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A few remarks on the generalized Novikov equation

Shouming Zhou[1*] and Rong Chen[2]

*Correspondence:
zhoushouming76@163.com
[1]College of Mathematics Science,
Chongqing Normal University,
Chongqing, 401331, P.R. China
Full list of author information is
available at the end of the article

Abstract

This paper deals with the Cauchy problem for a generalized Novikov equation $u_t - u_{xxt} + (b+1)u^2 u_x = buu_x u_{xx} + u^2 u_{xxx}$, where b is a constant. The local well-posedness in the critical Besov space $B_{2,1}^{3/2}$ is established. Moreover, a lower bound for the maximal existence time and lower semicontinuity of the existence are derived, the multi-peakon solutions are also obtained. Finally, the persistence properties in weighted spaces for the solution of this equation are considered.

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1 Introduction

The present paper focuses on the Cauchy problem for the following modified Novikov equation:

$$\begin{cases} u_t - u_{xxt} + (b+1)u^2 u_x = buu_x u_{xx} + u^2 u_{xxx}, & t > 0, x \in \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where b, k_1 and k_2 are arbitrary constants. Our main purpose of this paper is to establish the well-posedness in the critical Besov space $B_{2,1}^{3/2}$ and persistence in a weighted Sobolev space.

Note that when we take $b = 3$, Eq. (1.1) is the Novikov equation:

$$u_t - u_{xxt} + 4u^2 u_x = 3uu_x u_{xx} + u^2 u_{xxx}, \quad t > 0, x \in \mathbb{R}, \tag{1.2}$$

which was recently discovered by Novikov in a symmetry classification of nonlocal PDEs with quadratic or cubic nonlinearity [1]. The perturbative symmetry approach [2] yields necessary conditions for a PDE to admit infinitely many symmetries. Using this approach, Novikov was able to isolate Eq. (1.2) and find its first few symmetries, and he subsequently found a scalar Lax pair for it, proving that the equation is integrable. By using the prolongation algebra method, Hone and Wang [3] gave a matrix Lax pair and many conserved densities and a bi-Hamiltonian structure of the Novikov equation, and they showed how it was related by a reciprocal transformation to a negative flow in the Sawada-Kotera hierarchy. Then in [4], the authors calculated the explicit formulas for multi-peakon solutions of the Novikov equation.

Recently, by the transport equations theory and the classical Friedrichs regularization method, the authors proved that the Cauchy problem for the Novikov equation is locally well posed in the Besov spaces $B_{p,r}^s$ (with $1 \le p, r \le +\infty$ and $s > \max\{1 + 1/p, 3/2\}$ in [5, 6], and with the critical index $s = 3/2$, $p = 2$ in [7]). It was also shown in [7] that the Novikov equation associated with the initial value is locally well posed in the Sobolev space H^s with $s > 3/2$ by using the abstract Kato theorem. Two results about the persistence properties of the strong solution for Eq. (1.2) were established in [7]. A Galerkin-type approximation method was used in Himonas and Holliman's paper [8] to establish the well-posedness of Novikov equation (1.2) in the Sobolev space H^s with $s > 3/2$ on both the line and the circle, and in [9, 10] the authors proved that the data-to-solution map is not globally uniformly continuous on H^s for $s < 3/2$, this result supplements Himonas and Holliman's works. Tiglay [11] showed the local well-posedness of the problem in Sobolev spaces and the existence and uniqueness of solutions for all time using orbit invariants. For analytic initial data, the existence and uniqueness of analytic solutions for Eq. (1.2) were also obtained in [11]. Analogous to the Camassa-Holm equation, the Novikov equation possesses a blow-up phenomenon [10, 12] and global weak solutions [13, 14].

On the other hand, it is well known that the nonlinearity of the following b-equation is quadratic:

$$u_t - u_{xxt} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}, \quad t > 0, x \in \mathbb{R}, \tag{1.3}$$

which can be derived as the family of asymptotically equivalent shallow water wave equations that emerges at quadratic order accuracy for any $b \ne -1$ by an appropriate Kodama transformation. For the case $b = -1$, the corresponding Kodama transformation is singular and the asymptotic ordering is violated (see [15–17]). Equation (1.2) belongs to the following family of nonlinear dispersive partial differential equations:

$$u_t - \gamma u_{xxx} - \alpha^2 u_{xxt} = \left(c_1 u^2 + c_2 u_x^2 + c_3 uu_{xx}\right)_x,$$

where γ, α, c_1, c_2 and c_3 are real constants. By using Painlevé analysis in [18–20], there are only three asymptotically integrable within this family: the KdV equation, the Camassa-Holm (Eq. (1.3) with $b = 2$) equation and the Degasperis-Procesi equation (Eq. (1.3) with $b = 3$). The solutions of the b-equation were studied numerically for various values of b in [21, 22], where b was taken as a bifurcation parameter. The necessary conditions for integrability of the b-equation were investigated in [2]. The b-equation also admits peakon solutions for any $b \in \mathbb{R}$ (see [19, 21, 22]). The well-posedness, blow-up phenomena and global solutions for the b-equation were shown in [23–25].

Recently, Mi and Mu [26] studied the local well-posedness in the Besov space $B_{p,r}^s$ with $1 \le p, r \le +\infty$ and $s > \max\{1 + 1/p, 3/2\}$. It is well known that $B_{2,2}^s(\mathbb{R}) = H^s$ and for any $s' < 3/2 < s$: $H^s \hookrightarrow B_{2,1}^{\frac{3}{2}} \hookrightarrow H^{\frac{3}{2}} \hookrightarrow B_{2,\infty}^{\frac{3}{2}} \hookrightarrow H^{s'}$, which shows that H^s and $B_{2,2}^s$ are quite close, so here we first establish the local well-posedness in the critical Besov space $B_{2,1}^{3/2}$.

Theorem 1.1 *Assume that the initial data $u_0(x) \in B_{2,1}^{\frac{3}{2}}$. Then there exist a unique solution $u(x,t)$ and a maximal time $T = T(u_0) > 0$ to the Cauchy problem* (1.1) *such that*

$$u = u(\cdot, u_0) \in C\left([0,T]; B_{2,1}^{\frac{3}{2}}\right) \cap C^1\left([0,T]; B_{2,1}^{\frac{1}{2}}\right).$$

Moreover, the solution depends continuously on the initial data, i.e., the mapping

$$u_0 \mapsto u(\cdot, u_0) : B_{2,1}^{\frac{3}{2}} \mapsto C\big([0,T]; B_{2,1}^{\frac{3}{2}}\big) \cap C^1\big([0,T]; B_{2,1}^{\frac{1}{2}}\big)$$

is continuous.

Remark 1.1 Following the proof of Theorem 5.1 in [5] and Theorem 1.3 in [10], one can easily get that Eq. (1.1) is not locally well posed in $B_{2,\infty}^{\frac{3}{2}}$ in the following sense: There exists a global solution $u(t,x) \in L^\infty(\mathbb{R}^+; B_{2,\infty}^{\frac{3}{2}})$ to Eq. (1.1) such that for any $T > 0$ and $\epsilon > 0$, there exists a solution $v(t,x) \in L^\infty(\mathbb{R}^+; B_{2,\infty}^{\frac{3}{2}})$ to Eq. (1.1) with

$$\big\|u(0) - v(0)\big\|_{B_{2,\infty}^{\frac{3}{2}}} \leq \epsilon \quad \text{but} \quad \big\|u(t) - v(t)\big\|_{L^\infty(\mathbb{R}^+; B_{2,\infty}^{\frac{3}{2}})} \leq 1.$$

Remark 1.2 Theorem 1.1 improves the corresponding result in [26]. On the other hand, noting that the counterexample given in [10] cannot be applied to the case in $B_{2,r}^{3/2}$ with $1 < r < \infty$, the question of local well-posedness of Eq. (1.1) in $H^{3/2}$ remains open. Actually, this is still an open problem for the Camassa-Holm and Novikov equations.

Remark 1.3 Since

$$\|fg\|_{B_{p,\infty}^{-\frac{1}{p}}} \leq C\|f\|_{B_{p,\infty}^{-\frac{1}{p}}} \|g\|_{B_{p,\infty}^{\frac{1}{p}} \cap L^\infty} \quad \text{and} \quad \|f\|_{B_{p,1}^{\frac{1}{p}}} \leq C\|f\|_{B_{p,\infty}^{\frac{1}{p}}} \log\left(e + \frac{\|f\|_{B_{p,\infty}^{1+\frac{1}{p}}}}{\|f\|_{B_{p,\infty}^{\frac{1}{p}}}}\right)$$

holds for $1 \leq p \leq \infty$ (see [27]), Theorem 1.1 holds true in the case of $B_{p,1}^{1+1/p}$ with $1 \leq p < \infty$. Besides, using similar arguments in [28], Theorem 1.1 can also hold true in the case of $B_{p,r}^s$ with $s > \max(1 + 1/p, 3/2)$. Furthermore, the existence of solutions to Eq. (1.1) holds as the initial data belong to $B_{p,r}^s \cap Lip$ with $s > 1$, which covers the corresponding result in [26].

