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Generalized lacunary Δ^m-statistically convergent sequences of fuzzy numbers using a modulus function

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Abstract
In this paper, we introduce the space of lacunary strongly $\Delta^m_{(p)}$-summable sequences of fuzzy numbers and discuss relations between Δ^m-statistically convergent sequences and lacunary Δ^m-statistically convergent sequences of fuzzy numbers. We also study inclusion relations using different arbitrary lacunary sequences.
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1 Introduction

The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [1], and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as topological spaces, similarity relations and fuzzy orderings, fuzzy mathematical programming *etc.* Later on, various types of sequence spaces of fuzzy numbers have been constructed by several authors such as Matloka [2], Nanda [3], Nuray and Savaş [4], Mursaleen and Basarır [5], Malkowsky *et al.* [6], Tripathy and Chandra [7], Tripathy and Borgogain [8] and so on. Later on, the fuzzy sequence space got momentum after the introduction of new convergence methods and theories in the process as well as the requirement. Some of them are statistical convergence, lacunary statistical convergence *etc.*

Nuray and Savaş [4] introduced the idea of statistical convergence of fuzzy numbers and Nuray [9] introduced the related concept of convergence with the help of a lacunary sequence. Using their ideas, many authors such as Kwon and Shim [10], Bilgin [11], Altin *et al.* [12], Esi [13], Tripathy and Baruah [14, 15], Tripathy and Dutta [16, 17] and others constructed different types of sequence spaces.

2 Definitions and preliminaries

Definition 2.1 A fuzzy number X is a mapping $X : \mathbb{R} \to [0,1]$ associating each real number t with its grade of membership $X(t)$.

Definition 2.2 If there exists $t \in \mathbb{R}$ such that $X(t) = 1$, then the fuzzy number X is called normal.

Definition 2.3 A fuzzy number X is said to be convex if $X(t) \geq X(s) \wedge X(r) = \min\{X(s), X(r)\}$, where $s < t < r$.

Definition 2.4 A fuzzy number X is said to be upper semi-continuous if for each $\varepsilon > 0$, $X^{-1}([0, a + \varepsilon))$ for all $a \in [0,1]$ is open in the usual topology of $\mathbb{R}$.

$L(\mathbb{R})$ denotes the set of all upper semi-continuous, normal, convex fuzzy numbers such that $[X]^\alpha = \overline{\{t \in \mathbb{R} : X(t) > 0\}}$ is compact, where $\overline{\{t \in \mathbb{R} : X(t) > 0\}}$ denotes the closure of the set $\{t \in \mathbb{R} : X(t) > 0\}$ in the usual topology of $\mathbb{R}$.

Definition 2.5 The set $L(\mathbb{R})$ forms a linear space under addition and scalar multiplication in terms of α-level sets as defined below:

$$[X + Y]^\alpha = [X]^\alpha + [Y]^\alpha \quad \text{and} \quad [\lambda X]^\alpha = \lambda [X]^\alpha \quad \text{for each } 0 \leq \alpha \leq 1,$$

where X^α is given as

$$X^\alpha = \begin{cases} t : X(t) \geq \alpha & \text{if } \alpha \in (0,1], \\ t : X(t) > 0 & \text{if } \alpha = 0. \end{cases}$$

For each $\alpha \in [0,1]$, the set X^α is a closed, bounded and nonempty interval of $\mathbb{R}$.

Let D denote the set of all closed and bounded intervals $A = [a_1, a_2]$ on the real line $\mathbb{R}$. For $A, B \in D$, (D, d) is a complete metric space where the metric d is defined as

$$d(A, B) = \max\{|a_1 - b_1|, |a_2 - b_2|\}$$

for any $A = [a_1, a_2]$ and $B = [b_1, b_2]$.

It is easy to verify that $\overline{d} : L(\mathbb{R}) \times L(\mathbb{R}) \to \mathbb{R}$, defined by

$$\overline{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d(X^\alpha, Y^\alpha),$$

is a metric on $L(\mathbb{R})$.

Definition 2.6 Let $X = (X_k)$ be a sequence of fuzzy numbers. Then the sequence $X = (X_k)$ is said to be Δ^m-convergent to the fuzzy number X_0, denoted as $\lim_{k \to \infty} \Delta^m X_k = X_0$, if for every $\varepsilon > 0$, there exists a positive integer k_0 such that $\overline{d}(\Delta^m X_k, X_0) < \varepsilon$ for all $k > k_0$.

Definition 2.7 The sequence $X = ((X_k)_{k=1}^\infty)$ of fuzzy numbers is said to be Δ^m-statistically convergent to the fuzzy number ξ if for every $\varepsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{n} \left| \{k \leq n : \overline{d}(\Delta^m X_k, \xi) \geq \varepsilon\} \right| = 0.$$

In this case, we write $X_k \to \xi(S^F(\Delta^m))$ and $S^F(\Delta^m)$ denotes the set of all Δ^m-statistically convergent sequences of fuzzy numbers.

By a lacunary sequence we mean an increasing sequence of integers $\theta = (k_r)$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper, the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$.

Definition 2.8 Let θ be a lacunary sequence. A lacunary refinement of $\theta = (k_r)$ is a lacunary sequence $\theta' = (k'_r)$ satisfying $\{k_r\} \subseteq \{k'_r\}$.

Definition 2.9 The sequence $X = (X_k)$ of fuzzy numbers is said to be lacunary Δ^m-statistically convergent to the fuzzy number ξ if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} \bigl|\bigl\{k \in I_r : \overline{d}\bigl(\Delta^m X_k, \xi\bigr) \geq \varepsilon\bigr\}\bigr| = 0.$$

In this case, we write $X_k \to \xi(S_\theta^F(\Delta^m))$, and $S_\theta^F(\Delta^m)$ denotes the set of all lacunary Δ^m-statistically convergent sequences of fuzzy numbers.

Definition 2.10 A metric $\overline{d}$ on $L(\mathbb{R})$ is said to be translation invariant if $\overline{d}(X + Z, Y + Z) = \overline{d}(X, Y)$ for all $X, Y, Z \in L(\mathbb{R})$.

