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Stability of functional inequalities in matrix random normed spaces

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Abstract

In this paper, we prove the Hyers-Ulam stability of the Cauchy additive functional inequality and the Cauchy-Jensen additive functional inequality in matrix random normed spaces by using the fixed point method.

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1 Introduction

The abstract characterization given for linear spaces of bounded Hilbert space operators in terms of *matricially normed spaces* [1] implies that quotients, mapping spaces and various tensor products of operator spaces may again be regarded as operator spaces. The proof given in [1] appealed to the theory of ordered operator spaces [2]. Effros and Ruan [3] showed that one can give a purely metric proof of this important theorem by using a technique of Pisier [4] and Haagerup [5]. The theory of operator spaces has an increasingly significant effect on operator algebra theory (see [6, 7]).

The stability problem of functional equations originated from a question of Ulam [8] concerning the stability of group homomorphisms. The functional equation

$$f(x+y)=f(x)+f(y)$$

is called the *Cauchy additive functional equation*. In particular, every solution of the Cauchy additive functional equation is said to be an *additive mapping*. Hyers [9] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [10] for additive mappings and by Rassias [11] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [12] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

In [13], Gilányi showed that if f satisfies the functional inequality

$$\left\|2f(x)+2f(y)-f\left(xy^{-1}\right)\right\| \le \left\|f(xy)\right\|,$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x)+2f(y)=f(xy)+f\left(xy^{-1}\right).$$

See also [14]. Gilányi [15] and Fechner [16] proved the Hyers-Ulam stability of the above functional inequality.

Park *et al.* [17] proved the Hyers-Ulam stability of the following functional inequalities:

$$\|f(x)+f(y)+f(z)\| \le \|f(x+y+z)\|, \tag{1.1}$$

$$\|f(x)+f(y)+2f(z)\| \le \left\|2f\left(\frac{x+y}{2}+z\right)\right\|. \tag{1.2}$$

In the sequel, we adopt the usual terminology, notations and conventions of the theory of random normed spaces, as in [18–21]. Throughout this paper, Δ^+ is the space of distribution functions, that is, the space of all mappings $F:\mathbb{R}\cup\{-\infty,\infty\}\to[0,1]$ such that F is left-continuous and non-decreasing on $\mathbb{R}$, $F(0)=0$ and $F(+\infty)=1$. D^+ is a subset of Δ^+ consisting of all functions $F\in\Delta^+$ for which $l^-F(+\infty)=1$, where $l^-f(x)$ denotes the left limit of the function f at the point x, that is, $l^-f(x)=\lim_{t\to x^-}f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, *i.e.*, $F\le G$ if and only if $F(t)\le G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t)=\begin{cases}0 & \text{if } t\le 0,\\ 1 & \text{if } t>0.\end{cases}$$

Definition 1.1 ([20]) A mapping $T:[0,1]\times[0,1]\to[0,1]$ is a continuous triangular norm (briefly, a continuous t-norm) if T satisfies the following conditions:

(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a,1)=a$ for all $a\in[0,1]$;
(d) $T(a,b)\le T(c,d)$ whenever $a\le c$ and $b\le d$ for all $a,b,c,d\in[0,1]$.

Definition 1.2 ([21]) A *random normed space* (briefly, RN-space) is a triple (X,μ,T), where X is a vector space, T is a continuous t-norm and μ is a mapping from X into D^+ such that the following conditions hold:

(RN_1) $\mu_x(t)=\varepsilon_0(t)$ for all $t>0$ if and only if $x=0$;
(RN_2) $\mu_{\alpha x}(t)=\mu_x(\frac{t}{|\alpha|})$ for all $x\in X$, $\alpha\ne 0$;
(RN_3) $\mu_{x+y}(t+s)\ge T(\mu_x(t),\mu_y(s))$ for all $x,y\in X$ and all $t,s\ge 0$.

Every normed space $(X,\|\cdot\|)$ defines a random normed space (X,μ,T_M), where

$$\mu_x(t)=\frac{t}{t+\|x\|}$$

for all $t>0$, and T_M is the minimum t-norm. This space is called the induced random normed space.