Theorem 1.2 *Let $u_0 \in B_{p,r}^s$ with $1 \leq p, r \leq \infty$ and $s > \max(3/2, 1 + 1/p)$, then there exists a lifespan $T_{u_0}^* > 0$ such that*

$$T_{u_0}^* < \infty \quad \Longrightarrow \quad \int_0^{T_{u_0}^*} \big\|u\partial_x u(\tau)\big\|_{L^\infty} d\tau = \infty.$$

We now get a lower bound depending only on $\|u_0\|_{Lip}$ for the maximal existence time.

Theorem 1.3 *Assume that $u_0 \in Lip \cap B_{p,r}^s$, $s > \max\{3/2, 1 + 1/p\}$. Let T^* be the maximal existence time of the solution u to Eq. (1.1) with the initial data u_0. Then T^* satisfies*

$$T^* \geq \frac{1}{a(2\|u_0\|_{L^\infty} + \|u_{0,x}\|_{L^\infty})^2}$$

with $a = 4\max\{2/3, |b-2|, 2|b|/3, |6-b|/3\}$.

Next, we shall derive lower semicontinuity of the existence time, provided the initial data is smooth enough.

Theorem 1.4 *Let $v_0 \in B_{p,r}^s$, $s > \max\{3/2, 1 + 1/p\}$ and $u_0 \in B_{p,r}^{s+1}$. Assume that u, v are two solutions to Eq.* (1.1) *with the initial data u_0, v_0. Let T_{u_0}, T_{v_0} be the maximal existence time of the solution u, v. If there exist $T < T_{u_0}$ and a constant C such that*

$$\|u_0 - v_0\|_{B_{p,r}^s} < \frac{1}{2C \int_0^T (\|u(\tau)\|_{B_{p,r}^{s+1}} + \|u_0 - v_0\|_{B_{p,r}^s}) \exp\{2C \int_0^\tau \|u(\tau')\|_{B_{p,r}^{s+1}}^2 \, d\tau'\} \, d\tau},$$

then Eq. (1.1) *has a unique solution $v \in E_{p,r}^s(T)$.*

In [26], the authors consider the single peakon taking the form $u(t,x) = \sqrt{c}e^{-|x-ct-x_0|}$, $c > 0$. Moreover, this peakon solitary is a global weak solution to Eq. (1.1). Next, we discuss the existence of multi-peakon solutions to Eq. (1.1).

Theorem 1.5 *Equation* (1.1) *has peakon solutions of the form*:

$$u(t,x) = \sum_{i=1}^N p_i(t)e^{-|x-q_i(t)|}, \tag{1.4}$$

whose positions $q_t(t)$ and amplitudes $p_j(t)$ are according to the dynamical system

$$
\begin{aligned}
p_j' &= \left(\sum_{i=1}^N p_i e^{-|q_j - q_i(t)|} \right)^2, \\
q_j' &= (b-2)q_j \left(\sum_{i=1}^N p_i e^{-|q_j - q_i|} \right) \left(\sum_{i=1}^N p_i \operatorname{sgn}(q_j - q_i) e^{-|q_j - q_i|} \right).
\end{aligned}
\tag{1.5}
$$

In [7, 29–33], the spacial decay rates for the strong solutions to the Camassa-Holm equation, the b-equation, and the Novikov equation were established provided that the corresponding initial data decay at infinity. This kind of property is the so-called persistence property. Following the main idea of [33], we also prove the persistence properties in weighted spaces for the solution of Eq. (1.1). However, the hard question is that there are cubic nonlinearities in (1.1) which make the proof very difficult. First, we give the following definition of an admissible weight function.

Definition 1.1 An admissible weight function for Eq. (1.1) is a locally absolutely continuous function $\phi : \mathbb{R} \to \mathbb{R}$ such that, for some $A > 0$ and a.e. $x \in \mathbb{R}$, $|\phi'(x)| \le A|\phi(x)|$, and that is ν-moderate for some sub-multiplicative weight function ν satisfying $\inf_{\mathbb{R}} \nu > 0$ and

$$\int_{\mathbb{R}} \frac{\nu(x)}{e^{|x|}} \, dx < \infty. \tag{1.6}$$

We recall that a locally absolutely continuous function is a.e. differentiable in $\mathbb{R}$. Moreover, its a.e. derivative belongs to L_{loc}^1 and agrees with its distributional derivative. We can now state our main result on admissible weights.

Theorem 1.6 *Let $T > 0$, $s > 3/2$, and $2 \le p < \infty$. Let also $u \in C([0, T], H^s(\mathbb{R}))$ be a strong solution of the Cauchy problem for Eq.* (1.1) *such that $u|_{t=0} = u_0$ satisfies*

$$u_0 \phi \in L^p(\mathbb{R}) \quad \text{and} \quad (\partial_x u_0)\phi \in L^p(\mathbb{R}),$$

where ϕ is an admissible weight function for Eq. (1.1). Then, for all $t \in [0, T]$, we have the estimate

$$\|u(t)\phi\|_p + \|\partial_x u(t)\phi\|_p \geq (\|u_0\phi\|_p + \|\partial_x u_0 \phi\|_p)e^{CMt}$$

for some constant $C > 0$ depending only on v, ϕ (through the constants A, C_0, $\inf_{\mathbb{R}} v$, and $\int_{\mathbb{R}} \frac{v(x)}{e^{|x|}}\,dx < \infty$), and

$$M \equiv \sup_{t \in [0,T]} (\|u(t)\|_\infty + \|\partial_x u(t)\|_\infty) < \infty.$$

Remark 1.4 The basic example of the application of Theorem 1.6 is obtained by taking the standard weights $\phi = \phi_{a,b,c,d}(x) = e^{a|x|^b}(1 + |x|)^c \log(e + |x|)^d$ with the following conditions:

$$a \geq 0, \qquad c, d \in \mathbb{R}, \qquad 0 \leq b \leq 1, \qquad ab < 1.$$

(For $a < 0$, one has $\phi(x) \to 0$ as $|x| \to \infty$: the conclusion of the theorem remains true but it is not interesting in this case.) The restriction $ab < 1$ guarantees the validity of condition (1.6) for a multiplicative function $v(x) \geq 1$.

The limit case $a = b = 1$ is not covered by Theorem 1.1. The result holds true, however, for the weight $\phi = \phi_{1,1,c,d}$ with $c < 0$, $d \in \mathbb{R}$, and $\frac{1}{|c|} < p \leq \infty$, or, more generally, when $(1 + |\cdot|)^c \log(e + |\cdot|)^d \in L^p(\mathbb{R})$. See Theorem 1.6 below, which covers the case of such fast growing weights.

Remark 1.5 Let us consider a few particular cases:

(1) Take $\phi = \phi_{0,0,c,0}$ with $c > 0$, and choose $p = \infty$. In this case, Theorem 1.6 states that the condition

$$|u_0(x)| + |\partial_x u_0(x)| \leq C(1 + |x|)^{-c}$$

implies the uniform algebraic decay in $[0, T]$:

$$|u(x, t)| + |\partial_x u(x, t)| \leq C(1 + |x|)^{-c}.$$

It is worth pointing out that this is a new result for the Novikov equation.

(2) Choose $\phi = \phi_{a,1,0,0}$ if $x \geq 0$ and $\phi(x) = 1$ if $x \leq 0$ with $0 \leq a \leq 1$. Such weight clearly satisfies the admissibility conditions of Definition 1.1. Applying Theorem 1.6 with $p = \infty$, we conclude that the pointwise decay $O(e^{-ax})$ as $x \to +\infty$ is conserved during the evolution. Similarly, we have persistence of the decay $O(e^{-ax})$ as $x \to -\infty$. Hence, our Theorem 1.6 encompasses also Theorem 4.1 of [7].

