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Strong and △-convergence theorems for total asymptotically nonexpansive nonself mappings in CAT(0) spaces

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Abstract

The purpose of this paper is to study the existence theorems of fixed points, △-convergence and strong convergence theorems for total asymptotically nonexpansive nonself mappings in the framework of CAT(0) spaces. The convexity and closedness of a fixed point set of such mappings are also studied. Our results generalize, unify and extend several comparable results in the existing literature.

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1 Introduction

In recent years, CAT(0) spaces have attracted the attention of many authors as they have played a very important role in different aspects of geometry [1]. Kirk [2, 3] showed that a nonexpansive mapping defined on a bounded closed convex subset of a complete CAT(0) space has a fixed point. Since then, the fixed point theory in CAT(0) spaces has been rapidly developed and many papers have appeared (see, *e.g.*, [4–8]).

In 1976, the concept of △-convergence in general metric spaces was coined by Lim [9]. In 2008, Kirk *et al.* [8] specialized this concept to CAT(0) spaces and proved that it is very similar to the weak convergence in the Banach space setting. Dhompongsa *et al.* [6] and Abbas *et al.* [10] obtained △-convergence theorems for the Mann and Ishikawa iterations in the CAT(0) space setting.

Motivated by the work going on in this direction, the purpose of this paper is twofold. We investigate the existence theorems of fixed points, the convexity and closedness of a fixed point set in CAT(0) spaces for total asymptotically nonexpansive nonself mappings which is essentially wider than that of the asymptotically nonexpansive nonself mappings and the asymptotically nonexpansive mapping in the intermediate sense. We also study sufficient conditions for △-convergence and strong convergence of a sequence generated by finite or an infinite family of total asymptotically nonexpansive nonself mappings in CAT(0) spaces.

2 Preliminaries

Let (X,d) be a metric space and $x,y \in X$ with $d(x,y) = l$. A *geodesic path* from x to y is an isometry $c:[0,l] \to X$ such that $c(0) = x$ and $c(l) = y$. The image of a geodesic path is called a *geodesic segment*. A metric space X is a (uniquely) *geodesic space* if every two points of X are joined by only one geodesic segment. A *geodesic triangle* $\triangle(x_1,x_2,x_3)$ in a geodesic space X consists of three points x_1, x_2, x_3 of X and three geodesic segments joining each pair of vertices. A *comparison triangle* of a geodesic triangle $\triangle(x_1,x_2,x_3)$ is the triangle $\bar{\triangle}(x_1,x_2,x_3) := \triangle(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ in the Euclidean space R^2 such that

$$d(x_i,x_j) = d_{R^2}(\bar{x}_i,\bar{x}_j), \quad \forall i,j = 1,2,3.$$

A geodesic space X is a CAT(0) *space* if for each geodesic triangle $\triangle(x_1,x_2,x_3)$ in X and its comparison triangle $\bar{\triangle} := \triangle(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ in R^2, the CAT(0) inequality

$$d(x,y) \le d_{R^2}(\bar{x},\bar{y}) \tag{2.1}$$

is satisfied for all $x,y \in \triangle$ and $\bar{x},\bar{y} \in \bar{\triangle}$.

A thorough discussion of these spaces and their important role in various branches of mathematics are given in [11–15].

Let C be a nonempty subset of a metric space (X,d). Recall that a mapping $T : C \to X$ is said to be *nonexpansive* if

$$d(Tx, Ty) \le d(x,y), \quad \forall x,y \in C. \tag{2.2}$$

T is said to be an *asymptotically nonexpansive nonself mapping* if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ such that

$$d\bigl(T(PT)^{n-1}x, T(PT)^{n-1}y\bigr) \le k_n d(x,y), \quad \forall x,y \in C, n \ge 1. \tag{2.3}$$

Let (X,d) be a metric space, and C be a nonempty and closed subset of X. Recall that C is said to be a *retract* of X if there exists a continuous map $P : X \to C$ such that $Px = x$, $\forall x \in C$. A map $P : X \to C$ is said to be a *retraction* if $P^2 = P$. If P is a retraction, then $Py = y$ for all y in the range of P.