Lemma 2.1 (Mursaleen and Basarır [5]) *If $\overline{d}$ is a translation invariant metric on $L(\mathbb{R})$, then*

(i) $\overline{d}(X + Y, \overline{0}) \leq \overline{d}(X, \overline{0}) + \overline{d}(Y, \overline{0})$,
(ii) $\overline{d}(\lambda X, \overline{0}) \leq |\lambda| \overline{d}(X, \overline{0})$, $|\lambda| > 1$.

Lemma 2.2 (Maddox [18]) *Let a_k, b_k for all k be sequences of complex numbers, and let (p_k) be a bounded sequence of positive real numbers, then*

$$|a_k + b_k|^{p_k} \leq C\bigl(|a_k|^{p_k} + |b_k|^{p_k}\bigr)$$

and

$$|\lambda|^{p_k} \leq \max\bigl(1, |\lambda|^H\bigr),$$

where $C = \max(1, 2^{H-1})$, $H = \sup p_k$ and λ is any complex number.

Lemma 2.3 (Maddox [18]) *Let $a_k \geq 0$, $b_k \geq 0$ for all k be sequences of complex numbers and $1 \leq p_k \leq \sup p_k < \infty$, then*

$$\left(\sum_k |a_k + b_k|^{p_k}\right)^{\frac{1}{M}} \leq \left(\sum_k |a_k|^{p_k}\right)^{\frac{1}{M}} + \left(\sum_k |b_k|^{p_k}\right)^{\frac{1}{M}},$$

where $M = \max(1, H)$, $H = \sup p_k$.

3 Main results

Now, we introduce the space $N_\theta^F(\Delta_{(p)}^m, f)$ as follows:

$$N_\theta^F\bigl(\Delta_{(p)}^m, f\bigr) = \left\{X = (X_k) : X_k \in L(\mathbb{R}) \text{ such that } \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \bigl(f\bigl(\overline{d}\bigl(\Delta^m X_k, \xi\bigr)\bigr)\bigr)^{p_k} = 0\right\},$$

where $\theta = (k_r)$ is a lacunary sequence, f is any modulus function and $p = (p_k)$ is any sequence of strictly positive real numbers.

Now we define a lacunary strongly Δ_p^m-summable sequence of fuzzy numbers as follows.

The sequence $X = (X_k)$ is said to be lacunary strongly $\Delta_{(p)}^m$-summable to the fuzzy number $\xi \in L(\mathbb{R})$ if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \bigl(f\bigl(\overline{d}\bigl(\Delta^m X_k, \xi\bigr)\bigr)\bigr)^{p_k} = 0.$$

In this case, we write $X_k \to \xi(N_\theta^F(\Delta_{(p)}^m, f))$. Thus the class $N_\theta^F(\Delta_{(p)}^m, f)$ denotes the set of all lacunary strongly $\Delta_{(p)}^m$-summable sequences of fuzzy numbers.

For suitable choices of m, f and p_k, some of the known sequence spaces, which are obtained from the space $N_\theta^F(\Delta_{(p)}^m, f)$, are as follows.

(i) Let $f(x) = x$, $m = 0$ and $p_k \equiv p$ for all $k \in \mathbb{N}$, the sequence space $N_\theta^F(\Delta_{(p)}^m, f)$ represents the space N_θ studied by Kwon and Shim [10].

(ii) Let $f(x) = x$, $m = 1$ and $p_k \equiv 1$ for all $k \in \mathbb{N}$, the sequence space $N_\theta^F(\Delta_{(p)}^m, f)$ denotes the space $N_\theta(\Delta)$ investigated by Bilgin [11].

(iii) Let $f(x) = x$, the sequence space $N_\theta^F(\Delta_{(p)}^m, f)$ denotes the space $N_\theta(\Delta_p^m)$ studied by Altin *et al.* [12].

(iv) Let $f(x) = x$ and $p_k \equiv 1$ for all $k \in \mathbb{N}$, the sequence space $N_\theta^F(\Delta_{(p)}^m, f)$ reduces to the space $N_\theta(\Delta^m, F)$ investigated by Esi [13].

Thus, the study of the sequence space $N_\theta^F(\Delta_{(p)}^m, f)$ gives a unified approach to many of the earlier known spaces.

Theorem 3.1 *Let (p_k) be a bounded sequence of positive real numbers. Then the class $N_\theta^F(\Delta_{(p)}^m, f)$ is a linear space over $\mathbb{R}$.*

Proof Using Lemma 2.1, Lemma 2.2, the subadditivity property of a modulus function f and the result $f(\lambda x) \le (1 + [|\lambda|])f(x)$, it is easy to show that $N_\theta^F(\Delta_{(p)}^m, f)$ is a linear space over the real field $\mathbb{R}$. □

Theorem 3.2 *Let (p_k) be a bounded sequence of positive real numbers such that* $\inf p_k > 0$. *Then the sequence space $N_\theta^F(\Delta_{(p)}^m, f)$ is a complete metric space with respect to the metric*

$$h(X, Y) = \sum_{i=1}^{m} f\bigl(\overline{d}(X_i, Y_i)\bigr) + \sup_r \biggl(\frac{1}{h_r} \sum_{k\in I_r} \bigl(f\bigl(\overline{d}\bigl(\Delta^m X_k, \Delta^m Y_k\bigr)\bigr)\bigr)^{p_k}\biggr)^{\frac{1}{M}},$$

where $M = \max\{1, \sup p_k\}$.