Since 'peakon' solution $u(t, x) = \sqrt{c}e^{-|x-ct|}$, $c > 0$ does not satisfy the asymptotic behavior in Theorem 1.2 (see Remark 1.4), the purpose of the next theorem is to establish a variant of this theorem that can be applied to some v-moderate weights ϕ for which condition (1.6) does not hold. Instead of assuming (1.6), we now put the weaker condition

$$ve^{-|\cdot|} \in L^p(\mathbb{R}). \tag{1.7}$$

It is easily checked that, for any continuous sub-multiplicative weight function v, we have

$$ve^{-|\cdot|} \in L^1(\mathbb{R}) \quad \Longrightarrow \quad ve^{-|\cdot|} \in L^p(\mathbb{R}), \quad \forall 1 \le p \le \infty,$$

so that condition (1.7) is indeed weaker than condition (1.6) (see [33] for the details).

Theorem 1.7 *Let $2 \le p \le \infty$ and ϕ be a v-moderate weight function as in Definition* 1.1 *satisfying condition* (1.7) *instead of* (1.6). *Let also* $u|_{t=0} = u_0$ *satisfy*

$$u_0\phi \in L^p(\mathbb{R}), \qquad u_0\phi^{\frac{1}{3}} \in L^3(\mathbb{R})$$

and

$$(\partial_x u_0)\phi \in L^p(\mathbb{R}), \qquad (\partial_x u_0)\phi^{\frac{1}{3}} \in L^3(\mathbb{R}).$$

Let also $u \in C([0, T], H^s(\mathbb{R}))$, $s > 3/2$ *be the strong solution of the Cauchy problem for Eq.* (1.1), *emanating from* u_0. *Then*

$$\sup_{t\in[0,T]} \left(\|u(t)\phi\|_{L^p} + \|\partial_x u(t)\phi\|_{L^p} \right) < \infty$$

and

$$\sup_{t\in[0,T]} \left(\|u(t)\phi^{\frac{1}{3}}\|_{L^3} + \|\partial_x u(t)\phi^{\frac{1}{3}}\|_{L^3} \right) < \infty.$$

Remark 1.6 Like Remark 1.5, Theorem 1.7 not only recovers Theorem 4.2 in [7], but also gives a new asymptotic behavior of solutions to Eq. (1.1).

The plan of this paper is organized as follows. In the next section, the local well-posedness in the critical Besov space $B_{2,1}^{3/2}$ is considered and Theorem 1.1 is proved. The blow-up criteria and multi-peakon solutions are obtained in Section 3 and Theorems 1.2-1.5 are proved. In the last section, the persistence properties in weighted spaces for the solution of Eq. (1.1) are considered, and Theorems 1.6-1.7 are proved.

2 Local well-posedness in critical Besov spaces

In this section, we shall establish the local well-posedness of Eq. (1.1) in critical Besov spaces. More precisely, we give the proof of Theorem 1.1. First, we rewrite model (1.1) in the following transports equation form:

$$\begin{cases} u_t + u^2 u_x + \frac{b-2}{2}(1-\partial_x^2)^{-1}(\partial_x u)^3 + (1-\partial_x^2)^{-1}\partial_x(\frac{b}{3}u^3 + \frac{6-b}{2}u(\partial_x u)^2) = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{2.1}$$

We can easily get the following two lemmas.

Lemma 2.1 *Let* $u_0(x) \in B_{2,1}^{\frac{3}{2}}$. *Then there exists a time* $T > 0$ *such that the Cauchy problem* (1.1) *has a solution* $u \in C([0, T]; B_{2,1}^{\frac{3}{2}}) \cap C^1([0, T]; B_{2,1}^{\frac{1}{2}})$.

Lemma 2.2 *Assume that u_0 (respectively v_0) $\in B_{2,1}^{\frac{3}{2}}$ such that u (respectively v) $\in L^\infty([0,T];$
$B_{2,1}^{\frac{3}{2}}) \cap C([0,T];B_{2,1}^{\frac{1}{2}})$ is a solution to the Cauchy problem* (1.1) *with the initial data u_0 (respectively v_0). Let $w = u - v$ and $w_0 = u_0 - v_0$. Then, for every $t \in [0,T]$,*

$$\|u(t) - v(t)\|_{B_{p,r}^{s-1}} \le \|u_0 - v_0\|_{B_{p,r}^{s-1}} \times \exp\left\{ C \int_0^T \left(\|u(\tau)\|_{B_{p,r}^s}^2 + \|v(\tau)\|_{B_{p,r}^s}^2 \right) d\tau \right\}. \qquad (2.2)$$

Proof of Lemmas 2.1-2.2 The proof is much similar to the case $u_0 \in B_{p,r}^s$, $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$
(see [5, 26]), thus we omit it here. □

Lemma 2.3 *For any $u_0 \in B_{2,1}^{\frac{3}{2}}$, there exist a neighborhood V of u_0 in $B_{2,1}^{\frac{3}{2}}$ and a time $T > 0$
such that for any solution of the Cauchy problem* (1.1) *$v \in V$, the map*

$$\Phi : v_0 \mapsto v(\cdot, v_0) : V \subset B_{2,1}^{\frac{3}{2}} \mapsto C\left([0,T];B_{2,1}^{\frac{3}{2}}\right) \cap C^1\left([0,T];B_{2,1}^{\frac{1}{2}}\right)$$

is continuous.

Proof Firstly, we prove the continuity of the map Φ in $C([0,T];B_{2,1}^{\frac{1}{2}})$. Fix $u_0 \in B_{2,1}^{\frac{3}{2}}$ and
$\delta > 0$. Now we claim that there exist $T > 0$ and $M > 0$ such that for any $v_0 \in B_{2,1}^{\frac{3}{2}}$ with $\|v_0 - u_0\|_{B_{2,1}^{\frac{3}{2}}} \le \delta$, the solution $v = \Phi(v)$ of the Cauchy problem (1.1) belongs to $C([0,T];B_{2,1}^{\frac{3}{2}})$ and
satisfies $\|v\|_{L^\infty([0,T];B_{2,1}^{\frac{3}{2}})} \le M$. In fact, according to the proof of the local well-posedness,
we have that if we fix $T > 0$ such that

$$0 < T < \min\left\{ \frac{1}{4C\|v_0\|_{B_{2,1}^{\frac{3}{2}}}^2}, \frac{1}{2C} \right\},$$

then

$$\|v(t)\|_{B_{2,1}^{\frac{3}{2}}} \le \frac{\|v_0\|_{B_{2,1}^{\frac{3}{2}}}}{(1 - 4C\|v_0\|_{B_{2,1}^{\frac{3}{2}}}^2 t)^{\frac{1}{2}}} \quad \text{for all } t \in [0,T]. \qquad (2.3)$$

As $\|v_0 - u_0\|_{B_{2,1}^{\frac{3}{2}}} \le \delta$, then $\|v_0\|_{B_{2,1}^{\frac{3}{2}}} \le \|u_0\|_{B_{2,1}^{\frac{3}{2}}} + \delta$. Here, one can choose some suitable
constant C such that

$$T = \frac{3}{16C(\|u_0\|_{B_{2,1}^{\frac{3}{2}}} + \delta + 1)^2} < \min\left\{ \frac{1}{4C\|v_0\|_{B_{2,1}^{\frac{3}{2}}}^2}, \frac{1}{2C} \right\}$$

and

$$M = 2\left(\|u_0\|_{B_{2,1}^{\frac{3}{2}}} + \delta \right).$$

Now, combining the above uniform bounds with Lemma 2.2, we get that

$$\|\Phi(v_0) - \Phi(u_0)\|_{L^\infty(0,T;B_{2,1}^{\frac{3}{2}})} \le \delta e^{C(2M^2+1)T}.$$

Hence Φ is Holder continuous from $B_{2,1}^{\frac{3}{2}}$ into $C([0,T];B_{2,1}^{\frac{1}{2}})$.

Next we prove the continuity of the map Φ in $C([0,T]; B_{2,1}^{\frac{3}{2}})$. Let $u_0^\infty \in B_{2,1}^{\frac{3}{2}}$ and $(u_0^{(n)})_{n\in\mathbb{N}} \to u_0^\infty$ in $B_{2,1}^{\frac{3}{2}}$. Let $u^{(n)}$ be the solution of the Cauchy problem (1.1) with the initial data $u_0^{(n)}$. From the above argument, we deduce that for any $n \in \mathbb{N}$, $t \in T$,

$$\sup_{n\in\mathbb{N}} \|u^{(n)}\|_{L_T^\infty(B_{2,1}^{\frac{3}{2}})} \le M. \tag{2.4}$$

Note that to prove $u^{(n)} \to u^\infty$ in $C([0,T]; B_{2,1}^{\frac{3}{2}})$ means to prove $u_x^{(n)} \to u_x^\infty$ in $C([0,T]; B_{2,1}^{\frac{1}{2}})$.