Definition 2.1 [16] Let X and C be the same as above. A mapping $T : C \to X$ is said to be $(\{\mu_n\},\{\nu_n\},\zeta)$*-total asymptotically nonexpansive nonself mapping* if there exist nonnegative sequences $\{\mu_n\}$, $\{\nu_n\}$ with $\mu_n \to 0$, $\nu_n \to 0$ and a strictly increasing continuous function $\zeta : [0,\infty) \to [0,\infty)$ with $\zeta(0) = 0$ such that

$$d\bigl(T(PT)^{n-1}x, T(PT)^{n-1}y\bigr) \le d(x,y) + \nu_n \zeta\bigl(d(x,y)\bigr) + \mu_n, \quad \forall n \ge 1, x,y \in C, \tag{2.4}$$

where P is a nonexpansive retraction of X onto C.

Remark 2.2 From the definitions, it is to know that each nonexpansive mapping is an asymptotically nonexpansive nonself mapping with a sequence $\{k_n = 1\}$, and each asymptotically nonexpansive nonself mapping is a $(\{\mu_n\},\{\nu_n\},\zeta)$-total asymptotically nonexpansive nonself mapping with $\mu_n = 0$, $\nu_n = k_n - 1$, $\forall n \ge 1$ and $\zeta(t) = t$, $t \ge 0$.

Definition 2.3 [16] A nonself mapping $T : C \to X$ is said to be *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$d\bigl(T(PT)^{n-1}x, T(PT)^{n-1}y\bigr) \le Ld(x,y), \quad \forall n \ge 1, x,y \in C. \tag{2.5}$$

The following lemma plays an important role in our paper.

In this paper, we write $(1-t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that

$$d(z,x) = td(x,y), \qquad d(z,y) = (1-t)d(x,y). \tag{2.6}$$

We also denote by $[x,y]$ the geodesic segment joining from x to y, that is, $[x,y] = \{(1-t)x \oplus ty : t \in [0,1]\}$.

A subset C of a CAT(0) space is convex if $[x,y] \subset C$ for all $x, y \in C$.

Lemma 2.4 [17] *A geodesic space X is a* CAT(0) *space if and only if the following inequality holds*:

$$d^2\big((1-t)x \oplus ty, z\big) \le (1-t)d^2(x,z) + td^2(y,z) - t(1-t)d^2(x,y) \tag{2.7}$$

for all $x, y, z \in X$ and all $t \in [0,1]$. In particular, if x, y, z are points in a CAT(0) *space and $t \in [0,1]$, then*

$$d\big((1-t)x \oplus ty, z\big) \le (1-t)d(x,z) + td(y,z). \tag{2.8}$$

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r\big(x, \{x_n\}\big) = \limsup_{n\to\infty} d(x, x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r\big(\{x_n\}\big) = \inf\big\{r\big(x, \{x_n\}\big) : x \in X\big\}. \tag{2.9}$$

The *asymptotic radius* $r_C(\{x_n\})$ *of* $\{x_n\}$ *with respect to* $C \subset X$ is given by

$$r_C\big(\{x_n\}\big) = \inf\big\{r\big(x, \{x_n\}\big) : x \in C\big\}. \tag{2.10}$$

The *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A\big(\{x_n\}\big) = \big\{x \in X : r\big(x, \{x_n\}\big) = r\big(\{x_n\}\big)\big\}. \tag{2.11}$$

And the *asymptotic center* $A_C(\{x_n\})$ *of* $\{x_n\}$ *with respect to* $C \subset X$ is the set

$$A_C\big(\{x_n\}\big) = \big\{x \in C : r\big(x, \{x_n\}\big) = r_C\big(\{x_n\}\big)\big\}. \tag{2.12}$$

Recall that a bounded sequence $\{x_n\}$ in X is said to be regular if $r(\{x_n\}) = r(\{u_n\})$ for every subsequence $\{u_n\}$ of $\{x_n\}$.

