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A note on 'n-tuplet fixed point theorems for contractive type mappings in partially ordered metric spaces'

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Abstract
In this note, we show that multidimensional fixed point theorems established in the recent report [M. Ertürk and V. Karakaya, *n-tuplet fixed point theorems for contractive type mappings in partially ordered metric spaces*, Journal of Inequalities and Applications 2013, 2013:196] have gaps. Furthermore, the results of the mentioned paper can be reduced to unidimensional (existing) fixed point theorems.

MSC: 47H10; 54H25

Keywords: multidimensional fixed point; partially ordered metric space

1 Introduction and preliminaries

Throughout this manuscript, X will be a non-empty set and $\preceq$ will denote a partial order on X. Given $n \in \mathbb{N}$ with $n \geq 2$, let us denote by X^n the product space $X \times X \times \overset{(n)}{\cdots} \times X$ of n identical copies of X.

The study of multidimensional fixed point theorems was initiated by Guo and Lakshmikantham in [1] in the coupled case.

Definition 1.1 (Guo and Lakshmikantham [1]) Let $F : X \times X \to X$ be a given mapping. We say that $(x, y) \in X \times X$ is a *coupled fixed point of* F if

$$F(x,y) = x \quad \text{and} \quad F(y,x) = y.$$

In 2006, Bhaskar and Lakshmikantham [2] proved some coupled fixed point theorems for a mapping $F : X \times X \to X$ (where X is a partially ordered metric space) by introducing the notion of *mixed monotone mapping*.

Definition 1.2 (See [2]) Let $(X, \preceq)$ be a partially ordered set. A mapping $F : X \times X \to X$. F is said to have the *mixed monotone property* if $F(x, y)$ is monotone nondecreasing in x and is monotone non-increasing in y, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad x_1 \preceq x_2 \quad \Rightarrow \quad F(x_1, y) \preceq F(x_2, y) \quad \text{and}$$
$$y_1, y_2 \in X, \quad y_1 \preceq y_2 \quad \Rightarrow \quad F(x, y_2) \preceq F(x, y_1).$$

Following this paper, Lakshmikantham and Ćirić [3] established coupled fixed/coincidence point theorems for mappings $F : X \times X \to X$ and $g : X \to X$ by defining the concept of the *mixed g-monotone property*. Later, Berinde and Borcut studied the tripled case.

Definition 1.3 (Berinde and Borcut [4]) Let $F : X^3 \to X$ be a given mapping. We say that $(x, y, x) \in X^3$ is a *tripled fixed point of F* if

$$F(x,y,z) = x, \qquad F(y,x,y) = y \quad \text{and} \quad F(z,x,y) = z.$$

Definition 1.4 (See [4]) Let $(X, \preceq)$ be a partially ordered set and $F : X^3 \to X$. We say that F has the *mixed monotone property* if $F(x, y, z)$ is monotone non-decreasing in x and z, and it is monotone non-increasing in y, that is, for any $x, y, z \in X$,

$$\begin{aligned}
&x_1, x_2 \in X, \quad x_1 \preceq x_2 \quad \Rightarrow \quad F(x_1, y, z) \preceq F(x_2, y, z),\\
&y_1, y_2 \in X, \quad y_1 \preceq y_2 \quad \Rightarrow \quad F(x, y_1, z) \succeq F(x, y_2, z) \quad \text{and}\\
&z_1, z_2 \in X, \quad z_1 \preceq z_2 \quad \Rightarrow \quad F(x, y, z_1) \preceq F(x, y, z_2).
\end{aligned}$$

Karapınar and Luong studied the quadruple case.

Definition 1.5 (See [5–7]) An element $(x, y, z, w) \in X^4$ is called a *quadruple fixed point of* $F : X^4 \to X$ if

$$F(x,y,z,w) = x, \qquad F(y,z,w,x) = y, \qquad F(z,w,x,y) = z \quad \text{and} \quad F(w,x,y,z) = w.$$

Definition 1.6 (See [5]) Let $(X, \preceq)$ be a partially ordered set and $F : X^4 \to X$. We say that F has the *mixed monotone property* if $F(x, y, z, w)$ is monotone non-decreasing in x and z, and it is monotone non-increasing in y and w, that is, for any $x, y, z, w \in X$,

$$\begin{aligned}
&x_1, x_2 \in X, \quad x_1 \preceq x_2 \quad \Rightarrow \quad F(x_1, y, z, w) \preceq F(x_2, y, z, w),\\
&y_1, y_2 \in X, \quad y_1 \preceq y_2 \quad \Rightarrow \quad F(x, y_1, z, w) \succeq F(x, y_2, z, w),\\
&z_1, z_2 \in X, \quad z_1 \preceq z_2 \quad \Rightarrow \quad F(x, y, z_1, w) \preceq F(x, y, z_2, w) \quad \text{and}\\
&w_1, w_2 \in X, \quad w_1 \preceq w_2 \quad \Rightarrow \quad F(x, y, z, w_1) \succeq F(x, y, z, w_2).
\end{aligned}$$

When a mapping $g : X \to X$ is involved, we have the notion of *coincidence point*. We will only recall the corresponding definitions in the quadruple case since they are similar in other dimensions.