Recall that $(v^{(n)}) \doteq \partial_x u^{(n)}$ solves the linear transport equation:

$$\begin{cases} \partial_t(v^{(n)}) + (u^{(n)})^2 \partial_x(v^{(n)}) = f^{(n)}, \\ v^{(n)}|_{t=0} = \partial_x(u_0^{(n)}), \end{cases}$$

where

$$f^{(n)} = 2(u^{(n)})(\partial_x u^{(n)})^2 - \frac{(b-2)}{2}\partial_x(1-\partial_x^2)^{-1}(\partial_x u^{(n)})^3$$
$$- (1-\partial_x^2)^{-1}\partial_x^2\left(\frac{b}{3}(u^{(n)})^3 + \frac{6-b}{2}(u^{(n)})(\partial_x u^{(n)})^2\right).$$

Thanks to the Kato theory [28], we decompose $v^{(n)}$ into $v^{(n)} = z^{(n)} + w^{(n)}$ with

$$\begin{cases} \partial_t(z^{(n)}) + (u^{(n)})^2 \partial_x(z^{(n)}) = f^{(n)} - f^\infty, \\ v^{(n)}|_{t=0} = \partial_x(u_0^{(n)}) - \partial_x(u_0^\infty) \end{cases} \tag{2.5}$$

and

$$\begin{cases} \partial_t(w^{(n)}) + (w^{(n)})^2 \partial_x(w^{(n)}) = f^\infty, \\ w^{(n)}|_{t=0} = \partial_x(u_0^\infty). \end{cases}$$

According to the first step, we have that the sequence $(u^{(n)})_{n\in\overline{\mathbb{N}}}$ ($\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$) is uniformly bounded in $C([0,T]; B_{2,1}^{\frac{3}{2}})$ and tends to (u^∞) in $C([0,T]; B_{2,1}^{\frac{1}{2}})$, thus we can use Proposition 3 in [27], which implies that $w^{(n)}$ tends to w^∞ in $C([0,T]; B_{2,1}^{\frac{1}{2}})$, i.e., for any $\varepsilon > 0$, $\|w^{(n)} - w^\infty\|_{B_{2,1}^{\frac{1}{2}}} \le \varepsilon$.

On the other hand, applying Lemma 2.3 in [7] and the product law in the Besov spaces to Eq. (2.5), one may get that

$$\|z^{(n)}\|_{B_{2,1}^{\frac{1}{2}}} \le \exp\left\{ C \int_0^t \|(u^{(n)})^2(\tau)\|_{B_{2,1}^{\frac{3}{2}}} \, d\tau \right\}$$
$$\cdot \left(\|\partial_x(u_0^{(n)}) - \partial_x(u_0^\infty)\|_{B_{2,1}^{\frac{1}{2}}} + \int_0^t \|f^{(n)} - f^\infty\|_{B_{2,1}^{\frac{1}{2}}} \, d\tau \right). \tag{2.6}$$

Using the properties of Besov spaces exhibited in [27], one easily checks that $(f^{(n)})_{n\in\overline{\mathbb{N}}}$ is uniformly bounded in $C([0,T]; B_{1,2}^{\frac{1}{2}})$. Moreover,

$$\|f^{(n)} - f^\infty\|_{B_{2,1}^{\frac{1}{2}}} \le C\big(\|u^{(n)}\|_{B_{2,1}^{\frac{3}{2}}}^2 + \|u^{(n)}\|_{B_{2,1}^{\frac{3}{2}}}\|u^\infty\|_{B_{2,1}^{\frac{3}{2}}}\big)\|\partial_x(u^{(n)}) - \partial_x(u^\infty)\|_{B_{2,1}^{\frac{1}{2}}}$$
$$+ C\|u^{(n)} - u^\infty\|_{B_{2,1}^{\frac{1}{2}}}\|u^\infty\|_{B_{2,1}^{\frac{3}{2}}}^2. \tag{2.7}$$

Hence, combining the convergence of $z^{(n)}$ in $C([0, T]; B_{2,1}^{\frac{1}{2}})$ with estimates (2.4)-(2.7), we deduce that for large enough $n \in \mathbb{N}$,

$$\left\| \partial_x(u^{(n)}) - \partial_x(u^\infty) \right\|_{B_{2,1}^{\frac{1}{2}}} \leq \varepsilon + C(2M^2 + 1)e^{C(2M^2+1)} \Bigg[\left\| \partial_x(u_0^{(n)}) - \partial_x(u_0^\infty) \right\|_{B_{2,1}^{\frac{1}{2}}}$$
$$+ \int_0^t \left\| u^{(n)} - u^\infty \right\|_{B_{2,1}^{\frac{1}{2}}} d\tau + \int_0^t \left\| \partial_x(u^{(n)}) - \partial_x(u^\infty) \right\|_{B_{2,1}^{\frac{1}{2}}} d\tau \Bigg].$$

Thanks to Gronwall's inequality, we have

$$\left\| \partial_x(u^{(n)}) - \partial_x(u^\infty) \right\|_{L^\infty(0,T;B_{2,1}^{\frac{1}{2}})} \leq C(M, m, T) \left(\left\| \partial_x(u_0^{(n)}) - \partial_x(u_0^\infty) \right\|_{B_{2,1}^{\frac{1}{2}}} + \varepsilon \right)$$

for some constant C depending only on M and b. We have completed the continuity of the map Φ in $C([0, T]; B_{2,1}^{\frac{3}{2}})$.

Now, applying ∂_t to Eq. (1.1) and by the same argument to the resulting equation in terms of $\partial_t u$, we may check the continuity of the map Φ in $C^1([0, T]; B_{2,1}^{\frac{1}{2}})$. □

Proof of Theorem 1.1 Combining the result in Lemma 2.1 with that in Lemma 2.2, one gets the existence and uniqueness of the solution of the Cauchy problem (1.1). And Lemma 2.3 shows that the solution of the Cauchy problem (1.1) depends continuously on the initial data. This completes the proof of Theorem 1.1. □

3 Blow-up criterion and multi-peakon solutions

This section is devoted to the proof of Theorems 1.2-1.5. Theorems 1.2-1.3 are based on the following lemma.

Lemma 3.1 *Let $u_0 \in B_{p,r}^s$ with $1 \leq p, r \leq \infty$ and $s > 1$. Let $u \in L^\infty([0, T]; B_{p,r}^s)$ solve Eq. (1.1) on $[0, T) \times \mathbb{R}$ with the initial data u_0. There exist a constant C_1 depending only on s and p and a universal constant C_2 such that for all $t \in [0, T)$, we have*

$$\left\| u(t) \right\|_{B_{p,r}^s} \leq \left\| u_0 \right\|_{B_{p,r}^s} \exp\left(C_1 \int_0^t \left\| u(\tau) \right\|_{Lip}^2 d\tau \right), \tag{3.1}$$

$$\left\| u(t) \right\|_{Lip} \leq \left\| u_0 \right\|_{Lip} \exp\left(C_2 \int_0^t \left\| u \partial_x u(\tau) \right\|_{L^\infty} d\tau \right). \tag{3.2}$$

Proof Applying the last of Lemma 2.3 in [6] to the Novikov equation and using the fact that $(1 - \partial_x^2)^{-1}$ is a multiplier of order -2 yields

$$\exp\left(-C \int_0^t \left\| u \partial_x u(\tau) \right\|_{L^\infty} d\tau \right) \left\| u(t) \right\|_{B_{p,r}^s}$$
$$\leq \left\| u_0 \right\|_{B_{p,r}^s} + C \int_0^t \exp\left(-C \int_0^\tau \left\| u \partial_x u(\tau) \right\|_{L^\infty} d\tau' \right)$$
$$\cdot \left(\left\| u^3 \right\|_{B_{p,r}^{s-1}} + \left\| u u_x^2 \right\|_{B_{p,r}^{s-1}} + \left\| u_x^3 \right\|_{B_{p,r}^{s-2}} \right) d\tau.$$

As $s - 1 > 0$, according to Lemma 2.2(5) in [6], one gets

$$\left\| u^3 \right\|_{B_{p,r}^{s-1}} + \left\| u u_x^2 \right\|_{B_{p,r}^{s-1}} + \left\| u_x^3 \right\|_{B_{p,r}^{s-2}} \leq C \left\| u \right\|_{Lip}^2 \left\| u \right\|_{B_{p,r}^s}.$$

Therefore

$$
\exp\left(-C\int_0^t \left\|u\partial_x u(\tau)\right\|_{L^\infty} d\tau\right) \left\|u(t)\right\|_{B_{p,r}^s}
$$

$$
\leq \|u_0\|_{B_{p,r}^s} + C\int_0^t \exp\left(-C\int_0^\tau \left\|u\partial_x u(\tau)\right\|_{L^\infty} d\tau'\right) \|u\|_{Lip}^2 \|u\|_{B_{p,r}^s}\, d\tau.
$$

Applying Gronwall's lemma completes the proof of (3.1).