Proof It is easy to verify that h gives a metric on $N_\theta^F(\Delta_{(p)}^m, f)$. To show completeness, let us assume that (X^u) is a Cauchy sequence in $N_\theta^F(\Delta_{(p)}^m, f)$, where $X^u = (X_k^u)_{k=1}^\infty \in N_\theta^F(\Delta_{(p)}^m, f)$ for each $u \in \mathbb{N}$. So, for given $\varepsilon > 0$, there exists $u_0 \in \mathbb{N}$ such that

$$g\bigl(X^u, X^v\bigr) < \varepsilon \quad \text{for all } u, v \ge u_0,$$

i.e.,

$$\sum_{i=1}^{m} f\bigl(\overline{d}\bigl(X_i^u, X_i^v\bigr)\bigr) + \sup_r \biggl(\frac{1}{h_r} \sum_{k\in I_r} \bigl(f\bigl(\overline{d}\bigl(\Delta^m X_k^u, \Delta^m X_k^v\bigr)\bigr)\bigr)^{p_k}\biggr)^{\frac{1}{M}} < \varepsilon \quad \text{for all } u, v \ge u_0.$$

This means that

$$\sum_{i=1}^{m} f\big(\overline{d}\big(X_i^u, X_i^v\big)\big) < \varepsilon \quad \text{for all } u, v \geq u_0 \tag{3.1}$$

as well as

$$\frac{1}{h_r}\sum_{k\in I_r}\big(f\big(\overline{d}\big(\Delta^m X_k^u, \Delta^m X_k^v\big)\big)\big)^{p_k} < \varepsilon \quad \text{for all } u, v \geq u_0 \text{ and for all } r. \tag{3.2}$$

Since f is a modulus function, so equation (3.1) gives $\overline{d}(X_i^u, X_i^v) < \varepsilon_1$ for all $u, v \geq u_0$, for some ε_1 such that $\varepsilon > \varepsilon_1 > 0$ and for each $i = 1, 2, \ldots, m$, *i.e.*,

$$\big(X_i^u\big) \text{ is a Cauchy sequence in } L(\mathbb{R}) \quad \text{for each } i = 1, 2, \ldots, m. \tag{3.3}$$

Similarly, as f is a modulus function, so by choosing suitable $\varepsilon_2 > 0$, equation (3.2) gives $\overline{d}(\Delta^m X_k^u, \Delta^m X_k^v) < \varepsilon_2$ for all $u, v \geq u_0$ and for each k, *i.e.*,

$$\big(\Delta^m X_k^u\big) \text{ is a Cauchy sequence in } L(\mathbb{R}) \quad \text{for each } k \in \mathbb{N}. \tag{3.4}$$

Using equation (3.3) and equation (3.4), it can be easily shown that (X_k^u) is a Cauchy sequence in $L(\mathbb{R})$. But $L(\mathbb{R})$ is complete, so the sequence (X_k^u) converges to $X = (X_k)$ in $L(\mathbb{R})$ as $u \to \infty$.

Keeping u fixed and letting $v \to \infty$ in equation (3.1) and equation (3.2), we get

$$\sum_{i=1}^{m} f\big(\overline{d}\big(X_i^u, X_i\big)\big) < \varepsilon \quad \text{for all } u \geq u_0$$

and

$$\frac{1}{h_r}\sum_{k\in I_r}\big(f\big(\overline{d}\big(\Delta^m X_k^u, \Delta^m X_k\big)\big)\big)^{p_k} < \varepsilon \quad \text{for all } u \geq u_0 \text{ and for all } r, \tag{3.5}$$

i.e.,

$$g\big(X^u, X\big) < \varepsilon \quad \text{for all } u \geq u_0.$$

Next we show that the limit point $X \in N_\theta^F(\Delta_{(p)}^m, f)$, for which the proof is as follows.

Since $X^u = (X_k^u) \in N_\theta^F(\Delta_{(p)}^m, f)$ for $u \in \mathbb{N}$, so for each u, there exist $L^u \in L(\mathbb{R})$ and $r_u \in \mathbb{N}$ such that

$$\frac{1}{h_r}\sum_{k\in I_r}\big(f\big(\overline{d}\big(\Delta^m X_k^u, L^u\big)\big)\big)^{p_k} < \varepsilon \quad \text{for all } r \geq r_u. \tag{3.6}$$

Similarly, for each v, there exist $L^v \in L(\mathbb{R})$ and $r_v \in \mathbb{N}$ such that

$$\frac{1}{h_r}\sum_{k\in I_r}\big(f\big(\overline{d}\big(\Delta^m X_k^v, L^v\big)\big)\big)^{p_k} < \varepsilon \quad \text{for all } r \geq r_v. \tag{3.7}$$

Now let $u, v \geq u_0$ and $r_0 = \max(r_u, r_v)$. Then from equation (3.2), equation (3.6), equation (3.7) and using Lemma 2.2, we have

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r}\bigl(f\bigl(\overline{d}\bigl(L^u, L^v\bigr)\bigr)\bigr)^{p_k} &\leq C\frac{1}{h_r}\sum_{k\in I_r}\bigl(f\bigl(\overline{d}\bigl(L^u, \Delta^m X_k^u\bigr)\bigr)\bigr)^{p_k} \\ &\quad + C\frac{1}{h_r}\sum_{k\in I_r}\bigl(f\bigl(\overline{d}\bigl(\Delta^m X_k^u, \Delta^m X_k^v\bigr)\bigr)\bigr)^{p_k} \\ &\quad + C\frac{1}{h_r}\sum_{k\in I_r}\bigl(f\bigl(\overline{d}\bigl(\Delta^m X_k^v, L^v\bigr)\bigr)\bigr)^{p_k} \\ &< 3C\varepsilon \quad \text{for all } u, v \geq u_0 \text{ and for all } r \geq r_0. \end{aligned} \tag{3.8}$$

Now using the fact that the modulus function is monotone and for a suitable choice of $\varepsilon_3 > 0$, we get

$$\overline{d}\bigl(L^u, L^v\bigr) < \varepsilon_3 \quad \text{for all } u, v \geq u_0,$$

i.e., (L^u) is a Cauchy sequence in $L(\mathbb{R})$. Since $L(\mathbb{R})$ is complete, so there exists $\xi \in L(\mathbb{R})$ such that $L^u \to \xi$ as $u \to \infty$. Keeping u fixed and letting $v \to \infty$ in equation (3.8), we get

$$\frac{1}{h_r}\sum_{k\in I_r}\bigl(f\bigl(\overline{d}\bigl(L^u, \xi\bigr)\bigr)\bigr)^{p_k} < 3C\varepsilon \quad \text{for } u \geq u_0 \text{ and for } r \geq r_0. \tag{3.9}$$