Proposition 2.5 [18] *Let X be a complete* CAT(0) *space, $\{x_n\}$ be a bounded sequence in X and C be a closed convex subset of X. Then*

(1) *there exists a unique point* $u \in C$ *such that*

$$r(u, \{x_n\}) = \inf_{x \in C} r(x, \{x_n\});$$

(2) $A(\{x_n\})$ *and* $A_C(\{x_n\})$ *both are singleton.*

Definition 2.6 [8, 19] Let X be a CAT(0) space. A sequence $\{x_n\}$ in X is said to $\triangle$*-converge to* $q \in X$ if q is the unique asymptotic center of $\{u_n\}$ for each subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\triangle\text{-}\lim_{n\to\infty} x_n = q$ and call q the $\triangle$*-limit of* $\{x_n\}$.

Lemma 2.7 (1) *Every bounded sequence in a complete* CAT(0) *space always has a* $\triangle$-convergent *subsequence* [8].

(2) *Let* X *be a complete* CAT(0) *space,* C *be a closed convex subset of* X*. If* $\{x_n\}$ *is a bounded sequence in* C*, then the asymptotic center of* $\{x_n\}$ *is in* C [20].

Remark 2.8 (1) Let X be a CAT(0) space and C be a closed convex subset of X. Let $\{x_n\}$ be a bounded sequence in C. In what follows, we define

$$\{x_n\} \rightharpoonup w \quad \Leftrightarrow \quad \Phi(w) = \inf_{x \in C} \Phi(x), \tag{2.13}$$

where $\Phi(x) := \limsup_{n\to\infty} d(x_n, x)$.

(2) It is easy to know that $\{x_n\} \rightharpoonup w$ if and only if $A_C(\{x_n\}) = \{w\}$.

Nanjaras *et al.* [21] established the following relation between $\triangle$-convergence and weak convergence in a CAT(0) space.

Lemma 2.9 [21] *Let* $\{x_n\}$ *be a bounded sequence in a* CAT(0) *space* X*, and let* C *be a closed convex subset of* X *which contains* $\{x_n\}$*. Then*

(i) $\triangle\text{-}\lim_{n\to\infty} x_n = q$ *implies that* $\{x_n\} \rightharpoonup q$;

(ii) *the converse of* (i) *is true if* $\{x_n\}$ *is regular.*

Lemma 2.10 [16] *Let* C *be a closed and convex subset of a complete* CAT(0) *space* X*, and let* $T : C \to X$ *be a uniformly* L*-Lipschitzian and* $(\{\mu_n\}, \{\nu_n\}, \zeta)$*-total asymptotically nonexpansive nonself mapping. Let* $\{x_n\}$ *be a bounded sequence in* C *such that* $\{x_n\} \rightharpoonup q$ *and* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$*. Then* $Tq = q$.

Lemma 2.11 [16] *Let* C *be a closed and convex subset of a complete* CAT(0) *space* X*, and let* $T : C \to X$ *be an asymptotically nonexpansive nonself mapping with a sequence* $\{k_n\} \subset [1, \infty)$, $k_n \to 1$*. Let* $\{x_n\}$ *be a bounded sequence in* C *such that* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ *and* $\triangle\text{-}\lim_{n\to\infty} x_n = q$*. Then* $Tq = q$.

Lemma 2.12 [22] *Let* X *be a* CAT(0) *space,* $x \in X$ *be a given point and* $\{t_n\}$ *be a sequence in* $[b, c]$ *with* $b, c \in (0, 1)$ *and* $0 < b(1-c) \le \frac{1}{2}$*. Let* $\{x_n\}$ *and* $\{y_n\}$ *be any sequences in* X *such that*

$$\limsup_{n\to\infty} d(x_n, x) \le r, \qquad \limsup_{n\to\infty} d(y_n, x) \le r,$$

and

$$\lim_{n\to\infty} d\big((1-t_n)x_n \oplus t_n y_n, x\big) = r$$

for some $r \geq 0$. *Then*

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{2.14}$$

Lemma 2.13 [22] *Let* $\{a_n\}$, $\{\lambda_n\}$ *and* $\{c_n\}$ *be the sequences of nonnegative numbers such that*

$$a_{n+1} \leq (1+\lambda_n)a_n + c_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $\sum_{n=1}^{\infty} c_n < \infty$, *then the limit* $\lim_{n\to\infty} a_n$ *exists. If there exists a subsequence of* $\{a_n\}$ *which converges to* 0, *then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.14 [6] *Let* X *be a complete* CAT(0) *space,* $\{x_n\}$ *be a bounded sequence in* X *with* $A(\{x_n\}) = \{p\}$, *and let* $\{u_n\}$ *be a subsequence of* $\{x_n\}$ *with* $A(\{u_n\}) = \{u\}$ *and let the sequence* $\{d(x_n, u)\}$ *converge, then* $p = u$.