Definition 1.3 Let (X,μ,T) be an RN-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $\epsilon>0$ and $\lambda>0$, there exists a positive integer N such that $\mu_{x_n-x}(\epsilon)>1-\lambda$ whenever $n\ge N$.

(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(\epsilon) > 1 - \lambda$ whenever $n \geq m \geq N$.
(3) An RN-space (X, μ, T) is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X.

Theorem 1.4 ([20]) *If (X, μ, T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

We introduce the concept of matrix random normed space.

Definition 1.5 Let (X, μ, T) be a random normed space. Then
(1) $(X, \{\mu^{(n)}\}, T)$ is called a *matrix random normed space* if for each positive integer n, $(M_n(X), \mu^{(n)}, T)$ is a random normed space and $\mu^{(k)}_{AxB}(t) \geq \mu^{(n)}_x(\frac{t}{\|A\| \cdot \|B\|})$ for all $t > 0$, $A \in M_{k,n}(\mathbb{R})$, $x = [x_{ij}] \in M_n(X)$ and $B \in M_{n,k}(\mathbb{R})$ with $\|A\| \cdot \|B\| \neq 0$.
(2) $(X, \{\mu^{(n)}\}, T)$ is called a *matrix random Banach space* if (X, μ, T) is a random Banach space and $(X, \{\mu^{(n)}\}, T)$ is a matrix random normed space.

Let E, F be vector spaces. For a given mapping $h : E \to F$ and a given positive integer n, define $h_n : M_n(E) \to M_n(F)$ by

$$h_n([x_{ij}]) = [h(x_{ij})]$$

for all $[x_{ij}] \in M_n(E)$.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies
(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.6 ([22, 23]) *Let (X, d) be a complete generalized metric space, and let $J : X \to X$ be a strictly contractive mapping with a Lipschitz constant $\alpha < 1$. Then, for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that
(1) $d(J^n x, J^{n+1} x) < \infty$, $\forall n \geq n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ for all $y \in Y$.*

The stability problem in a random normed space was considered by Mihet and Radu [24]; next some authors proved some stability results in random normed spaces by different methods (see [25–27]).

In 1996, Isac and Rassias [28] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [29–34]).

Throughout this paper, let X be a normed space and $(Y, \{\mu^{(n)}\}, T)$ be a matrix random Banach space. In Section 2, we prove the Hyers-Ulam stability of the Cauchy additive functional inequality (1.1) in matrix normed spaces by using the direct method. In Section 3, we prove the Hyers-Ulam stability of the Cauchy-Jensen additive functional inequality (1.2) in matrix normed spaces by using the fixed point method.

2 Hyers-Ulam stability of the Cauchy additive functional inequality

In this section, we prove the Hyers-Ulam stability of the Cauchy additive functional inequality (1.1) in matrix random normed spaces by using the fixed point method.

Theorem 2.1 *Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists $\alpha < 1$ with*

$$\varphi(a,b,c) \le \frac{\alpha}{2}\varphi(2a,2b,2c)$$

for all $a,b,c \in X$. Let $f : X \to Y$ be an odd mapping satisfying

$$\mu^{(n)}_{f_n([x_{ij}])+f_n([y_{ij}])+f_n([z_{ij}])}(t) \ge \min\left\{\mu^{(n)}_{f_n([x_{ij}+y_{ij}+z_{ij}])}\left(\frac{t}{2}\right), \frac{t}{t+\sum_{i,j=1}^{n}\varphi(x_{ij},y_{ij},z_{ij})}\right\} \tag{2.1}$$

for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}], z = [z_{ij}] \in M_n(X)$. Then $A(a) := \lim_{l\to\infty} 2^l f(\frac{a}{2^l})$ exists for each $a \in X$ and defines an additive mapping $A : X \to Y$ such that

$$\mu_{f_n([x_{ij}])-A_n([x_{ij}])}(t) \ge \frac{2(1-\alpha)t}{2(1-\alpha)t + n^2\alpha\sum_{i,j=1}^{n}\varphi(x_{ij},x_{ij},-2x_{ij})} \tag{2.2}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof Let $n = 1$. Then (2.1) is equivalent to

$$\mu_{f(a)+f(b)+f(c)}(t) \ge \min\left\{\mu_{f(a+b+c)}\left(\frac{t}{2}\right), \frac{t}{t+\varphi(a,b,c)}\right\} \tag{2.3}$$

for all $t > 0$ and $a,b,c \in X$.