By differentiating once Eq. (1.1) with respect to x, and applying the L^∞ estimate for transport equations, we easily prove that

$$
\exp\left(-C\int_0^t \left\|u\partial_x u(\tau)\right\|_{L^\infty} d\tau\right) \left\|u(t)\right\|_{Lip}
$$

$$
\leq \|u_0\|_{Lip} + C\int_0^t \exp\left(-C\int_0^\tau \left\|u\partial_x u(\tau)\right\|_{L^\infty} d\tau'\right)
$$

$$
\cdot \left\|\frac{b-2}{2}\left(1-\partial_x^2\right)^{-1}(\partial_x u)^3 + \left(1-\partial_x^2\right)^{-1}\partial_x\left(\frac{b}{3}u^3 + \frac{6-b}{2}u(\partial_x u)^2\right)\right\|_{Lip} d\tau.
$$

Since $(1-\partial_x^2)^{-1}f = \frac{1}{2}e^{-|x|} * f$ and the Young inequality, we get

$$
\left\|\frac{b-2}{2}\left(1-\partial_x^2\right)^{-1}(\partial_x u)^3 + \left(1-\partial_x^2\right)^{-1}\partial_x\left(\frac{b}{3}u^3 + \frac{6-b}{2}u(\partial_x u)^2\right)\right\|_{Lip} \leq C'\|u\|_{Lip}^2 \|\partial_x u\|_{L^\infty}
$$

for some universal constant C'. Hence Gronwall's lemma gives inequality (3.2). □

Proof of Theorem 1.2 Let $u \in \bigcap_{T<T^*} E_{p,r}^s(T)$ be such that $\int_0^{T^*} \|u\partial_x u(\tau)\|_{L^\infty}\, d\tau$ is finite. According to inequality (3.2), $\int_0^{T^*} \|u(\tau)\|_{L^\infty}^2\, d\tau$ is also finite. Hence, (3.1) insures that

$$
\|u(t)\|_{B_{p,r}^s} \leq M_{T^*} := \|u_0\|_{B_{p,r}^s} \exp\left(C_1\int_0^t \|u(\tau)\|_{Lip}^2\, d\tau\right) < \infty, \quad \forall t \in [0, T^*). \tag{3.3}
$$

Let $\epsilon > 0$ be such that $2C^2\epsilon M_{T^*} < 1$, where C stands for the constants used in the proof of Lemma 2.1 in [26]. We then have a solution $\tilde{u}(t) \in E_{p,r}^s(\epsilon)$ to Eq. (1.1) with the initial data $u(T^* - \epsilon/2)$. For the sake of uniqueness, $\tilde{u}(t) = u(t + T^* - \epsilon/2)$ on $[0, \epsilon/2)$ so that $\tilde{u}$ extends the solution u beyond T^*. We conclude that $T^* < T_{u_0}^*$ and Theorem 1.2 is proved. □

Proof of Theorem 1.3 Multiplying Eq. (2.1) by u^{2n-1} with $n \in \mathbb{Z}^+$ and integrating by parts, we obtain

$$
\int_{\mathbb{R}} u^{2n-1}\left(u_t + u^2 u_x + F\right) dx = 0
$$

with $F = \frac{b-2}{2}(1-\partial_x^2)^{-1}(\partial_x u)^3 + (1-\partial_x^2)^{-1}\partial_x(\frac{b}{3}u^3 + \frac{6-b}{2}u(\partial_x u)^2)$. Note that the estimates

$$
\int_{\mathbb{R}} u^{2n-1}u_t\, dx = \frac{1}{2n}\frac{d}{dt}\left\|u(x,t)\right\|_{L^{2n}}^{2n} = \left\|u(x,t)\right\|_{L^{2n}}^{2n-1}\frac{d}{dt}\left\|u(x,t)\right\|_{L^{2n}}
$$

and

$$
\left|\int_{\mathbb{R}} u^{2n-1}\left(u^2 u_x\right) dx\right| \leq \left\|uu_x(x,t)\right\|_{L^\infty}\left\|u(x,t)\right\|_{L^{2n}}^{2n}
$$

are true. Moreover, using Hölder's inequality

$$\left| \int_{\mathbb{R}} u^{2n-1} F \, dx \right| \leq \|u(x,t)\|_{L^{2n}}^{2n-1} \|F\|_{L^{2n}},$$

from (2.2) we can obtain

$$\frac{d}{dt} \|u(x,t)\|_{L^{2p}} \leq \|uu_x(x,t)\|_{L^\infty} \|u(x,t)\|_{L^{2p}} + \|F\|_{L^{2p}}.$$

Since $\|f\|_{L^p} \to \|f\|_{L^\infty}$ as $p \to \infty$ for any $f \in L^\infty \cap L^2$ and the operator $(1 - \partial_x^2)^{-1} = \frac{1}{2} e^{-|x|}$, from the above inequality we deduce that

$$\frac{d}{dt} \|u(x,t)\|_{L^\infty} \leq \|u_x\|_{L^\infty} \|u\|_{L^\infty}^2 + \frac{|b-2|}{2} \|u_x(\tau)\|_{L^\infty}^3$$

$$+ \frac{|b|}{3} \|u(\tau)\|_{L^\infty}^3 + \frac{|6-b|}{2} \|u(\tau)\|_{L^\infty} \|u_x(\tau)\|_{L^\infty}^2. \tag{3.4}$$

Next, we give estimates on $\|u_x(x,t)\|_{L^\infty}$. Differentiating (2.1) with respect to x-variable produces the equation

$$u_{xt} + u^2 u_{xx} + 2u u_x^2 + \partial_x F = 0.$$

Similar to the estimate of (3.4), we deduce that

$$\frac{d}{dt} \|u_x(t)\|_{L^\infty} \leq 2\|u_x\|_{L^\infty} \|u\|_{L^\infty}^2 + \frac{|b-2|}{2} \|u_x(\tau)\|_{L^\infty}^3$$

$$+ \frac{2|b|}{3} \|u(\tau)\|_{L^\infty}^3 + |6-b| \|u(\tau)\|_{L^\infty} \|u_x(\tau)\|_{L^\infty}^2. \tag{3.5}$$

Choose

$$H(t) := 2\|u(t)\|_{L^\infty} + \|u_x(t)\|_{L^\infty}.$$

Combining (3.4) with (3.5), we get

$$\frac{d}{dt} H(t) \leq \frac{a}{2} H^3(t), \tag{3.6}$$

with $a = 4\max\{2/3, |b-2|, 2|b|/3, |6-b|/3\}$. Define $T := \frac{1}{a(2\|u_0\|_{L^\infty} + \|u_{0,x}\|_{L^\infty})^2}$. By (3.6), then for all $t < \min\{T, T^*\}$, one can easily get

$$H(t) \leq \frac{H(0)}{\sqrt{1 - aH^2(0)t}}. \tag{3.7}$$

Theorem 1.2 yields that $T^* \geq T$. This completes the proof of Theorem 1.3. □

Proof of Theorem 1.4 Let $w = u - v$. In view of Eq. (2.1), one can get

$$w_t + (u + w)^2 w_x = -(2u - w)w u_x + f + g, \tag{3.8}$$

with

$$f = \left(1 - \partial_x^2\right)^{-1} \partial_x \left[\frac{b}{3} \left(u^3 - v^3\right) + \frac{6-b}{2} \left(u(\partial_x u)^2 - v(\partial_x v)^2\right) \right],$$

$$g = \frac{b-2}{2} \left(1 - \partial_x^2\right)^{-1} \left[(\partial_x u)^3 - (\partial_x v)^3 \right].$$

Using standard energy arguments and integration by parts, we end up with

$$\left\| w(t) \right\|_{B_{p,r}^s} \le \|w_0\|_{B_{p,r}^s} + c \int_0^t \left\| (u+w)^2(\tau) \right\|_{B_{p,r}^s} \left\| w(\tau) \right\|_{B_{p,r}^s} \, d\tau$$

$$+ \int_0^t \left(\left\| (2u-w)wu_x(\tau) \right\|_{B_{p,r}^s} + \left\| (f+g)(\tau) \right\|_{B_{p,r}^s} \right) d\tau. \tag{3.9}$$

