}{e^p - 1}\left[1 + O\left(\frac{1}{n}\right)\right] \cdot R^{np+2}\|P_n\|^p_{A_p(|z|<R_1)}.$$ □

3 Proof of the theorem

Proof First of all, let us convince ourselves that for the proof of (1.5) it is sufficient to show the fulfilment of estimation

$$\|P_n\|_{A_p(G_R \setminus G_{R_1})} \le cR^{n+\frac{2}{p}} \|P_n\|_{A_p(G_{R_1}\setminus G)} \tag{3.1}$$

for some constant $c = c(p, R_1) > 0$ independent of R and n. Really, let (3.1) be true. Then

$$\|P_n\|^p_{A_p(G_R \setminus G_{R_1})} \le c^p R^{np+2} \|P_n\|^p_{A_p(G_{R_1}\setminus G)}. \tag{3.2}$$

Now, we will add to both sides $\|P_n\|^p_{A_p(G_{R_1})}$:

$$\begin{aligned}\|P_n\|^p_{A_p(G_R)} &\le c^p R^{np+2} \|P_n\|^p_{A_p(G_{R_1}\setminus G)} + \|P_n\|^p_{A_p(G_{R_1})}\\&\le c^p R^{np+2} \|P_n\|^p_{A_p(G_{R_1}\setminus G)} + c^p R^{np+2} \|P_n\|^p_{A_p(G_{R_1})}\\&= 2c^p R^{np+2} \|P_n\|^p_{A_p(G_{R_1})}.\end{aligned} \tag{3.3}$$

Therefore,

$$\|P_n\|_{A_p(G_R)} \le 2^{\frac{1}{p}} cR^{n+\frac{2}{p}} \|P_n\|_{A_p(G_{R_1})}.$$

Now, let us make a proof of (3.1).

For the $p > 0$, let us set

$$f_n(w) := P_n\big(\Psi(w)\big)\big[\Psi'(w)\big]^{\frac{2}{p}}, \quad w = \Phi(z).$$

The function f_n is analytic in Δ and has a pole of degree at most n at $w = \infty$. Then, according to Lemma 2.1, we have

$$\|f_n\|^p_{A_p(R_1<|w|<R)} \le S(R, R_1, n, p)\|f_n\|^p_{A_p(1<|w|<R_1)},$$

where

$$S^p := \frac{R^{np+2} - R_1^{np+2}}{R_1^{np+2} - 1} = R^{np+2} \cdot \frac{1 - (\frac{R_1}{R})^{np+2}}{R_1^{np+2} - 1}.$$

Then

$$\begin{aligned}\iint_{G_R \setminus G_{R_1}} \big|P_n(z)\big|^p \, d\sigma_z &= \iint_{R_1<|w|<R} \big|f_n(w)\big|^p \, d\sigma_w \\ &\le S^p \iint_{1<|w|<R_1} \big|f_n(w)\big|^p \, d\sigma_w \\ &\le R^{np+2} \cdot \frac{1}{R_1^{np+2} - 1} \iint_{G_{R_1} \setminus G} \big|P_n(z)\big|^p \, d\sigma_z.\end{aligned}$$

Therefore,

$$\iint_{G_R} \big|P_n(z)\big|^p \, d\sigma_z \le 2R^{np+2} \cdot \frac{1}{R_1^{np+2} - 1} \iint_{G_{R_1}} \big|P_n(z)\big|^p \, d\sigma_z. \tag{3.4}$$

Taking $R_1 = 1 + \frac{1}{n}$, from (2.9) and (2.11) we get

$$\frac{1}{R_1^{np+2} - 1} = \frac{1}{e^p - 1}\left[1 + O\left(\frac{1}{n}\right)\right], \quad n \to \infty. \tag{3.5}$$

Now, from (3.4) and (3.5) we complete the proof. □

3.1 Proof of the remark

Proof Let $P_n^* = z^n$, $G^* = B := \{z : |z| < 1\}$ and $R \le \frac{8e^p}{e^p - 1}$. Then

$$\begin{aligned}\big\|P_n^*\big\|^p_{A_p(G_R^*)} &= \iint_{|z|<R} \big|z^n\big|^p \, d\sigma_z \\ &= R^{np+2} \cdot R_1^{-(np+2)} \big\|P_n^*\big\|^p_{A_p(G_{R_1}^*)} \\ &= \frac{R}{R_1^2 \cdot R_1^{np}} \cdot R^{np+2} \big\|P_n^*\big\|^p_{A_p(G_{R_1}^*)}.\end{aligned} \tag{3.6}$$

For $R_1 = 1 + \frac{1}{n}$, from (2.9) we obtain

$$\left(1+\frac{1}{n}\right)^{np} \leq \left(e - \frac{e}{2n+2}\right)^{p} \leq e^p,$$
$$\left(1+\frac{1}{n}\right)^{2} \leq 4.$$

Then

$$\frac{R}{R_1^{np+2}} \geq \frac{R}{4e^p}$$

and

$$\|P_n^*\|^p_{A_p(G_R^*)} \geq \frac{R}{4e^p} \cdot R^{np+2} \|P_n^*\|^p_{A_p(G_{R_1}^*)}.$$

In particular, for $R = \frac{8e^p}{e^p - 1}$ we have

$$\|P_n^*\|^p_{A_p(G_R^*)} \geq \frac{2}{e^p - 1} \cdot R^{np+2} \|P_n^*\|^p_{A_p(G_{R_1}^*)}. \qquad \square$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Received: 17 May 2013 Accepted: 7 November 2013 Published: 02 Dec 2013

References
1. Walsh, JL: Interpolation and Approximation by Rational Functions in the Complex Domain. Am. Math. Soc., Providence (1960)
2. Hille, E, Szegö, G, Tamarkin, JD: On some generalization of a theorem of A Markoff. Duke Math. J. **3**, 729-739 (1937)
3. Lehto, O, Virtanen, KI: Quasiconformal Mapping in the Plane. Springer, Berlin (1973)
4. Abdullayev, FG: On the some properties of the orthogonal polynomials over the region of the complex plane (Part III). Ukr. Math. J. **53**(12), 1934-1948 (2001)
5. Duren, PL: Theory of H^p Spaces. Academic Press, San Diego (1970)
6. Polya, G, Szegö, G: Problems and Theorems in Analysis I. Nauka, Moscow (1978) (Russian edition)

10.1186/1029-242X-2013-570
Cite this article as: Abdullayev and Özkartepe: An analogue of the Bernstein-Walsh lemma in Jordan regions of the complex plane. *Journal of Inequalities and Applications* 2013, 2013:570