Letting $b = a$ and $c = -2a$ in (2.3), we get

$$\mu_{f(2a)-2f(a)}(t) \ge \frac{t}{t+\varphi(a,a,-2a)}, \tag{2.4}$$

and so

$$\mu_{f(a)-2f(\frac{a}{2})}(t) \ge \frac{t}{t+\varphi(\frac{a}{2},\frac{a}{2},-a)} \ge \frac{t}{t+\frac{\alpha}{2}\varphi(a,a,-2a)} \tag{2.5}$$

for all $t > 0$ and $a \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\nu \in \mathbb{R}_+ : \mu_{g(a)-h(a)}(\nu t) \geq \frac{t}{t+\varphi(a,a,-2a)}, \forall a \in X, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see the proof of [24, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(a) := 2g\left(\frac{a}{2}\right)$$

for all $a \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\mu_{g(a)-h(a)}(\varepsilon t) \geq \frac{t}{t+\varphi(a,a)}$$

for all $a \in X$ and $t > 0$. Hence

$$\begin{aligned}
\mu_{Jg(a)-Jh(a)}(\alpha\varepsilon t) &= \mu_{2g(\frac{a}{2})-2h(\frac{a}{2})}(\alpha\varepsilon t) = \mu_{g(\frac{a}{2})-h(\frac{a}{2})}\left(\frac{\alpha}{2}\varepsilon t\right) \\
&\geq \frac{\frac{\alpha t}{2}}{\frac{\alpha t}{2} + \varphi(\frac{a}{2}, \frac{a}{2}, -a)} \geq \frac{\frac{\alpha t}{2}}{\frac{\alpha t}{2} + \frac{\alpha}{2}\varphi(a,a,-2a)} \\
&= \frac{t}{t+\varphi(a,a,-2a)}
\end{aligned}$$

for all $a \in X$ and $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg,Jh) \leq \alpha\varepsilon$. This means that

$$d(Jg,Jh) \leq \alpha d(g,h)$$

for all $g, h \in S$.

It follows from (2.5) that $d(f,Jf) \leq \frac{\alpha}{2}$.

By Theorem 1.6, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, *i.e.*,

$$A\left(\frac{a}{2}\right) = \frac{1}{2}A(a)$$

for all $a \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

(2) $d(J^l f, A) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 2^l f\left(\frac{a}{2^l}\right) = A(a)$$

for all $a \in X$.

(3) $d(f,A) \le \frac{1}{1-\alpha} d(f,Jf)$, which implies the inequality

$$d(f,A) \le \frac{\alpha}{2-2\alpha}. \tag{2.6}$$

By (2.3),

$$\mu_{2^l f(\frac{a+b}{2^l}) - 2^l f(\frac{a}{2^l}) - 2^l f(\frac{b}{2^l})}(2^l t) \ge \frac{t}{t + \varphi(\frac{a}{2^l}, \frac{b}{2^l}, -\frac{a+b}{2^l})}$$

for all $a, b \in X$ and $t > 0$. So

$$\mu_{2^l f(\frac{a+b}{2^l}) - 2^l f(\frac{a}{2^l}) - 2^l f(\frac{b}{2^l})}(t) \ge \frac{\frac{t}{2^l}}{\frac{t}{2^l} + \frac{\alpha^l}{2^l}\varphi(a, b, -a-b)}$$

for all $a, b \in X$ and $t > 0$. Since $\lim_{l\to\infty} \frac{\frac{t}{2^l}}{\frac{t}{2^l} + \frac{\alpha^l}{2^l}\varphi(a,b,-a-b)} = 1$ for all $a, b \in X$ and $t > 0$,

$$\mu_{A(a+b)-A(a)-A(b)}(t) = 1$$

for all $a, b \in X$ and $t > 0$. Thus $A(a+b) - A(a) - A(b) = 0$. So the mapping $A : X \to Y$ is additive.