Hence, from equation (3.5), equation (3.6) and equation (3.9), we have

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r}\bigl(f\bigl(\overline{d}\bigl(\Delta^m X_k, \xi\bigr)\bigr)\bigr)^{p_k} &\leq C\frac{1}{h_r}\sum_{k\in I_r}\bigl(f\bigl(\overline{d}\bigl(\Delta^m X_k, \Delta^m X_k^{u_0}\bigr)\bigr)\bigr)^{p_k} \\ &\quad + C\frac{1}{h_r}\sum_{k\in I_r}\bigl(f\bigl(\overline{d}\bigl(\Delta^m X_k^{u_0}, L^{u_0}\bigr)\bigr)\bigr)^{p_k} \\ &\quad + C\frac{1}{h_r}\sum_{k\in I_r}\bigl(f\bigl(\overline{d}\bigl(L^{u_0}, \xi\bigr)\bigr)\bigr)^{p_k} \\ &< 2C\varepsilon + 3C^2\varepsilon \equiv \varepsilon_1 \quad \text{for all } r \geq r_0, \end{aligned}$$

which implies $\frac{1}{h_r}\sum_{k\in I_r}(f(\overline{d}(\Delta^m X_k, \xi)))^{p_k} \to 0$ as $r \to \infty$, *i.e.*, $X = (X_k) \in N_\theta^F(\Delta_{(p)}^m, f)$ and hence the sequence space $N_\theta^F(\Delta_{(p)}^m, f)$ is a complete metric space. □

Theorem 3.3 *Let $\theta = (k_r)$ be a lacunary sequence and $X = (X_k)$ be a sequence of fuzzy numbers. Then*:

(i) $X_k \to \xi(N_\theta^F(\Delta_{(p)}^m, f))$ *implies* $X_k \to \xi(S_\theta^F(\Delta^m))$.

(ii) *If f is a bounded modulus function and* $X_k \to \xi(S_\theta^F(\Delta^m))$, *then* $X_k \to \xi(N_\theta^F(\Delta_{(p)}^m, f))$.

Proof Easy, so omitted. □

Theorem 3.4 *Let $\theta = (k_r)$ be a lacunary sequence, and let $X = (X_k)$ be a sequence of fuzzy numbers. Then*:

(i) $W^F(\Delta^m, f) \subseteq N_\theta^F(\Delta^m, f)$ *if and only if* $\liminf_r q_r > 1$.

(ii) $N_\theta^F(\Delta^m, f) \subseteq W^F(\Delta^m, f)$ *if and only if* $\limsup_r q_r < \infty$, *where*

$$W^F(\Delta^m, f) = \left\{ (X_k) : \frac{1}{n} \sum_{k=1}^{n} f\left(\overline{d}\left(\Delta^m X_k, \xi\right)\right) \to 0 \text{ as } n \to \infty \right\}$$

and

$$N_\theta^F(\Delta^m, f) = \left\{ (X_k) : \frac{1}{h_r} \sum_{k \in I_r} f\left(\overline{d}\left(\Delta^m X_k, \xi\right)\right) \to 0 \text{ as } r \to \infty \right\}.$$

Proof (i) Assume that $\liminf_r q_r > 1$. Then there exists $\delta > 0$ such that $q_r \geq 1 + \delta$ for sufficiently large r.

Let $X_k \to \xi(W^F(\Delta^m, f))$. Then

$$\frac{1}{n} \sum_{k=1}^{n} f\left(\overline{d}\left(\Delta^m X_k, \xi\right)\right) \to 0 \quad \text{as } n \to \infty.$$

Consider

$$\begin{aligned} \frac{1}{h_r} \sum_{k \in I_r} f\left(\overline{d}\left(\Delta^m X_k, \xi\right)\right) &= \frac{1}{h_r} \sum_{k=1}^{k_r} f\left(\overline{d}\left(\Delta^m X_k, \xi\right)\right) - \frac{1}{h_r} \sum_{k=1}^{k_{r-1}} f\left(\overline{d}\left(\Delta^m X_k, \xi\right)\right) \\ &\geq \frac{k_r}{h_r} \left(\frac{1}{k_r} \sum_{k=1}^{k_r} f\left(\overline{d}\left(\Delta^m X_k, \xi\right)\right) \right) \\ &\quad - \frac{k_{r-1}}{h_r} \left(\frac{1}{k_{r-1}} \sum_{k=1}^{k_{r-1}} f\left(\overline{d}\left(\Delta^m X_k, \xi\right)\right) \right). \end{aligned}$$

Since $h_r = k_r - k_{r-1}$, we have

$$\frac{k_r}{h_r} = \frac{k_r}{k_r - k_{r-1}} = \frac{q_r}{q_r - 1} = 1 + \frac{1}{q_r - 1} \leq 1 + \frac{1}{\delta} = \frac{\delta + 1}{\delta}$$

and

$$\frac{k_{r-1}}{h_r} = \frac{k_{r-1}}{k_r - k_{r-1}} = \frac{1}{q_r - 1} \leq \frac{1}{\delta}.$$

Since $X \in W^F(\Delta^m, f)$. So both the terms $\frac{1}{k_r} \sum_{k=1}^{k_r} f(\overline{d}(\Delta^m X_k, \xi))$ and $\frac{1}{k_{r-1}} \sum_{k=1}^{k_{r-1}} f(\overline{d}(\Delta^m X_k, \xi))$ converge to 0 as $r \to \infty$ and hence $\frac{1}{h_r} \sum_{k_r \in I_r} f(\overline{d}(\Delta^m X_k, \xi))$ converges to 0 as $r \to \infty$, *i.e.*, $X \in N_\theta^F(\Delta^m, f)$ and hence $W^F(\Delta^m, f) \subseteq N_\theta^F(\Delta^m, f)$.

Conversely, let $X_k \to \xi(W^F(\Delta^m, f))$ imply $X_k \to \xi(N_\theta^F(\Delta^m, f))$ but $\liminf_r q_r = 1$.