Definition 1.7 (See [8]) An element $(x, y, z, w) \in X^4$ is called a *quadrupled coincident point of the mappings* $F : X^4 \to X$ *and* $g : X \to X$ if

$$gx = F(x,y,z,w), \qquad gy = F(y,z,w,x), \qquad gz = F(z,w,x,y) \quad \text{and} \quad gw = F(w,x,y,z).$$

Definition 1.8 (See [8]) Let $(X,\preceq)$ be a partially ordered set, and let $F: X^4 \to X$ and $g: X \to X$ be two mappings. We say that F has the *mixed g-monotone property* if $F(x,y,z,w)$ is g-non-decreasing in x and z and is g-non-increasing in y and w, that is, for any $x,y,z,w \in X$,

$$\begin{aligned} &x_1,x_2 \in X, \quad gx_1 \preceq gx_2 \quad \Rightarrow \quad F(x_1,y,z,w) \preceq F(x_2,y,z,w),\\ &y_1,y_2 \in X, \quad gy_1 \preceq gy_2 \quad \Rightarrow \quad F(x,y_1,z,w) \succeq F(x,y_2,z,w),\\ &z_1,z_2 \in X, \quad gz_1 \preceq gz_2 \quad \Rightarrow \quad F(x,y,z_1,w) \preceq F(x,y,z_2,w) \quad \text{and}\\ &w_1,w_2 \in X, \quad gw_1 \preceq gw_2 \quad \Rightarrow \quad F(x,y,z,w_1) \succeq F(x,y,z,w_2). \end{aligned}$$

It is very natural to extend the definition of two-dimensional fixed point (coupled fixed point), three-dimensional fixed point (tripled fixed point) and so on to multidimensional fixed point (n-tuple fixed point) (see, *e.g.*, [9–17]). In this paper, we give some remarks on the notion of n-tuple fixed point given by Ertürk and Karakaya in [18, 19]. Notice that the authors preferred to say 'n-tuplet fixed point' instead of 'n-tuple fixed point'.

Definition 1.9 (See [18]) An element $(x^1,x^2,x^3,\ldots,x^n) \in X^n$ is called an *n-tuple fixed point of the mapping* $F: X^n \to X$ if

$$\begin{aligned} &x^1 = F(x^1,x^2,x^3,\ldots,x^n),\\ &x^2 = F(x^2,x^3,\ldots,x^n,x^1),\\ &x^3 = F(x^3,\ldots,x^n,x^1,x^2),\\ &\vdots\\ &x^n = F(x^n,x^1,x^2,\ldots,x^{n-1}). \end{aligned}$$

Definition 1.10 (See [18]) Let $(X,\preceq)$ be a partially ordered set, and let $F: X^n \to X$ be a mapping. We say that F has the *mixed monotone property* if $F(x^1,x^2,x^3,\ldots,x^n)$ is non-decreasing in odd arguments and it is non-increasing in its even arguments, that is, for any $x^1,x^2,x^3,\ldots,x^n \in X$,

$$\begin{aligned} &y_1,z_1 \in X, \quad y_1 \preceq z_1 \quad \Rightarrow \quad F(y_1,x^2,x^3,\ldots,x^n) \preceq F(z_1,x^2,x^3,\ldots,x^n),\\ &y_2,z_2 \in X, \quad y_2 \preceq z_2 \quad \Rightarrow \quad F(x^1,y_2,x^3,\ldots,x^n) \succeq F(x^1,z_2,x^3,\ldots,x^n),\\ &\vdots\\ &y_n,z_n \in X, \quad y_n \preceq z_n \quad \Rightarrow \quad F(x^1,x^2,x^3,\ldots,y_n) \preceq F(x^1,x^2,x^3,\ldots,z_n) \quad \text{if } n \text{ is odd},\\ &y_n,z_n \in X, \quad y_n \preceq z_n \quad \Rightarrow \quad F((x^1,x^2,x^3,\ldots,y_n) \succeq F(x^1,x^2,x^3,\ldots,z_n) \quad \text{if } n \text{ is even}. \end{aligned}$$