3 Main results

Theorem 3.1 *Let* X *be a complete* CAT(0) *space,* C *be a nonempty bounded closed and convex subset of* X. *If* $T : C \to X$ *is a uniformly Lipschitzian and total asymptotically nonexpansive nonself mapping, then* T *has a fixed point in* C.

Proof For any given point $x_0 \in C$, define

$$\Psi(u) = \limsup_{n\to\infty} d\big(T(PT)^{n-1}x_0, u\big), \quad \forall u \in C,$$

where P is a nonexpansive retraction of X onto C.

Since T is a total asymptotically nonexpansive nonself mapping, ζ is a strictly increasing continuous function, one gets

$$\begin{aligned} d\big(T(PT)^{n+m-1}x_0, T(PT)^{m-1}u\big) &\leq d\big((PT)^n x_0, u\big) + \nu_m \zeta\big(d\big((PT)^n x_0, u\big)\big) + \mu_m \\ &\leq d\big(T(PT)^{n-1}x_0, u\big) + \nu_m \zeta\big(d\big(T(PT)^{n-1}x_0, u\big)\big) + \mu_m \end{aligned}$$

for any $n, m \geq 1$. Letting $n \to \infty$ and taking superior limit on the both sides of the above inequality, we have

$$\Psi\big(T(PT)^{m-1}u\big) \leq \Psi(u) + \nu_m \zeta\big(\Psi(u)\big) + \mu_m, \quad m \geq 1, \forall u \in C. \tag{3.1}$$

It is easy to know that the function $u \mapsto \Psi(u)$ is lower semi-continuous, and C is bounded closed and convex, there exists a point $w \in C$ such that $\Psi(w) = \inf_{u\in C} \Psi(u)$. Letting $u = w$ in (3.1), for each $n \geq 1$, we have

$$\Psi\big(T(PT)^{m-1}w\big) \leq \Psi(w) + \nu_m \zeta\big(\Psi(w)\big) + \mu_m, \quad m \geq 1. \tag{3.2}$$

By using (2.7) with $t = \frac{1}{2}$, for any positive integers $n, m \geq 1$, we obtain

$$\begin{aligned} d^2\left(T(PT)^{n-1}x_0, \frac{T(PT)^{m-1}w \oplus T(PT)^{k-1}w}{2}\right) &\leq \frac{1}{2}d^2\big(T(PT)^{n-1}x_0, T(PT)^{m-1}w\big) + \frac{1}{2}d^2\big(T(PT)^{n-1}x_0, T(PT)^{k-1}w\big) \\ &\quad - \frac{1}{4}d^2\big(T(PT)^{m-1}w, T(PT)^{k-1}w\big). \end{aligned} \tag{3.3}$$

Let $n \to \infty$ and take superior limit in (3.3). It follows from (3.2) that

$$\begin{aligned} \Psi^2(w) &\leq \Psi^2\left(\frac{T(PT)^{m-1}w \oplus T(PT)^{k-1}w}{2}\right) \\ &\leq \frac{1}{2}\Psi^2\big(T(PT)^{m-1}w\big) + \frac{1}{2}\Psi^2\big(T(PT)^{k-1}w\big) \\ &\quad - \frac{1}{4}d^2\big(T(PT)^{m-1}w, T(PT)^{k-1}w\big) \\ &\leq \frac{1}{2}\big\{\Psi(w) + \nu_m\zeta\big(\Psi(w)\big) + \mu_m\big\}^2 + \frac{1}{2}\big\{\Psi(w) + \nu_k\zeta\big(\Psi(w)\big) + \mu_k\big\}^2 \\ &\quad - \frac{1}{4}d^2\big(T(PT)^{m-1}w, T(PT)^{k-1}w\big). \end{aligned}$$