We note that $e_j \in M_{1,n}(\mathbb{R})$ is that jth component is 1 and the others are zero, $E_{ij} \in M_n(\mathbb{R})$ is that (i,j)-component is 1 and the others are zero, and $E_{ij} \otimes x \in M_n(X)$ is that (i,j)-component is x and the others are zero. Since $\mu^{(n)}_{E_{kl}\otimes x}(t) = \mu_x(t)$, we have

$$\begin{aligned}\mu^{(n)}_{[x_{ij}]}(t) &= \mu^{(n)}_{\sum_{i,j=1}^n E_{ij}\otimes x_{ij}}(t) \ge \min\left\{\mu^{(n)}_{E_{ij}\otimes x_{ij}}(t_{ij}) : i,j = 1,2,\dots,n\right\} \\ &= \min\left\{\mu_{x_{ij}}(t_{ij}) : i,j = 1,2,\dots,n\right\},\end{aligned}$$

where $t = \sum_{i,j=1}^n t_{ij}$. So $\mu^{(n)}_{[x_{ij}]}(t) \ge \min\{\mu_{x_{ij}}(\frac{t}{n^2}) : i,j = 1,2,\dots,n\}$.

By (2.6),

$$\begin{aligned}\mu^{(n)}_{f_n([x_{ij}])-A_n([x_{ij}])}(t) &\ge \min\left\{\mu_{f(x_{ij})-A(x_{ij})}\left(\frac{t}{n^2}\right) : i,j = 1,2,\dots,n\right\} \\ &\ge \min\left\{\frac{2(1-\alpha)t}{2(1-\alpha)t + n^2\alpha\varphi(x_{ij}, x_{ij}, -2x_{ij})} : i,j = 1,2,\dots,n\right\} \\ &\ge \frac{2(1-\alpha)t}{2(1-\alpha)t + n^2\alpha\sum_{i,j=1}^n \varphi(x_{ij}, x_{ij}, -2x_{ij})}\end{aligned}$$

for all $x = [x_{ij}] \in M_n(X)$. Thus $A : X \to Y$ is a unique additive mapping satisfying (2.2), as desired. □

Corollary 2.2 *Let r, θ be positive real numbers with $r > 1$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\begin{aligned}&\mu^{(n)}_{f_n([x_{ij}])+f_n([y_{ij}])+f_n([z_{ij}])}(t) \\ &\quad\ge \min\left\{\mu^{(n)}_{f_n([x_{ij}+y_{ij}+z_{ij}])}\left(\frac{t}{2}\right), \frac{t}{t + \sum_{i,j=1}^n \theta(\|x_{ij}\|^r + \|y_{ij}\|^r + \|z_{ij}\|^r)}\right\}\end{aligned} \tag{2.7}$$

for all $t > 0$ *and* $x = [x_{ij}], y = [y_{ij}], z = [z_{ij}] \in M_n(X)$. *Then* $A(a) := \lim_{l\to\infty} 2^l f(\frac{a}{2^l})$ *exists for each* $a \in X$ *and defines an additive mapping* $A : X \to Y$ *such that*

$$\mu_{f_n([x_{ij}])-A_n([x_{ij}])}(t) \geq \frac{(2^r - 2)t}{(2^r - 2)t + n^2(2 + 2^r)\sum_{i,j=1}^{n} \theta\|x_{ij}\|^r}$$

for all $t > 0$ *and* $x = [x_{ij}] \in M_n(X)$.