Definition 1.11 (See [18]) An element $(x^1,x^2,x^3,\ldots,x^n) \in X^n$ is called an *n-tuple coincidence point of the mappings* $F: X^n \to X$ and $g: X \to X$ if

$$\begin{aligned} &gx^1 = F(x^1,x^2,x^3,\ldots,x^n),\\ &gx^2 = F(x^2,x^3,\ldots,x^n,x^1), \end{aligned}$$

$$gx^3 = F(x^3, \ldots, x^n, x^1, x^2),$$
$$\vdots$$
$$gx^n = F(x^n, x^1, x^2, \ldots, x^{n-1}).$$

Definition 1.12 (See [18]) Let $(X, \preceq)$ be a partially ordered set, and let $F: X^n \to X$ and $g: X \to X$ be mappings. We say that F has the *mixed g-monotone property* if $F(x^1, x^2, x^3, \ldots, x^n)$ is g-non-decreasing in odd arguments and it is g-non-increasing in its even arguments, that is, for any $x^1, x^2, x^3, \ldots, x^n \in X$,

$$y_1, z_1 \in X, \quad gy_1 \preceq gz_1 \quad \Rightarrow \quad F(y_1, x^2, x^3, \ldots, x^n) \preceq F(z_1, x^2, x^3, \ldots, x^n),$$
$$y_2, z_2 \in X, \quad gy_2 \preceq gz_2 \quad \Rightarrow \quad F(x^1, y_2, x^3, \ldots, x^n) \succeq F(x^1, z_2, x^3, \ldots, x^n),$$
$$\vdots$$
$$y_n, z_n \in X, \quad gy_n \preceq gz_n \quad \Rightarrow \quad F(x^1, x^2, x^3, \ldots, y_n) \preceq F(x^1, x^2, x^3, \ldots, z_n) \quad \text{if } n \text{ is odd},$$
$$y_n, z_n \in X, \quad gy_n \preceq gz_n \quad \Rightarrow \quad F(x^1, x^2, x^3, \ldots, y_n) \succeq F(x^1, x^2, x^3, \ldots, z_n) \quad \text{if } n \text{ is even}.$$

2 Some remarks

Firstly we notice that in the case $n = 3$, Definitions 1.9 and 1.11,

$$gx^1 = F(x^1, x^2, x^3),$$
$$gx^2 = F(x^2, x^3, x^1),$$
$$gx^3 = F(x^3, x^1, x^2),$$

do not extend the notion of tripled coincidence point by Berinde and Borcut [4]. Therefore, their results are not extensions of the well-known results in the tripled case. This fact shows that the odd case is not well posed by Definitions 1.9 and 1.11 or, more precisely, the mixed monotone property is not useful to ensure the existence of coincidence points. In this sense, we have the following result.

Theorem 2.1 *Theorem* 1 *in* [18] *is not valid if n is odd.*

Proof In order not to complicate the proof, we only study the case $n = 3$, which is very illustrative and can be identically extrapolated to the case in which n is odd. Let us follow the lines in the proof of Theorem 1 in [18]. Using the initial points $x_0^1, x_0^2, x_0^3 \in X$, it is possible to construct three sequences $\{x_k^1\}$, $\{x_k^2\}$ and $\{x_k^3\}$ recursively defined by:

$$gx_k^1 = F(x_{k-1}^1, x_{k-1}^2, x_{k-1}^3),$$
$$gx_k^2 = F(x_{k-1}^2, x_{k-1}^3, x_{k-1}^1),$$
$$gx_k^3 = F(x_{k-1}^3, x_{k-1}^1, x_{k-1}^2) \quad \text{for all } k \in \mathbb{N}, k \geq 1.$$

By assumption, we have that

$$\begin{aligned} gx_0^1 &\preceq F\big(x_0^1,x_0^2,x_0^3\big) = gx_1^1, \\ gx_0^2 &\succeq F\big(x_0^2,x_0^3,x_0^1\big) = gx_1^2, \\ gx_0^3 &\preceq F\big(x_0^3,x_0^1,x_0^2\big) = gx_1^3. \end{aligned}$$

Then the authors affirmed that these sequences verify, for all $k \geq 1$,

$$\begin{aligned} gx_{k-1}^1 &\preceq gx_k^1, \\ gx_{k-1}^2 &\succeq gx_k^2, \\ gx_{k-1}^3 &\preceq gx_k^3. \end{aligned}$$

However, it is impossible to prove that $gx_1^2 \succeq gx_2^2$ because the mixed g-monotone property leads to contrary inequalities. At most, we can deduce the following properties:

$$gx_1^2 \preceq gx_0^2 \quad\Rightarrow\quad F\big(x_1^2,x_0^3,x_0^1\big) \preceq F\big(x_0^2,x_0^3,x_0^1\big) = gx_1^2.$$