Since θ is a lacunary sequence, we can select a subsequence $(k_{r(j)})$ of θ satisfying

$$\frac{k_{r(j)}}{k_{r(j)-1}} < 1 + \frac{1}{j} \quad \text{and} \quad \frac{k_{r(j)-1}}{k_{r(j-1)}} > j, \quad \text{where } r(j) \geq r(j-1) + 2.$$

Define

$$\Delta^m X_k = \begin{cases} \overline{1} & \text{if } k \in I_{r(j)} \text{ for some } j = 1, 2, \ldots, \\ \overline{0} & \text{otherwise.} \end{cases}$$

Now we prove that the sequence X for which $\Delta^m X_k$, which is defined as above, is strongly Δ^m-summable but not lacunary strongly Δ^m-summable. Since

$$\frac{1}{h_{r(j)}}\sum_{k\in I_{r(j)}} f\bigl(\overline{d}\bigl(\Delta^m X_k,\overline{0}\bigr)\bigr) = f(1) \quad \text{for } r = r(j)$$

and

$$\frac{1}{h_r}\sum_{k\in I_r} f\bigl(\overline{d}\bigl(\Delta^m X_k,\overline{0}\bigr)\bigr) = 0 \quad \text{for } r \neq r(j),$$

which implies $X_k \nrightarrow \overline{0}(N_\theta^F(\Delta^m,f))$, but if n is sufficiently large integer and j is the unique integer with $k_{r(j)-1} < n \le k_{r(j+1)-1}$, then, since $n \to \infty$ implies $j \to \infty$, we have

$$\frac{1}{n}\sum_{k=1}^{n} f\bigl(\overline{d}\bigl(\Delta^m X_k,\overline{0}\bigr)\bigr) \le \frac{k_{r(j)}}{k_{r(j)-1}} \le \frac{k_{r(j)-1}+h_{r(j)}}{k_{r(j)-1}} \le \frac{2}{j} \to 0 \quad \text{as } n \to \infty,$$

i.e., $X_k \to \overline{0}(W^F(\Delta^m,f))$. This shows that $W^F(\Delta^m,f) \notin N_\theta^F(\Delta^m,f)$, which leads to a contradiction. Therefore it proves that $\liminf q_r > 1$.

(ii) (Sufficiency) Suppose $\limsup_r q_r < \infty$. Then there exists a constant $A > 0$ such that $q_r < A$ for all $r \in \mathbb{N}$. Let $X_k \to \xi(N_\theta^F(\Delta^m,f))$. So $\lim_{r\to\infty} \frac{1}{h_r}\sum_{k\in I_r} f(\overline{d}(\Delta^m X_k,\xi)) = 0$.

Let $\varepsilon > 0$. Then we can find $R > 0$ and $K > 0$ such that

$$\sup_{r\ge R}\frac{1}{h_r}\sum_{r\in I_r} f\bigl(\overline{d}\bigl(\Delta^m X_k,\xi\bigr)\bigr) < \varepsilon$$

and

$$\frac{1}{h_r}\sum_{i\in I_j} f\bigl(\overline{d}\bigl(\Delta^m X_k,\xi\bigr)\bigr) < K \quad \text{for all } j = 1,2,\ldots.$$

Then if n is any integer with $k_{r-1} < n \le k_r$, where $r > R$, then we can write

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n} f\bigl(\overline{d}\bigl(\Delta^m X_k,\xi\bigr)\bigr) &\le \frac{1}{k_{r-1}}\sum_{k=1}^{k_r} f\bigl(\overline{d}\bigl(\Delta^m X_k,\xi\bigr)\bigr) \\
&= \frac{1}{k_{r-1}}\biggl(\sum_{I_1} f\bigl(\overline{d}\bigl(\Delta^m X_k,\xi\bigr)\bigr) + \sum_{I_2} f\bigl(\overline{d}\bigl(\Delta^m X_k,\xi\bigr)\bigr) + \cdots \\
&\quad + \sum_{I_r} f\bigl(\overline{d}\bigl(\Delta^m X_k,\xi\bigr)\bigr)\biggr), \quad \text{where } I_r = (k_{r-1},k_r] \\
&= \frac{k_1}{k_{r-1}}\frac{1}{h_1}\sum_{I_1} f\bigl(\overline{d}\bigl(\Delta^m X_k,\xi\bigr)\bigr) \\
&\quad + \frac{k_2-k_1}{k_{r-1}}\frac{1}{h_2}\sum_{I_2} f\bigl(\overline{d}\bigl(\Delta^m X_k,\xi\bigr)\bigr) + \cdots \\
&\quad + \frac{k_R-k_{R-1}}{k_{r-1}}\frac{1}{h_R}\sum_{I_R} f\bigl(\overline{d}\bigl(\Delta^m X_k,\xi\bigr)\bigr) + \cdots
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{k_r - k_{r-1}}{k_{r-1}} \frac{1}{h_r} \sum_{I_r} f\big(\bar{d}\big(\Delta^m X_k, \xi\big)\big) \\
&\le \frac{k_r}{k_{r-1}} \left(\sup_{1\le i\le R} \frac{1}{h_i} \sum_{i} f\big(\bar{d}\big(\Delta^m X_k, \xi\big)\big) \right) \\
&+ \frac{k_r - k_R}{k_{r-1}} \left(\sup_{i\ge R} \frac{1}{h_i} \sum_{i} f\big(\bar{d}\big(\Delta^m X_k, \xi\big)\big) \right) \\
&< K \frac{k_r}{k_{r-1}} + \varepsilon \left(q_r - \frac{k_R}{k_{r-1}} \right) \\
&< K \frac{k_r}{k_{r-1}} + \varepsilon q_r < K \frac{k_r}{k_{r-1}} + A\varepsilon,
\end{aligned}
$$

which implies $\frac{1}{n}\sum_{k=1}^{n} f(\bar{d}(\Delta^m X_k, \xi)) \to 0$ as $n \to \infty$ and it follows that $X_k \to \xi(W^F(\Delta^m, f))$.

(Necessity) Suppose that $X_k \to \xi(W_\theta^F(\Delta^m, f))$ implies $X_k \to \xi(N^F(\Delta^m, f))$ but $\limsup_r q_r = \infty$.