Note that

$$\left\| (u+w)^2 \right\|_{B_{p,r}^s} \le c \left(\|u\|_{B_{p,r}^{s+1}}^2 + \|u\|_{B_{p,r}^{s+1}} \|w\|_{B_{p,r}^s} + \|w\|_{B_{p,r}^s}^2 \right),$$

$$\left\| (2u-w)wu_x \right\|_{B_{p,r}^s} \le c \left(\|u\|_{B_{p,r}^{s+1}}^2 + \|u\|_{B_{p,r}^{s+1}} \|w\|_{B_{p,r}^s} \right) \|w\|_{B_{p,r}^s}$$

and

$$\|f+g\|_{B_{p,r}^s} \le \|f\|_{B_{p,r}^s} + \|g\|_{B_{p,r}^s}$$

$$\le c \left(\|u\|_{B_{p,r}^{s+1}}^2 + \|u\|_{B_{p,r}^{s+1}} \|w\|_{B_{p,r}^s} + \|w\|_{B_{p,r}^s}^2 \right) \|w\|_{B_{p,r}^s}.$$

Using the above inequalities, and applying Gronwall's inequality to (3.9), one can easily get that

$$\left\| w(t) \right\|_{B_{p,r}^s}$$

$$\le \|w_0\|_{B_{p,r}^s} \exp \left\{ c \int_0^t \left(\|u(\tau)\|_{B_{p,r}^{s+1}}^2 + \|u(\tau)\|_{B_{p,r}^{s+1}} \|w(\tau)\|_{B_{p,r}^s} + \|w(\tau)\|_{B_{p,r}^s}^2 \right) d\tau \right\}. \tag{3.10}$$

Let

$$P(t) = \exp \left\{ -2c \int_0^t \left(\|u(\tau)\|_{B_{p,r}^{s+1}} \|w(\tau)\|_{B_{p,r}^s} + \|w(\tau)\|_{B_{p,r}^s}^2 \right) d\tau \right\},$$

$$Q(t) = c \left(\|u(\tau)\|_{B_{p,r}^{s+1}} \|w_0\|_{B_{p,r}^s} + \|w_0\|_{B_{p,r}^s}^2 \right) \exp \left\{ 2c \int_0^t \|u(\tau)\|_{B_{p,r}^{s+1}}^2 \, d\tau \right\}.$$

According to (3.10), we obtain

$$\frac{dP(t)}{dt} \ge 2Q(t). \tag{3.11}$$

Integrating (3.11) on $[0,t]$ with $t < T$, by virtue of (3.10), we get

$$\left\| w(t) \right\|_{B_{p,r}^s}^2 \le \frac{\|w_0\|_{B_{p,r}^s}^2 \exp\{2c \int_0^t \|u(\tau)\|_{B_{p,r}^{s+1}}^2 \, d\tau\}}{1 - 2c \int_0^t (\|u(\tau)\|_{B_{p,r}^{s+1}} \|w_0\|_{B_{p,r}^s} + \|w_0\|_{B_{p,r}^s}^2) \exp\{2c \int_0^\tau \|u(\tau')\|_{B_{p,r}^{s+1}}^2 \, d\tau'\} \, d\tau}.$$

If $T_{v_0} \geq T$, for all $t \leq T_{v_0}$, the above inequality implies that

$$
\|w(t)\|_{B_{p,r}^s}^2 \leq \frac{\|w_0\|_{B_{p,r}^s}^2 \exp\{2c \int_0^{T_{v_0}} \|u(\tau)\|_{B_{p,r}^{s+1}}^2 \, d\tau\}}{1 - 2c \int_0^{T_{v_0}} (\|u(\tau)\|_{B_{p,r}^{s+1}} \|w_0\|_{B_{p,r}^s} + \|w_0\|_{B_{p,r}^s}^2) \exp\{2c \int_0^\tau \|u(\tau')\|_{B_{p,r}^{s+1}}^2 \, d\tau'\} \, d\tau}
$$

$$
\leq \infty.
$$

Therefore, $\|w(t)\|_{B_{p,r}^s}$ is uniformly bounded in $[0, T_{v_0}]$. In view of Theorem 1.2, the solution can be extended beyond T_{v_0}. This is in conflict with the definition of T_{v_0}. □

Remark 3.1 If $r = 1$, $s = 1 + 1/p$, in view of $B_{p,1}^{1/p}$ being an algebra, we have (3.10). Thus we also deduce the result of Theorem 1.4.

Proof of Theorem 1.5 We now derive the multi-peakon solutions of Eq. (1.1). Assume that Eq. (1.1) has an N-peakon solution of the form (1.4). It follows from the definition of a weak solution that for any $\psi(t,x) \in C_c^\infty([0,\infty) \times \mathbb{R})$, the solution (1.1) satisfies

$$
\int_0^\infty \int_{\mathbb{R}} \left[u_t + u^2 u_x + \frac{(b-2)}{2}(1 - \partial_x^2)^{-1}(\partial_x u)^3 \right.
$$
$$
\left. + (1 - \partial_x^2)^{-1} \partial_x \left(\frac{b}{3} u^3 + \frac{6-b}{2} u(\partial_x u)^2 \right) \right] \varphi(x) \, dx \, dt = 0, \tag{3.12}
$$

which is equivalent to the following equation:

$$
\int_0^\infty \int_{\mathbb{R}} \left[u_t(\phi - \phi_{xx}) + \frac{1}{3} u^3 \phi_{xxx} + \frac{(b-2)}{2}(\partial_x u)^3 \phi \right.
$$
$$
\left. - \phi_x \left(\frac{b+1}{3} u^3 + \frac{6-b}{2} u(\partial_x u)^2 \right) \right] dx \, dt = 0, \tag{3.13}
$$

where $\varphi = \phi - \phi_{xx}$, $\phi(t,x) \in C_c^\infty([0,\infty) \times \mathbb{R})$.

A straightforward computation gives

$$
\int_0^\infty \int_{\mathbb{R}} u_t(\phi - \phi_{xx}) \, dx \, dt = \sum_{i=1}^N \int_0^\infty \int_{-\infty}^{q_j(t)} (p_j' - p_j q_j') e^{x - q_j} (\phi - \phi_{xx}) \, dx \, dt
$$
$$
+ \sum_{i=1}^N \int_0^\infty \int_{q_j(t)}^\infty (p_j' + p_j q_j') e^{-(x - q_j)} (\phi - \phi_{xx}) \, dx \, dt
$$
$$
= 2 \int_0^\infty \sum_{i=1}^N (p_j' \phi(q_j) + p_j q_j' \phi_x(q_j)) \, dt \tag{3.14}
$$

and

$$
\frac{1}{3} \int_{\mathbb{R}} u^3 \phi_{xxx} \, dx = -\left(\int_{-\infty}^{q_1} + \sum_{j=1}^{N-1} \int_{q_j}^{q_{j+1}} + \int_{q_N}^\infty \right) u^2 u_x \phi_{xx} \, dx
$$
$$
= -u^2 u_x \phi_x \left(|_{-\infty}^{q_1} + \sum_{j=1}^{N-1} |_{q_j}^{q_{j+1}} + |_{q_N}^\infty \right) + \int_{\mathbb{R}} (2u u_x^2 + u^2 u_{xx}) \phi_x \, dx
$$

$$= \left[-u^2 u_x \phi_x + \left(2uu_x^2 + u^3 \right) \phi \right] \left(|_{-\infty}^{q_1} + \sum_{j=1}^{N-1} |_{q_j}^{q_{j+1}} + |_{q_N}^{\infty} \right)$$

$$- \int_{\mathbb{R}} \left(2u_x^3 + 4u^4 u_x + 4u^2 u_x \right) \phi \, dx \tag{3.15}$$

and

$$- \int_{\mathbb{R}} \left(\frac{b+1}{3} u^3 + \frac{6-b}{2} u(\partial_x u)^2 \right) \phi_x \, dx$$

$$= -\left[\left(\frac{b+1}{3} u^3 + \frac{6-b}{2} uu_x^2 \right) \phi \right] \left(|_{-\infty}^{q_1} + \sum_{j=1}^{N-1} |_{q_j}^{q_{j+1}} + |_{q_N}^{\infty} \right)$$

$$+ \int_{\mathbb{R}} \left((b+1)u^2 u_x + (6-b)u^2 u_x + \frac{6-b}{2} u_x^3 \right) dx. \tag{3.16}$$