Moreover,

$$gx_0^3 \preceq gx_1^3 \quad\Rightarrow\quad F\big(x_1^2,x_0^3,x_0^1\big) \succeq F\big(x_1^2,x_1^3,x_0^1\big).$$

Joining the two previous inequalities, we obtain

$$F\big(x_1^2,x_1^3,x_0^1\big) \preceq F\big(x_1^2,x_0^3,x_0^1\big) \preceq F\big(x_0^2,x_0^3,x_0^1\big) = gx_1^2.$$

However, in the third component, the inequality is on the contrary

$$gx_0^1 \preceq gx_1^1 \quad\Rightarrow\quad F\big(x_1^2,x_1^3,x_0^1\big) \preceq F\big(x_1^2,x_1^3,x_1^1\big) = gx_2^2.$$

Then we can deduce that

$$F\big(x_1^2,x_1^3,x_0^1\big) \preceq gx_1^2 \quad\text{and}\quad F\big(x_1^2,x_1^3,x_0^1\big) \preceq gx_2^2.$$

Since other possibilities yield similar incomparable cases, we cannot get the inequality $gx_1^2 \succeq gx_2^2$. □

For completeness and to conclude this paper, instead of Definitions 1.9 and 1.11, we recall here the concept of *multidimensional fixed/coincidence point* introduced by Roldán *et al.* in [11] (see also [12–14]), which is an extension of Berzig and Samet's notion given in [10].

Fix $n \in \mathbb{N}$ such that $n \geq 2$. To separate the variables, let $\{A,B\}$ be a partition of $\Lambda_n = \{1,2,\dots,n\}$, that is, $A \cup B = \Lambda_n$ and $A \cap B = \varnothing$, and suppose that A and B are non-empty. We will define

$$\begin{aligned} \Omega_{A,B} &= \big\{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq A \text{ and } \sigma(B) \subseteq B\big\} \quad \text{and} \\ \Omega'_{A,B} &= \big\{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq B \text{ and } \sigma(B) \subseteq A\big\}. \end{aligned}$$

Let $\sigma_1, \sigma_2, \ldots, \sigma_n : \Lambda_n \to \Lambda_n$ be n mappings from Λ_n into itself, and let Φ be the n-tuple $(\sigma_1, \sigma_2, \ldots, \sigma_n)$.

Let $F : X^n \to X$ and $g : X \to X$ be two mappings.

Definition 2.2 [11] A point $(x_1, x_2, \ldots, x_n) \in X^n$ is called a Φ-*coincidence point of the mappings F and g* if

$$F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}) = gx_i \quad \text{for all } i \in \{1, 2, \ldots, n\}.$$

If g is the identity mapping on X, then $(x_1, x_2, \ldots, x_n) \in X^n$ is called a Φ-*fixed point of the mapping F*.

Definition 2.3 [11] Let $(X, \preccurlyeq)$ be a partially ordered space. We say that F has the *mixed g-monotone property (w.r.t.* $\{A, B\}$*)* if F is g-monotone non-decreasing in arguments of A and g-monotone non-increasing in arguments of B, *i.e.*, for all $x_1, x_2, \ldots, x_n, y, z \in X$ and all i,

$$gy \preccurlyeq gz \quad \Rightarrow \quad \begin{cases} F(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \preccurlyeq F(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n) & \text{if } i \in A, \\ F(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \succcurlyeq F(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n) & \text{if } i \in B. \end{cases}$$

In order to ensure the existence of Φ-coincidence/fixed points, it is very important to assume that the mixed g-monotone property is compatible with the permutation of the variables, that is, the mappings of $\Phi = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ should verify:

$$\sigma_i \in \Omega_{A,B} \quad \text{if } i \in A \quad \text{and} \quad \sigma_i \in \Omega'_{A,B} \quad \text{if } i \in B.$$

Notice that, in fact, when n is even, Definitions 1.11 and 1.12 can be seen as particular cases of the previous definitions, when A is the set of all odd numbers and B is the family of all even numbers in $\{1, 2, \ldots, n\}$, and the mappings $\sigma_1, \sigma_2, \ldots, \sigma_n$ are appropriate permutations of the variables.

Finally, to be fair, we remark that most of multidimensional fixed point theorems can be reduced to one-dimensional (usual) fixed point results (see, *e.g.*, [14, 20]). More precisely, for instance in [14], the authors proved that the first coupled fixed point result, Theorem 2.1 in [2], is a consequence of Theorem 2.1 in [21]. In [20], the authors proved that the initial multidimensional fixed point result, Theorem 9 in [11], can be derived from Theorem 2.1 in [21] either.

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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