This implies that

$$\begin{aligned} d^2\big(T(PT)^{m-1}w, T(PT)^{k-1}w\big) &\leq 2\big\{\Psi(w) + \nu_m\zeta\big(\Psi(w)\big) + \mu_m\big\}^2 \\ &\quad + 2\big\{\Psi(w) + \nu_k\zeta\big(\Psi(w)\big) + \mu_k\big\}^2 - 4\Psi^2(w). \end{aligned}$$

As T is a total asymptotically nonexpansive nonself mapping, so

$$\limsup_{n\to\infty} d\big(T(PT)^{m-1}w, T(PT)^{k-1}w\big) \leq 0,$$

which implies that $\{T(PT)^{n-1}w\}$ is a Cauchy sequence in C. Since C is complete, let $\lim_{n\to\infty} T(PT)^{n-1}w = v \in C$. In view of the continuity of TP, we have

$$v = \lim_{n\to\infty} T(PT)^n w = \lim_{n\to\infty} TP\big(T(PT)^{n-1}w\big) = TPv.$$

Since $v \in C$, $Pv = v$, this shows that $v = Tv$, *i.e.*, v is a fixed point of T in C.

The proof is completed. □

Remark 3.2 Theorem 3.1 is a generalization of Kirk [2, 3] and Abbas *et al.* [10] from nonexpansive mappings and asymptotically nonexpansive mappings in the intermediate sense to total asymptotically nonexpansive nonself mappings.

Theorem 3.3 *Let X be a complete* CAT(0) *space, C be a nonempty bounded closed and convex subset of X. If $T : C \to X$ is a uniformly Lipschitzian and total asymptotically nonexpansive nonself mapping, then the fixed point set of T, $F(T)$, is closed and convex.*

Proof As T is continuous, so $F(T)$ is closed. In order to prove that $F(T)$ is convex, it is enough to prove that $\frac{1}{2}(x \oplus y) \in F(T)$ whenever $x, y \in F(T)$. Setting $w = \frac{1}{2}(x \oplus y)$, by using (2.7) with $t = \frac{1}{2}$, for any $n \geq 1$, we have

$$\begin{aligned} d^2\bigl(T(PT)^{n-1}w, w\bigr) &= d^2\left(T(PT)^{n-1}w, \frac{1}{2}(x \oplus y)\right) \\ &\leq \frac{1}{2}d^2\bigl(x, T(PT)^{n-1}w\bigr) + \frac{1}{2}d^2\bigl(y, T(PT)^{n-1}w\bigr) - \frac{1}{4}d^2(x,y). \end{aligned} \tag{3.4}$$

Since T is a total asymptotically nonexpansive nonself mapping, using (2.8), we obtain

$$\begin{aligned} d^2\bigl(x, T(PT)^{n-1}w\bigr) &= d^2\bigl(T(PT)^{n-1}x, T(PT)^{n-1}w\bigr) \\ &\leq \bigl\{d(x,w) + \nu_n\zeta\bigl(d(x,w)\bigr) + \mu_n\bigr\}^2 \\ &\leq \left\{d\left(x, \frac{1}{2}(x \oplus y)\right) + \nu_n\zeta\left(d\left(x, \frac{1}{2}(x \oplus y)\right)\right) + \mu_n\right\}^2 \\ &\leq \left\{\frac{1}{2}d(x,y) + \nu_n\zeta\left(\frac{1}{2}d(x,y)\right) + \mu_n\right\}^2. \end{aligned} \tag{3.5}$$

Similarly,

$$d^2\bigl(y, T(PT)^{n-1}w\bigr) \leq \left\{\frac{1}{2}d(x,y) + \nu_n\zeta\left(\frac{1}{2}d(x,y)\right) + \mu_n\right\}^2. \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.4) and simplifying, we have

$$d^2\bigl(T(PT)^{n-1}w, w\bigr) \leq \left\{\frac{1}{2}d(x,y) + \nu_n\zeta\left(\frac{1}{2}d(x,y)\right) + \mu_n\right\}^2 - \frac{1}{4}d^2(x,y)$$

for any $n \geq 1$. Hence $\lim_{n\to\infty} T(PT)^{n-1}w = w$, in view of the continuity of TP, we have

$$w = \lim_{n\to\infty} T(PT)^n w = \lim_{n\to\infty} TP\bigl(T(PT)^{n-1}w\bigr) = TPw.$$

Since C is convex, this shows that $w = \frac{1}{2}(x \oplus y) \in C$. Therefore $Pw = w$, which implies that $w = Tw$, *i.e.*, $w \in F(T)$.