Proof The proof follows from Theorem 2.1 by taking $\varphi(a, b, c) = \theta(\|a\|^r + \|b\|^r + \|c\|^r)$ for all $a, b, c \in X$. Then we can choose $\alpha = 2^{1-r}$ and we get the desired result. □

Theorem 2.3 *Let* $f : X \to Y$ *be an odd mapping satisfying* (2.1) *for which there exists a function* $\varphi : X^3 \to [0, \infty)$ *such that there exists* $\alpha < 1$ *with*

$$\varphi(a, b, c) \leq 2\alpha\varphi\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$$

for all $a, b, c \in X$. *Then* $A(a) := \lim_{l\to\infty} \frac{1}{2^l} f(2^l a)$ *exists for each* $a \in X$ *and defines an additive mapping* $A : X \to Y$ *such that*

$$\mu_{f_n([x_{ij}])-A_n([x_{ij}])}(t) \geq \frac{2(1 - \alpha)t}{2(1 - \alpha)t + n^2 \sum_{i,j=1}^{n} \varphi(x_{ij}, x_{ij}, -2x_{ij})}$$

for all $t > 0$ *and* $x = [x_{ij}] \in M_n(X)$.

Proof Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(a) := 2g\left(\frac{a}{2}\right)$$

for all $a \in X$.

It follows from (2.4) that $d(f, Jf) \leq \frac{1}{2}$. So

$$d(f, A) \leq \frac{1}{2 - 2\alpha}.$$

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.4 *Let* r, θ *be positive real numbers with* $r < 1$. *Let* $f : X \to Y$ *be a mapping satisfying* (2.7). *Then* $A(a) := \lim_{l\to\infty} 2^l f(\frac{a}{2^l})$ *exists for each* $a \in X$ *and defines an additive mapping* $A : X \to Y$ *such that*

$$\mu_{f_n([x_{ij}])-A_n([x_{ij}])}(t) \geq \frac{(2 - 2^r)t}{(2 - 2^r)t + n^2(2 + 2^r)\sum_{i,j=1}^{n} \theta\|x_{ij}\|^r}$$

for all $t > 0$ *and* $x = [x_{ij}] \in M_n(X)$.

Proof The proof follows from Theorem 2.3 by taking $\varphi(a, b, c) = \theta(\|a\|^r + \|b\|^r + \|c\|^r)$ for all $a, b, c \in X$. Then we can choose $\alpha = 2^{r-1}$ and we get the desired result. □

3 Hyers-Ulam stability of the Cauchy-Jensen additive functional inequality

In this section, we prove the Hyers-Ulam stability of Cauchy-Jensen additive functional inequality (1.2) in matrix random normed spaces by using the fixed point method.

Theorem 3.1 *Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists $\alpha < 1$ with*

$$\varphi(a,b,c) \le \frac{\alpha}{2}\varphi(2a,2b,2c)$$

for all $a,b,c \in X$. Let $f : X \to Y$ be an odd mapping satisfying

$$\mu^{(n)}_{f_n([x_{ij}])+f_n([y_{ij}])+f_n([2z_{ij}])}(t) \ge \min\left\{\mu^{(n)}_{2f_n([\frac{x_{ij}+y_{ij}}{2}+z_{ij}])}\left(\frac{2t}{3}\right), \frac{t}{t+\sum_{i,j=1}^{n}\varphi(x_{ij},y_{ij},z_{ij})}\right\} \tag{3.1}$$

for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}], z = [z_{ij}] \in M_n(X)$. Then $A(a) := \lim_{l\to\infty} 2^l f(\frac{a}{2^l})$ exists for each $a \in X$ and defines an additive mapping $A : X \to Y$ such that

$$\mu_{f_n([x_{ij}])-A_n([x_{ij}])}(t) \ge \frac{2(1-\alpha)t}{2(1-\alpha)t+n^2\alpha\sum_{i,j=1}^{n}\varphi(x_{ij},x_{ij},-x_{ij})} \tag{3.2}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof Let $n = 1$. Then (3.1) is equivalent to

$$\mu_{f(a)+f(b)+f(2c)}(t) \ge \min\left\{\mu_{2f(\frac{a+b}{2}+c)}\left(\frac{2t}{3}\right), \frac{t}{t+\varphi(a,b,c)}\right\} \tag{3.3}$$

for all $t > 0$ and $a,b,c \in X$.