Thus, combining (3.15) with (3.16), we get

$$\int_{\mathbb{R}} \left[\frac{1}{3} u^3 \phi_{xxx} + \frac{b-2}{2} (\partial_x u)^3 \phi - \phi_x \left(\frac{b+1}{3} u^3 + \frac{6-b}{2} u(\partial_x u)^2 \right) \right] dx$$

$$= \left(-u^2 u_x \phi_x + \frac{b-2}{2} uu_x^2 \phi \right) \left(|_{-\infty}^{q_1} + \sum_{j=1}^{N-1} |_{q_j}^{q_{j+1}} + |_{q_N}^{\infty} \right)$$

$$= -2 \sum_{j=1}^{N} \left[p_j \left(\sum_{i=1}^{N} p_i e^{-|q_j - q_i|} \right)^2 \phi_x(q_j) \right]$$

$$- 2(b-2) \sum_{j=1}^{N} \left[p_j \left(\sum_{i=1}^{N} p_i e^{-|q_j - q_i|} \right) \left(\sum_{i=1}^{N} p_i \operatorname{sgn}(q_j - q_i) e^{-|q_j - q_i|} \right) \phi(q_j) \right]. \tag{3.17}$$

Substituting (3.14), (3.17) into (3.13), we obtain the following system:

$$\begin{aligned} p_j' &= \left(\sum_{i=1}^{N} p_i e^{-|q_j - q_i(t)|} \right)^2, \\ q_j' &= (b-2) q_j \left(\sum_{i=1}^{N} p_i e^{-|q_j - q_i|} \right) \left(\sum_{i=1}^{N} p_i \operatorname{sgn}(q_j - q_i) e^{-|q_j - q_i|} \right), \end{aligned} \tag{3.18}$$

which leads to the conclusion of Theorem 1.5. □

4 Analysis of the Novikov equation in weighted spaces

In this section, for the convenience of the readers, we first present some standard definitions. In general, a weight function is simply a non-negative function. A weight function $v : \mathbb{R}^n \to \mathbb{R}$ is called sub-multiplicative if

$$v(x+y) \le v(x)v(y) \quad \text{for all } x, y \in \mathbb{R}^n.$$

Given a sub-multiplicative function v, a positive function ϕ is v-moderate if and only if

$$\exists C_0 > 0 : \phi(x+y) \le C_0 v(x) \phi(y) \quad \text{for all } x, y \in \mathbb{R}^n.$$

If ϕ is ν-moderate for some sub-multiplicative function ν, then we say that ϕ is moderate. This is the usual terminology in time-frequency analysis papers [34]. Let us recall the most standard examples of such weights. Let

$$\phi(x) = \phi_{a,b,c,d}(x) = e^{a|x|^b}\big(1 + |x|\big)^c \log\big(e + |x|\big)^d.$$

We have (see [33]) the following conditions:

(i) For $a, c, d \geq 0$ and $0 \leq b \leq 1$, such weight is sub-multiplicative.

(ii) If $a, c, d \in \mathbb{R}$ and $0 \leq b \leq 1$, then ϕ is moderate. More precisely, $\phi_{a,b,c,d}$ is $\phi_{\alpha,\beta,\gamma,\delta}$-moderate for $|a| \leq \alpha$, $|b| \leq \beta$, $|c| \leq \gamma$ and $|d| \leq \delta$.

The elementary properties of sub-multiplicative and moderate weights can be found in [33]. Next, we prove Theorem 1.6.

Proof of Theorem 1.6 We define

$$F(u) = \frac{b-2}{2}(\partial_x u)^3 + \partial_x\left(\frac{b}{3}u^3 + \frac{6-b}{2}u(\partial_x u)^2\right).$$

We also introduce the kernel $G(x) = \frac{1}{2}e^{-|x|}$. Then Eq. (1.1) can be rewritten as

$$u_t + u^2\partial_x u + G * F(u) = 0. \tag{4.1}$$

Note that from the assumption $u \in C([0, T], H^s)$, $s > 3/2$, we get

$$M \equiv \sup_{t \in [0,T]} \big(\|u(t)\|_\infty + \|\partial_x u(t)\|_\infty\big) < \infty.$$

For any $N \in \mathbb{Z}^+$, let us consider the N-truncations

$$f(x) = f_N(x) = \begin{cases} \phi(x) & \text{if } \phi(x) \leq N, \\ N & \text{if } \phi(x) > N. \end{cases}$$

Observe that $f : \mathbb{R} \to \mathbb{R}$ is a locally absolutely continuous function such that

$$\|f\|_\infty \leq N, \qquad |f'(x)| \leq A|f(x)| \quad \text{a.e.}$$

In addition, if $C_1 = \max\{C_0, \alpha^{-1}\}$, where $\alpha = \inf_{x \in \mathbb{R}} \nu(x) > 0$, then

$$f(x + y) \leq C_1\nu(x)f(y), \quad \forall x, y \in \mathbb{R}.$$

Indeed, let us introduce the set $U_N = \{x : \phi(x) \leq N\}$, if $y \in U_N$, then $f(x + y) \leq \phi(x + y) \leq C_0\nu(x)f(y)$; if $y \notin U_N$, then $f(x + y) \leq N = f(y) \leq \alpha^{-1}\nu(x)f(y)$.

The constant C^1 being independent on N shows that the N-truncations of a ν-moderate weight are uniformly ν-moderate with respect to N.

We start considering the case $2 \leq p < \infty$. Multiplying Eq. (4.1) by f and then by $|uf|^{p-2}(uf)$, we get, after integration,

$$\int_{\mathbb{R}} |uf|^{p-2}(uf)(\partial_t uf)\,dx + \int_{\mathbb{R}} |uf|^p(u\partial_x u)\,dx + \int_{\mathbb{R}} |uf|^{p-2}(uf)\big(f \cdot G * F(u)\big)\,dx = 0. \tag{4.2}$$

Note that the estimates

$$\int_{\mathbb{R}} |uf|^{p-2}(uf)(\partial_t uf)\,dx = \frac{1}{p}\frac{d}{dt}\|uf\|_{L^p}^p = \|uf\|_{L^p}^{p-1}\frac{d}{dt}\|uf\|_{L^p}$$

and

$$\left|\int_{\mathbb{R}} (uf)^p(u\partial_x u)\,dx\right| \le \|u\partial_x u\|_{L^\infty}\|uf\|_{L^p}^p \le M^2\|uf\|_{L^p}^p$$

are true. Moreover, we get

$$\left|\int_{\mathbb{R}} |uf|^{p-2}(uf)\big[f\cdot\big(G*F(u)\big)\big]\,dx\right|$$
$$\le \|uf\|_{L^p}^{p-1}\left\|f\cdot\left\{G*\left[\partial_x\left(\frac{b}{3}u^3+\frac{6-b}{2}u(\partial_x u)^2\right)+\frac{b-2}{2}(\partial_x u)^3\right]\right\}\right\|_{L^p}$$
$$\le \|uf\|_{L^p}^{p-1}\left\{\big\|(\partial_x G)v\big\|_{L^1}\left\|f\left(\frac{|b|}{3}u^3+\frac{|6-b|}{2}u(\partial_x u)^2\right)\right\|_{L^p}\right.$$
$$\left.+\frac{|b-2|}{2}\|Gv\|_{L^1}\big\|f(\partial_x u)^3\big\|_{L^p}\right\}$$
$$\le CM^2\|uf\|_{L^p}^{p-1}\big(\|uf\|_{L^p}+\big\|(\partial_x u)f\big\|_{L^p}\big).$$

In the first inequality we used Hölder's inequality, and in the second inequality we applied Proposition 3.2 in [33], and in the last one we used condition (1.6). Here, C depends only on v and ϕ. From (4.2) we can obtain

$$\frac{d}{dt}\|uf\|_{L^p} \le C_1 M^2\|uf\|_{L^p}+C_2 M^2\big\|(\partial_x u)f\big\|_{L^p}. \tag{4.3}$$

Next, we give estimates on $u_x f$. Differentiating (4.1) with respect to x-variable, next multiplying by f produces the equation

$$\partial_t\big[(\partial_x u)f\big]+u^2 f\partial_x^2 u+\big[(\partial_x u)f\big](u\partial_x u)+f\big[\partial_x G*F(u)\big]=0.$$