The proof is completed. □

Now we prove a $\triangle$-convergence theorem for the following implicit iterative scheme:

$$x_n = P\bigl((1-\alpha_n)x_{n-1} \oplus \alpha_n T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n\bigr), \quad \forall n \geq 1, \tag{3.7}$$

where C is a nonempty closed and convex subset of a complete CAT(0) space X for each $i = 1, 2, \ldots, N$, $T_i : C \to X$ is a uniformly L_i-Lipschitzian and $(\{\mu_n^{(i)}\}, \{\nu_n^{(i)}\}, \zeta^{(i)})$-total asymptotically nonexpansive nonself mapping defined by (2.4), and for each positive integer n, $i(n)$ and $k(n)$ are the solutions to the positive integer equation $n = (k(n)-1)N + i(n)$. It is easy to see that $k(n) \to \infty$ (as $n \to \infty$).

Remark 3.4 Letting $L = \max\{L_i, i = 1,2,\dots,N\}$, $\nu_n = \max\{\nu_n^{(i)}, i = 1,2,\dots,N\}$, $\mu_n = \max\{\mu_n^{(i)}, i = 1,2,\dots,N\}$ and $\zeta = \max\{\zeta^{(i)}, i = 1,2,\dots,N\}$, then $\{T_i\}_{i=1}^N$ is a finite family of uniformly L-Lipschitzian and $(\{\mu_n\},\{\nu_n\},\zeta)$-total asymptotically nonexpansive nonself mappings defined by (2.4).

Theorem 3.5 *Let X be a complete* CAT(0) *space, C be a nonempty bounded closed and convex subset of X. If $\{T_i\}_{i=1}^N : C \to X$ is a finite family of uniformly L-Lipschitzian and $(\{\mu_n\},\{\nu_n\},\zeta)$-total asymptotically nonexpansive nonself mappings satisfying the following conditions*:

(i) $\sum_{n=1}^{\infty} \nu_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$;
(ii) *there exists a constant $M^* > 0$ such that $\zeta(r) \le M^* r$, $\forall r \ge 0$;*
(iii) *there exist constants $a, b \in (0,1)$ with $0 < b(1-a) \le \frac{1}{2}$ such that $\{\alpha_n\} \subset [a,b]$.*

If $\mathcal{F} := \bigcap_{i=1}^N F(T_i) \neq \emptyset$, then the sequence $\{x_n\}$ defined by (3.7) *$\triangle$-converges to some point $q^* \in \mathcal{F}$.*

Proof Since for each $i = 1,2,\dots,N$, $T_i : C \to X$ is a $(\{\mu_n\},\{\nu_n\},\zeta)$-total asymptotically nonexpansive nonself mapping, by condition (ii), for each $i = 1,2,\dots,N$ and any $x, y \in C$, we have

$$\begin{aligned} d\bigl(T_i(PT_i)^{n-1}x, T_i(PT_i)^{n-1}y\bigr) &\le d(x,y) + \nu_n \zeta\bigl(d(x,y)\bigr) + \mu_n \\ &\le \bigl(1 + \nu_n M^*\bigr) d(x,y) + \mu_n, \quad \forall n \ge 1, \end{aligned} \tag{3.8}$$

where P is a nonexpansive retraction of X onto C.