Letting $b = a$ and $c = -a$ in (3.3), we get

$$\mu_{f(2a)-2f(a)}(t) \ge \frac{t}{t+\varphi(a,a,-a)}, \tag{3.4}$$

and so

$$\mu_{f(a)-2f(\frac{a}{2})}(t) \ge \frac{t}{t+\varphi(\frac{a}{2},\frac{a}{2},-\frac{a}{2})} \ge \frac{t}{t+\frac{\alpha}{2}\varphi(a,a,-a)} \tag{3.5}$$

for all $t > 0$ and $a \in X$.

Let (S,d) be the generalized metric space defined in the proof of Theorem 2.1.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(a) := 2g\left(\frac{a}{2}\right)$$

for all $a \in X$.

Let $g,h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\mu_{g(a)-h(a)}(\varepsilon t) \ge \frac{t}{t+\varphi(a,a,-a)}$$

for all $a \in X$ and $t > 0$. Hence

$$\begin{aligned}\mu_{Jg(a)-Jh(a)}(\alpha\varepsilon t) &= \mu_{2g(\frac{a}{2})-2h(\frac{a}{2})}(\alpha\varepsilon t) = \mu_{g(\frac{a}{2})-h(\frac{a}{2})}\left(\frac{\alpha}{2}\varepsilon t\right) \\ &\geq \frac{\frac{\alpha t}{2}}{\frac{\alpha t}{2} + \varphi(\frac{a}{2}, \frac{a}{2}, -\frac{a}{2})} \geq \frac{\frac{\alpha t}{2}}{\frac{\alpha t}{2} + \frac{\alpha}{2}\varphi(a, a, -a)} \\ &= \frac{t}{t + \varphi(a, a, -a)}\end{aligned}$$

for all $a \in X$ and $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq \alpha\varepsilon$. This means that

$$d(Jg, Jh) \leq \alpha d(g, h)$$

for all $g, h \in S$.

It follows from (3.5) that $d(f, Jf) \leq \frac{\alpha}{2}$.

By Theorem 1.6, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, *i.e.*,

$$A\left(\frac{a}{2}\right) = \frac{1}{2}A(a)$$

for all $a \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

(2) $d(J^l f, A) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 2^l f\left(\frac{a}{2^l}\right) = A(a)$$

for all $a \in X$.

(3) $d(f, A) \leq \frac{1}{1-\alpha} d(f, Jf)$, which implies the inequality

$$d(f, A) \leq \frac{\alpha}{2 - 2\alpha}. \tag{3.6}$$

By (3.3),

$$\mu_{2^l f(\frac{a+b}{2^l}) - 2^l f(\frac{a}{2^l}) - 2^l f(\frac{b}{2^l})}(2^l t) \geq \frac{t}{t + \varphi(\frac{a}{2^l}, \frac{b}{2^l}, -\frac{a+b}{2^{l+1}})}$$

for all $a, b \in X$ and $t > 0$. So

$$\mu_{2^l f(\frac{a+b}{2^l}) - 2^l f(\frac{a}{2^l}) - 2^l f(\frac{b}{2^l})}(t) \geq \frac{\frac{t}{2^l}}{\frac{t}{2^l} + \frac{\alpha^l}{2^l}\varphi(a, b, -\frac{a+b}{2})}$$

for all $a, b \in X$ and $t > 0$. Since $\lim_{l\to\infty} \frac{\frac{t}{2^l}}{\frac{t}{2^l} + \frac{\alpha^l}{2^l}\varphi(a,b,-a-b)} = 1$ for all $a, b \in X$ and $t > 0$,

$$\mu_{A(a+b)-A(a)-A(b)}(t) = 1$$

for all $a, b \in X$ and $t > 0$. Thus $A(a+b) - A(a) - A(b) = 0$. So the mapping $A : X \to Y$ is additive.