Multiplying this equation by $|f\partial_x u|^{p-2}(f\partial_x u)$ with $p\in\mathbb{Z}^+$, integrating the result in the x-variable, we note that

$$\int_{\mathbb{R}} |f\partial_x u|^{p-2}(f\partial_x u)\partial_t\big[(\partial_x u)f\big]\,dx = \|f\partial_x u\|_{L^p}^{p-1}\frac{d}{dt}\|f\partial_x u\|_{L^p}$$

and

$$\left|\int_{\mathbb{R}} |f\partial_x u|^{p-2}(f\partial_x u)\big[f\partial_x\big(G*F(u)\big)\big]\,dx\right|$$
$$\le \|f\partial_x u\|_{L^p}^{p-1}\big\|f\partial_x\big(G*F(u)\big)\big\|_{L^p}$$
$$\le CM^2\|f\partial_x u\|_{L^p}^{p-1}\big(\|uf\|_{L^p}+\big\|(\partial_x u)f\big\|_{L^p}\big).$$

In the third inequality we applied the pointwise bound $|\partial_x G(x)| \le \frac{1}{2} e^{-|x|}$ and the condition

$$\left| \int_{\mathbb{R}} |f \partial_x u|^{p-2} (f \partial_x u) u^2 f \partial_x^2 u \, dx \right|$$

$$= \left| \int_{\mathbb{R}} |f \partial_x u|^{p-2} (f \partial_x u) u^2 \big[\partial_x (f \partial_x u) - (\partial_x u)(\partial_x f) \big] \, dx \right|$$

$$= \left| \int_{\mathbb{R}} u^2 \partial_x \left(\frac{|f \partial_x u|^p}{p} \right) - \int_{\mathbb{R}} |f \partial_x u|^{p-2} (f \partial_x u) u^2 (\partial_x u)(\partial_x f) \, dx \right|$$

$$\le 2/p M^2 \|f \partial_x u\|_{L^p}^p + A M^2 \|f \partial_x u\|_{L^p}^p.$$

In the last inequality we used $|\partial_x f(x)| \le A f(x)$ for a.e. x. Thus, we get

$$\frac{d}{dt} \|f \partial_x u\|_{L^p} \le C_3 M^2 \|u f\|_{L^p} + C_4 M^2 \big\| (\partial_x u) f \big\|_{L^p}. \tag{4.4}$$

Now, combing inequalities (4.3) and (4.4) and then integrating yields

$$\big\| u(t) f \big\|_{L^p} + \big\| (\partial_x u)(t) f \big\|_{L^p} \le \big(\|u_0 f\|_{L^p} + \|\partial_x u_0 f\|_{L^p} \exp(C M^2 t) \big) \quad \text{for all } t \in [0, T].$$

Since $f(x) = f_N(x) \uparrow \phi(x)$ as $N \to \infty$ for a.e. $x \in \mathbb{R}$. Recalling that $u_0 \phi \in L^p(\mathbb{R})$ and $\partial_x u_0 \phi \in L^p(\mathbb{R})$, we get

$$\big\| u(t) \phi \big\|_{L^p} + \big\| (\partial_x u)(t) \phi \big\|_{L^p} \le \big(\|u_0 \phi\|_{L^p} + \|\partial_x u_0 \phi\|_{L^p} \exp(C M^2 t) \big) \quad \text{for all } t \in [0, T].$$

At last, we treat the case $p = \infty$. We have $u_0, \partial_x u_0 \in L^2 \cap L^\infty$ and $f(x) = f_N(x) \in L^\infty$. Hence, we have

$$\big\| u(t) f \big\|_{L^q} + \big\| (\partial_x u)(t) f \big\|_{L^q} \le \big(\|u_0 f\|_{L^q} + \|\partial_x u_0 f\|_{L^q} \big) \exp(C M^2 t), \quad q \in [2, \infty). \tag{4.5}$$

The last factor on the right-hand side is independent of q. Since $\|f\|_{L^p} \to \|f\|_{L^\infty}$ as $p \to \infty$ for any $f \in L^\infty \cap L^2$ implies that

$$\big\| u(t) f \big\|_{L^\infty} + \big\| (\partial_x u)(t) f \big\|_{L^\infty} \le \big(\|u_0 f\|_{L^\infty} + \|\partial_x u_0 f\|_{L^\infty} \big) \exp(C M^2 t).$$

The last factor on the right-hand side is independent of N. Now taking $N \to \infty$ implies that estimate (4.5) remains valid for $p = \infty$. □

Proof of Theorem 1.7 We start observing that $\phi^{1/3}$ is a $\nu^{1/3}$-moderate weight such that $(\phi^{1/3})'(x) \le \frac{A}{3} \phi^{1/3}(x)$. Moreover, $\inf_{\mathbb{R}} \nu^{1/3} > 0$. By condition (1.7), $\nu^{1/3} e^{-|x|/3} \in L^{3p}(\mathbb{R})$, hence Hölder's inequality implies that $\nu^{1/3} e^{-|x|} \in L^1(\mathbb{R})$. Then Theorem 1.6 applied with $p = 3$ to the weight $\phi^{1/3}$ yields

$$\big\| u(t) \phi^{1/3} \big\|_{L^3} + \big\| (\partial_x u)(t) \phi^{1/3} \big\|_{L^3} \le \big(\big\| u_0 \phi^{1/3} \big\|_{L^3} + \big\| \partial_x u_0 \phi^{1/3} \big\|_{L^3} \exp(C M^2 t) \big).$$

Arguing as in the proof of Theorem 1.6, we get

$$\frac{d}{dt} \|u f\|_{L^p} \le M^2 \|u f\|_{L^p} + \big\| f \big(G * F(u) \big) \big\|_{L^p} \quad \text{for } p < \infty, \tag{4.6}$$

where $f(x) = f_N(x) = \min\{\phi(x), N\}$.

On the other hand,

$$\frac{d}{dt}\|f\partial_x u\|_{L^p} \le CM^2\|(\partial_x u)f\|_{L^p} + \|f(\partial_x G * F(u))\|_{L^p} \quad \text{for } p < \infty. \tag{4.7}$$

Note that $F(u) = \frac{b-2}{2}(\partial_x u)^3 + \partial_x(\frac{b}{3}u^3 + \frac{6-b}{2}u(\partial_x u)^2)$ and

$$\|f(G * F(u))\|_{L^p}$$

$$\le C\left(\|(\partial_x G)v\|_{L^p}\left\|\phi\left(\frac{b}{3}u^3 + \frac{6-b}{2}u(\partial_x u)^2\right)\right\|_{L^1} + \frac{|b-2|}{2}\|Gv\|_{L^p}\|\phi(\partial_x u)^3\|_{L^1}\right)$$

$$\le C\big(\|\phi^{1/3}u\|_{L^3}^3 + \|\phi u(\partial_x u)^2\|_{L^1} + \|\phi^{1/3}(\partial_x u)\|_{L^3}^3\big)$$

$$\le C\big(\|\phi^{1/3}u\|_{L^3}^3 + \|\phi^{1/3}u\|_{L^3}\|\phi^{2/3}(\partial_x u)^2\|_{L^{3/2}} + \|\phi^{1/3}(\partial_x u)\|_{L^3}^3\big)$$

$$\le K_0 \exp(3CM^2 t).$$

The constant on the right-hand side is dependent on N. Similarly, recalling that $\partial_x^2 G = G - \delta$, we obtain

$$\|f(\partial_x G * F(u))\|_{L^p} \le \left\|f\left(G * \left(u^3 + \frac{3}{2}u(\partial_x u)^2\right)\right)\right\|_{L^p} + \left\|f\left(u^3 + \frac{3}{2}u(\partial_x u)^2\right)\right\|_{L^p}$$

$$+ \|f(\partial_x G * (\partial_x u)^3)\|_{L^p}$$

$$\le K_0 \exp(3CM^2 t) + CM^2\big(\|uf\|_{L^p} + \|(\partial_x u)f\|_{L^p}\big).$$

Plugging the two last estimates in (4.6)-(4.7) and summing, we obtain

$$\frac{d}{dt}\big(\|u(t)f\|_{L^p} + \|(\partial_x u)(t)f\|_{L^p}\big) \le K_1 M^2\big(\|u_0 f\|_{L^p} + \|\partial_x u_0 f\|_{L^p}\big) + 2K_0 \exp(3CM^2 t).$$

Integrating and finally letting $N \to \infty$ yields the conclusion in the case $2 \le p < \infty$. The constants throughout the proof are independent on p. Therefore, for $p = \infty$, one can rely on the result established for finite exponents q, and then let $q \to \infty$. The rest argument is fully similar to that of Theorem 1.6. □

Authors' contributions
This paper is the result of joint work of all authors who contributed equally to the final version of this paper. All authors read and approved the final manuscript.

Author details
[1]College of Mathematics Science, Chongqing Normal University, Chongqing, 401331, P.R. China. [2]College of Continuing Education, Chongqing Normal University, Chongqing, 400047, P.R. China.

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