(I) We first prove that the following limits exist:

$$\lim_{n\to\infty} d(x_n, q) \quad \text{for each } q \in \mathcal{F} \quad \text{and} \quad \lim_{n\to\infty} d(x_n, \mathcal{F}). \tag{3.9}$$

In fact, since $q \in \mathcal{F}$ and T_i, $i = 1,2,\dots,N$, is a total asymptotically nonexpansive nonself mapping, $P : X \to C$ is nonexpansive, it follows from Lemma 2.4 and (3.8) that

$$\begin{aligned} d(x_n, q) &= d\bigl(p\bigl((1-\alpha_n)x_{n-1} \oplus \alpha_n T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n\bigr), q\bigr) \\ &\le d\bigl((1-\alpha_n)x_{n-1} \oplus \alpha_n T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n, q\bigr) \\ &\le (1-\alpha_n) d(x_{n-1}, q) + \alpha_n d\bigl(T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n, q\bigr) \\ &\le (1-\alpha_n) d(x_{n-1}, q) + \alpha_n \bigl(\bigl(1 + \nu_{k(n)} M^*\bigr) d(x_n, q) + \mu_{k(n)}\bigr). \end{aligned} \tag{3.10}$$

Simplifying it and using condition (iii), we have

$$\begin{aligned} d(x_n, q) &\le \frac{1-\alpha_n}{1-\alpha_n(1+\nu_{k(n)}M^*)} d(x_{n-1}, q) + \frac{\alpha_n \mu_{k(n)}}{1-\alpha_n(1+\nu_{k(n)}M^*)} \\ &= \left(1 + \frac{\alpha_n \nu_{k(n)} M^*}{1-\alpha_n(1+\nu_{k(n)}M^*)}\right) d(x_{n-1}, q) + \frac{\alpha_n \mu_{k(n)}}{1-\alpha_n(1+\nu_{k(n)}M^*)} \\ &\le \left(1 + \frac{b \nu_{k(n)} M^*}{1-b(1+\nu_{k(n)}M^*)}\right) d(x_{n-1}, q) + \frac{b \mu_{k(n)}}{1-b(1+\nu_{k(n)}M^*)}. \end{aligned}$$

Since $1 - b(1 + \nu_{k(n)}M^*) \to 1 - b$ (as $n \to \infty$), there exists a positive integer n_0 such that $1 - b(1 + \nu_{k(n)}M^*) \geq \frac{1-b}{2}$ for all $n \geq n_0$. Therefore one has

$$d(x_n, q) \leq (1 + \sigma_n)d(x_{n-1}, q) + \xi_n, \quad \forall n \geq n_0, q \in \mathcal{F}, \tag{3.11}$$

and so

$$d(x_n, \mathcal{F}) \leq (1 + \sigma_n)d(x_{n-1}, \mathcal{F}) + \xi_n, \quad \forall n \geq n_0, \tag{3.12}$$

where $\sigma_n = \frac{2b\nu_{k(n)}M^*}{1-b}$ and $\xi_n = \frac{2b\mu_{k(n)}}{1-b}$. By condition (i), $\sum_{n=1}^{\infty} \sigma_n < \infty$ and $\sum_{n=1}^{\infty} \xi_n < \infty$. Therefore it follows from Lemma 2.13 that the limits $\lim_{n\to\infty} d(x_n, \mathcal{F})$ and $\lim_{n\to\infty} d(x_n, q)$ exist for each $q \in \mathcal{F}$.

(II) Next we prove that for each $i = 1, 2, \ldots, N$,

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0. \tag{3.13}$$

For each $q \in \mathcal{F}$, from the proof of (I), we know that $\lim_{n\to\infty} d(x_n, q)$ exists, we may assume that

$$\lim_{n\to\infty} d(x_n, q) = r \geq 0. \tag{3.14}$$

From (3.11) we get

$$\begin{aligned} r = \lim_{n\to\infty} d(x_n, q) &= \lim_{n\to\infty} d\big(p\big((1 - \alpha_n)x_{n-1} \oplus \alpha_n T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n\big), q\big) \\ &\leq \lim_{n\to\infty} d\big((1 - \alpha_n)x_{n-1} \oplus \alpha_n T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n, q\big) \\ &\leq \lim_{n\to\infty} \big((1 + \sigma_n)d(x_{n-1}, q) + \xi_n\big) = r, \end{aligned}$$

which implies that

$$\lim_{n\to\infty} d\big((1 - \alpha_n)x_{n-1} \oplus \alpha_n T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n, q\big) = r. \tag{3.15}$$