By (2.6),

$$\begin{aligned}\mu^{(n)}_{f_n([x_{ij}])-A_n([x_{ij}])}(t) &\geq \min\left\{\mu_{f(x_{ij})-A(x_{ij})}\left(\frac{t}{n^2}\right) : i,j = 1,2,\ldots,n\right\}\\ &\geq \min\left\{\frac{2(1-\alpha)t}{2(1-\alpha)t + n^2\alpha\varphi(x_{ij}, x_{ij}, -x_{ij})} : i,j = 1,2,\ldots,n\right\}\\ &\geq \frac{2(1-\alpha)t}{2(1-\alpha)t + n^2\alpha\sum_{i,j=1}^{n}\varphi(x_{ij}, x_{ij}, -x_{ij})}\end{aligned}$$

for all $x = [x_{ij}] \in M_n(X)$. Thus $A : X \to Y$ is a unique additive mapping satisfying (3.2), as desired. □

Corollary 3.2 *Let r, θ be positive real numbers with $r > 1$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\begin{aligned}&\mu^{(n)}_{f_n([x_{ij}])+f_n([y_{ij}])+f_n([2z_{ij}])}(t)\\ &\quad\geq \min\left\{\mu^{(n)}_{f_n([\frac{x_{ij}+y_{ij}}{2}+z_{ij}])}\left(\frac{2t}{3}\right), \frac{t}{t + \sum_{i,j=1}^{n}\theta(\|x_{ij}\|^r + \|y_{ij}\|^r + \|z_{ij}\|^r)}\right\}\end{aligned} \tag{3.7}$$

for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}], z = [z_{ij}] \in M_n(X)$. Then $A(a) := \lim_{l\to\infty} 2^l f(\frac{a}{2^l})$ exists for each $a \in X$ and defines an additive mapping $A : X \to Y$ such that

$$\mu_{f_n([x_{ij}])-A_n([x_{ij}])}(t) \geq \frac{(2^r - 2)t}{(2^r - 2)t + 3n^2\sum_{i,j=1}^{n}\theta\|x_{ij}\|^r}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof The proof follows from Theorem 3.1 by taking $\varphi(a,b,c) = \theta(\|a\|^r + \|b\|^r + \|c\|^r)$ for all $a, b, c \in X$. Then we can choose $\alpha = 2^{1-r}$ and we get the desired result. □

Theorem 3.3 *Let $f : X \to Y$ be an odd mapping satisfying* (3.1) *for which there exists a function $\varphi : X^3 \to [0,\infty)$ such that there exists $\alpha < 1$ with*

$$\varphi(a,b,c) \leq 2\alpha\varphi\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$$

for all $a, b, c \in X$. Then $A(a) := \lim_{l\to\infty} \frac{1}{2^l} f(2^l a)$ exists for each $a \in X$ and defines an additive mapping $A : X \to Y$ such that

$$\mu_{f_n([x_{ij}])-A_n([x_{ij}])}(t) \geq \frac{2(1-\alpha)t}{2(1-\alpha)t + n^2\sum_{i,j=1}^{n}\varphi(x_{ij}, x_{ij}, -x_{ij})}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(a) := 2g\left(\frac{a}{2}\right)$$

for all $a \in X$.

It follows from (3.4) that $d(f, Jf) \leq \frac{1}{2}$. So

$$d(f, A) \leq \frac{1}{2 - 2\alpha}.$$

The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.4 *Let r, θ be positive real numbers with $r < 1$. Let $f : X \to Y$ be a mapping satisfying* (3.7). *Then $A(a) := \lim_{l\to\infty} 2^l f(\frac{a}{2^l})$ exists for each $a \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$\mu_{f_n([x_{ij}])-A_n([x_{ij}])}(t) \geq \frac{(2-2^r)t}{(2-2^r)t + 3n^2 \sum_{i,j=1}^{n} \theta \|x_{ij}\|^r}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof The proof follows from Theorem 3.3 by taking $\varphi(a, b, c) = \theta(\|a\|^r + \|b\|^r + \|c\|^r)$ for all $a, b, c \in X$. Then we can choose $\alpha = 2^{r-1}$ and we get the desired result. □

Competing interests

The author declares that she has no competing interests.

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