Since θ is any lacunary sequence, we can select a subsequence $(k_{r(j)})$ of θ satisfying $q_{r(j)} > j$ and define a bounded difference sequence $X = (X_k)$ by

$$
\Delta^m X_k = \begin{cases} \bar{1} & \text{if } k_{r_{(j)-1}} < k \le 2k_{r_{(j)-1}} \text{ for some } j = 1, 2, \ldots, \\ \bar{0} & \text{otherwise.} \end{cases}
$$

Let $\varepsilon > 0$ be given. Then we have if $r = r(j)$, then

$$
\frac{1}{h_{r(j)}} \sum_{I_{r(j)}} f\big(\bar{d}\big(\Delta^m X_k, \bar{0}\big)\big) = \frac{2k_{r(j)-1} - k_{r(j)-1}}{k_{r(j)} - k_{r(j)-1}} f(1) = \frac{k_{r(j)-1}}{k_{r(j)} - k_{r(j)-1}} f(1) < \frac{f(1)}{j-1},
$$

and if $r \ne r(j)$, then

$$
\frac{1}{h_{r(j)}} \sum_{I_{r(j)}} f\big(\bar{d}\big(\Delta^m X_k, \bar{0}\big)\big) = 0,
$$

which implies that $X_k \to \bar{0}(N_\theta^F(\Delta^m, f))$.

For the above sequence and for $k = 1, 2, \ldots, k_{r(j)}$,

$$
\frac{1}{k_{r(j)}} \sum_{k=1}^{k_{r(j)}} f\big(\bar{d}\big(\Delta^m X_k, \bar{1}\big)\big) \ge \frac{1}{k_{r(j)}} (k_{r(j)} - 2k_{r(j)-1}) f(1) = \left(1 - \frac{2}{q_{r(j)}}\right) f(1) > \left(1 - \frac{2}{j}\right) f(1),
$$

and for $k = 1, 2, \ldots, 2k_{r(j)-1}$,

$$
\frac{1}{2k_{r(j)-1}} \sum_{k=1}^{2k_{r(j)-1}} f\big(\bar{d}\big(\Delta^m X_k, \bar{1}\big)\big) \ge \frac{k_{r(j)-1}}{2k_{r(j)-1}} f(1) = \frac{f(1)}{2}.
$$

It proves that $X = (X_k) \notin (W^F(\Delta^m, f))$ and hence the result follows immediately. □

Theorem 3.5 *Let $\theta = (k_r)$ be a lacunary sequence and $X = (X_k)$ be a sequence of fuzzy numbers. Then*:

(i) $S_\theta^F(\Delta^m) \subseteq S^F(\Delta^m)$ *if and only if* $\limsup_r q_r < \infty$.
(ii) $S^F(\Delta^m) \subseteq S_\theta^F(\Delta^m)$ *if and only if* $\liminf_r q_r > 1$.

Proof The proof can be established following the technique used in Theorem 3.4. □

Theorem 3.6 *If* $\theta' = (k_r')$ *is a lacunary refinement of* $\theta = (k_r)$ *and* $X_k \to \xi(S_{\theta'}^F(\Delta^m))$, *then* $X_k \to \xi(S_\theta^F(\Delta^m))$.

Theorem 3.7 *Let* $\beta = (l_i)$ *be a lacunary refinement of the lacunary sequence* $\theta = (k_i)$. *Let* $I_i = (k_{i-1}, k_i]$ *and* $J_i = (l_{i-1}, l_i]$, $i = 1, 2, 3, \ldots$, *be the corresponding intervals of* θ *and* β, *respectively, and* $h_i = k_i - k_{i-1}$ *and* $g_i = l_i - l_{i-1}$, *respectively. If there exists* $\delta > 0$ *such that*

$$\frac{g_j}{h_i} \geq \delta \quad \text{for every } J_j \subseteq I_i,$$

then $X_k \to \xi(S_\theta^F(\Delta^m))$ *implies* $X_k \to \xi(S_\beta^F(\Delta^m))$.

Proof Let $\varepsilon > 0$ be given, $X_k \to \xi(S_\theta^F(\Delta^m))$ and for every J_j, we can find I_i such that $J_j \subseteq I_i$, then we have

$$\begin{aligned} \frac{1}{g_j}\bigl|\bigl\{k \in J_j : \overline{d}\bigl(\Delta^m X_k, \xi\bigr) \geq \varepsilon\bigr\}\bigr| &= \frac{h_i}{g_j}\frac{1}{h_i}\bigl|\bigl\{k \in J_j : \overline{d}\bigl(\Delta^m X_k, \xi\bigr) \geq \varepsilon\bigr\}\bigr| \\ &\leq \frac{h_i}{g_j}\frac{1}{h_i}\bigl|\bigl\{k \in I_i : \overline{d}\bigl(\Delta^m X_k, \xi\bigr) \geq \varepsilon\bigr\}\bigr| \quad (\text{as } J_j \subseteq I_i) \\ &\leq \frac{1}{\delta}\frac{1}{h_i}\bigl|\bigl\{k \in I_i : \overline{d}\bigl(\Delta^m X_k, \xi\bigr) \geq \varepsilon\bigr\}\bigr|, \end{aligned}$$

which implies that $X_k \to \xi(S_\beta^F(\Delta^m))$. □

Theorem 3.8 *Let* $\beta = (l_i)$ *and* $\theta = (k_i)$ *be two lacunary sequences. Let* $I_i = (k_{i-1}, k_i]$ *and* $J_i = (l_{i-1}, l_i]$, $i = 1, 2, 3, \ldots$, *be the corresponding intervals of* θ *and* β, *respectively, and* $I_{ij} = I_i \cap J_j$, $i, j = 1, 2, 3, \ldots$, *and* $h_i = k_i - k_{i-1}$ *and* $g_i = l_i - l_{i-1}$. *If there exists* $\delta > 0$ *such that*

$$\frac{|I_{ij}|}{h_i} \geq \delta \quad \text{for every } i, j = 1, 2, 3, \ldots \text{ provided } I_{i,j} \neq \emptyset,$$

then $X_k \to \xi(S_\theta^F(\Delta^m))$ *implies* $X_k \to \xi(S_\beta^F(\Delta^m))$.

Proof Let $\alpha = \beta \cup \theta$. Then α is a lacunary refinement of both the lacunary sequences β and θ. The interval sequence of α is $\{I_{ij} = I_j \cap J_j : I_{ij} \neq \phi\}$.

Let $\varepsilon > 0$ be given, $X_k \to \xi(S_\theta^F(\Delta^m))$, and for every $I_{i,j}$, we can find I_i such that $I_{ij} \subseteq I_i$ and $I_{ij} \neq \emptyset$. Then we have

$$\begin{aligned} \frac{1}{|I_{ij}|}\bigl|\bigl\{k \in I_{ij} : \overline{d}\bigl(\Delta^m X_k, \xi\bigr) \geq \varepsilon\bigr\}\bigr| &= \frac{h_i}{|I_{ij}|}\frac{1}{h_i}\bigl|\bigl\{k \in I_{ij} : \overline{d}\bigl(\Delta^m X_k, \xi\bigr) \geq \varepsilon\bigr\}\bigr| \\ &\leq \frac{h_i}{|I_{ij}|}\frac{1}{h_i}\bigl|\bigl\{k \in I_i : \overline{d}\bigl(\Delta^m X_k, \xi\bigr) \geq \varepsilon\bigr\}\bigr| \quad (\text{as } I_{ij} \subseteq I_i) \\ &\leq \frac{1}{\delta}\frac{1}{h_i}\bigl|\bigl\{k \in I_i : \overline{d}\bigl(\Delta^m X_k, \xi\bigr) \geq \varepsilon\bigr\}\bigr|, \end{aligned}$$

which implies that $X_k \to \xi(S_\alpha^F(\Delta^m))$.

Since α is a lacunary refinement of β, so by Theorem 3.6 it follows that $X_k \to \xi(S_\alpha^F(\Delta^m))$ implies $X_k \to \xi(S_\beta^F(\Delta^m))$ and hence the result follows. □

Theorem 3.9 *Let $\beta = (l_i)$ and $\theta = (k_i)$ be two lacunary sequences. Let $I_i = (k_{i-1}, k_i]$, $J_i = (l_{i-1}, l_i]$, $i = 1, 2, 3, \ldots$, and $I_{ij} = I_i \cap J_j$, $i, j = 1, 2, 3, \ldots$. If there exists $\delta > 0$ such that*

$$\frac{|I_{ij}|}{(h_i + g_j)} \geq \delta \quad \textit{for every } i, j = 1, 2, 3, \ldots \textit{ provided } I_{i,j} \neq \emptyset,$$

then $X_k \to \xi(S_\theta^F(\Delta^m))$ if and only if $X_k \to \xi(S_\beta^F(\Delta^m))$.

Definition 3.1 Let $\beta = (l_i)$ and $\theta = (k_r)$ be two lacunary sequences. Let $J_i = (l_{i-1}, l_i]$, $I_i = (k_{i-1}, k_i]$, $i = 1, 2, \ldots$. There exist a sequence X and a fuzzy number L such that $X_k \to \xi(S_\beta^F(\Delta^m))$ and $X_k \nrightarrow \xi(S_\theta^F(\Delta^m))$ if and only if there exist $(s_r), (t_r) \subseteq \mathbb{N}$ and $\delta > 0$ which satisfy the following conditions:

(i) $J_{s_r} \cap I_{t_r} \neq \phi$;
(ii) $\lim_{r\to\infty} \frac{|J_{s_r}|}{|I_{t_r}|} = \infty$;
(iii) $\frac{|J_{s_r} \cap I_{t_r}|}{|I_{t_r}|} \geq \delta$, $r = 1, 2, \ldots$.

Proof Let there exist a sequence $X = (X_k)$ and a fuzzy number ξ such that $X_k \to \xi(S_\beta^F(\Delta^m))$ and $X_k \nrightarrow \xi(S_\theta^F(\Delta^m))$. Then we can find a subsequence $(t_r) \subseteq \mathbb{N}$, $\varepsilon > 0$ and $\delta > 0$ such that

$$\left(\frac{1}{|I_{t_r}|}\right)\left|\left\{k \in I_{t_r} : \overline{d}\left(\Delta^m X_k, \xi\right) \geq \varepsilon\right\}\right| \geq \delta, \quad r = 1, 2, \ldots.$$

For each t_r, there exist a positive integer s_r and a whole number u_r such that

$$J_{s_r+j} \cap I_{t_r} \neq \phi \quad \text{for every } j = 0, 1, 2, \ldots, r = 1, 2, \ldots.$$

Then we can write

$$\begin{aligned}
\delta &\leq \left(\frac{1}{|I_{t_r}|}\right)\left|\left\{k \in I_{t_r} : \overline{d}\left(\Delta^m X_k, \xi\right) \geq \varepsilon\right\}\right| \\
&= \left(\frac{1}{|I_{t_r}|}\right)\left|\left\{k \in \left(\bigcup_{j=0}^{n} J_{s_r+j}\right) \cap I_{t_r} : \overline{d}\left(\Delta^m X_k, \xi\right) \geq \varepsilon\right\}\right| \\
&= \left(\frac{1}{|I_{t_r}|}\right)\sum_{j=0}^{u_r}\left|\left\{k \in J_{s_r+j} \cap I_{t_r} : \overline{d}\left(\Delta^m X_k, \xi\right) \geq \varepsilon\right\}\right| \\
\Rightarrow \quad \delta &\leq \left(\frac{1}{|I_{t_r}|}\right)\sum_{j=0,u_r}\left|\left\{k \in J_{s_r+j} \cap I_{t_r} : \overline{d}\left(\Delta^m X_k, \xi\right) \geq \varepsilon\right\}\right| \\
&\quad + \left(\frac{1}{|I_{t_r}|}\right)\sum_{0<j<u_r}\left|\left\{k \in J_{s_r+j} \cap I_{t_r} : \overline{d}\left(\Delta^m X_k, \xi\right) \geq \varepsilon\right\}\right| \\
&= \left(\frac{1}{|I_{t_r}|}\right)\sum_{j=0,u_r}\left|\left\{k \in J_{s_r+j} \cap I_{t_r} : \overline{d}\left(\Delta^m X_k, \xi\right) \geq \varepsilon\right\}\right| \\
&\quad + \left(\frac{1}{|I_{t_r}|}\right)\sum_{0<j<u_r}\left|\left\{k \in J_{s_r+j} : \overline{d}\left(\Delta^m X_k, \xi\right) \geq \varepsilon\right\}\right|
\end{aligned}$$

$$= \left(\frac{1}{|I_{t_r}|}\right)\sum_{j=0,u_r}\left|\left\{k\in J_{s_r+j}\cap I_{t_r}:\overline{d}\left(\Delta^m X_k,\xi\right)\geq\varepsilon\right\}\right|$$
$$+\sum_{0<j<u_r}\left(\frac{|J_{s_r+j}|}{|I_{t_r}|}\right)\left(\frac{1}{|J_{s_r+j}|}\right)\left|\left\{k\in J_{s_r+j}:\overline{d}\left(\Delta^m X_k,\xi\right)\geq\varepsilon\right\}\right|. \tag{3.10}$$

Since $X_k \to \xi(S_\beta^F(\Delta^m))$, so for sufficiently large $s_r + j$, we have

$$\left(\frac{1}{|J_{s_r+j}|}\right)\left|\left\{k\in J_{s_r+j}:\overline{d}\left(\Delta^m X_k,\xi\right)\geq\varepsilon\right\}\right| < \frac{\delta}{2}. \tag{3.11}$$

It can be seen that $\sum_{0<j<u_r}(\frac{|J_{s_r+j}|}{|I_{t_r}|}) \leq 1$. Hence, by using equation (3.11) in equation (3.10), we get

$$\delta \leq \left(\frac{1}{|I_{t_r}|}\right)\left|\left\{k\in I_{t_r}:\overline{d}\left(\Delta^m X_k,\xi\right)\geq\varepsilon\right\}\right|$$
$$\leq \left(\frac{1}{|I_{t_r}|}\right)\sum_{j=0,u_r}\left|\left\{k\in J_{s_r+j}\cap I_{t_r}:\overline{d}\left(\Delta^m X_k,\xi\right)\geq\varepsilon\right\}\right| + \frac{\delta}{2}.$$

This implies that $(\frac{1}{|I_{t_r}|})\sum_{j=0,u_r}|\{k\in J_{s_r+j}\cap I_{t_r}:\overline{d}(\Delta^m X_k,\xi)\geq\varepsilon\}| \geq \frac{\delta}{2}$, which implies that at least one of the following two inequalities holds:

$$\left(\frac{1}{|I_{t_r}|}\right)\left|\left\{k\in J_{s_r}\cap I_{t_r}:\overline{d}\left(\Delta^m X_k,\xi\right)\geq\varepsilon\right\}\right| \geq \frac{\delta}{4} \tag{3.12}$$

or

$$\left(\frac{1}{|I_{t_r}|}\right)\left|\left\{k\in J_{s_r+u_r}\cap I_{t_r}:\overline{d}\left(\Delta^m X_k,\xi\right)\geq\varepsilon\right\}\right| \geq \frac{\delta}{4}. \tag{3.13}$$

Suppose that equation (3.12) holds. Since $|\{k\in J_{s_r}\cap I_{t_r}:\overline{d}(\Delta^m X_k,\xi)\geq\varepsilon\}| \leq |J_{s_r}\cap I_{t_r}|$, so using equation (3.12), we conclude that $\frac{\delta}{4} \leq \frac{|J_{s_r}\cap I_{t_r}|}{|I_{t_r}|}$, which proves (iii).

For such s_r, t_r chosen in the proof of (iii), from equation (3.12), we have

$$\frac{\delta}{4} \leq \left(\frac{1}{|I_{t_r}|}\right)\left|\left\{k\in J_{s_r}\cap I_{t_r}:\overline{d}\left(\Delta^m X_k,\xi\right)\geq\varepsilon\right\}\right|$$
$$\leq \left(\frac{|J_{s_r}|}{|I_{t_r}|}\right)\left(\frac{1}{|J_{s_r}|}\right)\left|\left\{k\in J_{s_r}\cap I_{t_r}:\overline{d}\left(\Delta^m X_k,\xi\right)\geq\varepsilon\right\}\right|. \tag{3.14}$$

Since $(\frac{1}{|J_{s_r}|})|\{k\in J_{s_r}\cap I_{t_r}:\overline{d}(\Delta^m X_k,\xi)\geq\varepsilon\}| \to 0$ as $r\to\infty$, from equation (3.14) we have $\frac{|J_{s_r}|}{|I_{t_r}|} \to \infty$ as $r\to\infty$, which implies that condition (ii) is satisfied.

Conversely, let for the two lacunary sequences $\beta = (l_i)$ and $\theta = (k_i)$ there exist sequences $(s_r), (t_r) \subseteq \mathbb{N}$ and $\delta > 0$ which satisfy the above three conditions. Define

$$\Delta^m X_k = \begin{cases} \overline{1} & \text{if } k\in J_{s_r}\cap I_{t_r}, \\ \overline{0} & \text{otherwise.} \end{cases}$$

Then, for any $0<\varepsilon<1$, if $j\neq s_r$ for any $r = 1,2,3,\ldots$, $\frac{1}{|J_r|}|\{k\in J_r:\overline{d}(\Delta^m X_k,\overline{0})\geq\varepsilon\}| = 0$ and if $j = s_r$ for some r, $\frac{1}{J_{s_r}}|\{k\in J_{s_r}:\overline{d}(\Delta^m X_k,\overline{0})\geq\varepsilon\}| = \frac{|J_{s_r}\cap I_{t_r}|}{|J_{s_r}|} \leq \frac{|I_{t_r}|}{|J_{s_r}|} \to 0$ as $r\to\infty$. Hence $X_k \to \overline{0}(S_\beta^F(\Delta^m))$.

But $(\frac{1}{|I_{t_r}|})|\{k \in I_{t_r} : \overline{d}(\Delta^m X_k, \overline{0}) \geq \varepsilon\}| = \frac{|J_{s_r} \cap I_{t_r}|}{|I_{t_r}|} \geq \delta$ for $r = 1, 2, \ldots$, which implies $X_k \nrightarrow \overline{0}(S_\theta^F(\Delta^m))$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
PDS in this paper introduced the space of lacunary strongly $\Delta_m^{(p)}$-summable sequences of fuzzy numbers by using a modulus function, defined a suitable metric for the completeness property and gave the idea to introduce the relation between two arbitrary lacunary sequences. SM proved the relation between the spaces of lacunary strongly Δ^m-summable sequences using a modulus function and Δ^m-summable sequences using a modulus function of fuzzy numbers and does some inclusion relations between any two arbitrary lacunary sequences. All authors read and approved the final manuscript.

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