In addition, since

$$\begin{aligned} d\big(T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n, q\big) &\leq d(x_n, q) + \nu_{k(n)}\zeta\big(d(x_n, q)\big) + \mu_{k(n)} \\ &\leq \big(1 + \nu_{k(n)}M^*\big)d(x_n, q) + \mu_{k(n)}, \quad \forall n \geq 1. \end{aligned}$$

From (3.14), we have

$$\limsup_{n\to\infty} d\big(T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n, q\big) \leq r. \tag{3.16}$$

It follows from (3.14)-(3.16) and Lemma 2.12 that

$$\lim_{n\to\infty} d\big(x_{n-1}, T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n\big) = 0. \tag{3.17}$$

Now, by Lemma 2.4, we obtain

$$\begin{aligned} d(x_n, x_{n-1}) &= d\big(p\big((1-\alpha_n)x_{n-1} \oplus \alpha_n T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n\big), x_{n-1}\big) \\ &\leq d\big((1-\alpha_n)x_{n-1} \oplus \alpha_n T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n, x_{n-1}\big) \\ &\leq \alpha_n d\big(T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n, x_{n-1}\big) \to 0 \quad (\text{as } n \to \infty). \end{aligned} \tag{3.18}$$

Hence, from (3.17) and (3.18), one gets

$$\lim_{n\to\infty} d\big(x_n, T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n\big) = 0, \tag{3.19}$$

and for each $j = 1, 2, \dots, N$,

$$\lim_{n\to\infty} d(x_n, x_{n+j}) = 0. \tag{3.20}$$

Since T_i, $i = 1, 2, \dots, N$, is uniformly L-Lipschitzian and for each $n > N$ and P is a nonexpansive retraction of X onto C, we have $n = i(n) + (k(n) - 1)N$, where $i(n) \in \{1, 2, \dots, N\}$, $i(n) = i(n + N)$ and $k(n) + 1 = k(n + N)$. Hence it follows from (3.19) and (3.20) that

$$\begin{aligned} d(x_n, T_n x_n) &\leq d(x_n, x_{n+N}) + d\big(x_{n+N}, T_{i(n+N)}(PT_{i(n+N)})^{k(n+N)-1}x_{n+N}\big) \\ &\quad + d\big(T_{i(n+N)}(PT_{i(n+N)})^{k(n+N)-1}x_{n+N}, T_{i(n+N)}(PT_{i(n+N)})^{k(n+N)-1}x_n\big) \\ &\quad + d\big(T_{i(n+N)}(PT_{i(n+N)})^{k(n+N)-1}x_n, T_n x_n\big) \\ &\leq d(x_n, x_{n+N}) + d\big(x_{n+N}, T_{i(n+N)}(PT_{i(n+N)})^{k(n+N)-1}x_{n+N}\big) \\ &\quad + d\big(T_{i(n+N)}(PT_{i(n+N)})^{k(n+N)-1}x_{n+N}, T_{i(n+N)}(PT_{i(n+N)})^{k(n+N)-1}x_n\big) \\ &\quad + d\big(T_{i(n)}(PT_{i(n)})^{k(n)}x_n, T_{i(n)}x_n\big) \\ &\leq (1+L)d(x_n, x_{n+N}) + d\big(x_{n+N}, T_{i(n+N)}(PT_{i(n+N)})^{k(n+N)-1}x_{n+N}\big) \\ &\quad + Ld\big((PT_{i(n)})^{k(n)}x_n, x_n\big) \\ &\leq (1+L)d(x_n, x_{n+N}) + d\big(x_{n+N}, T_{i(n+N)}(PT_{i(n+N)})^{k(n+N)-1}x_{n+N}\big) \\ &\quad + Ld\big(T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n, x_n\big) \to 0 \quad (\text{as } n \to \infty), \end{aligned} \tag{3.21}$$

where $T_n = T_{n(\operatorname{mod} N)}$. Consequently, for any $j = 1, 2, \dots, N$, from (3.20) and (3.21) it follows that

$$\begin{aligned} d(x_n, T_{n+j}x_n) &\leq d(x_n, x_{n+j}) + d(x_{n+j}, T_{n+j}x_{n+j}) + d(T_{n+j}x_{n+j}, T_{n+j}x_n) \\ &\leq (1+L)d(x_n, x_{n+j}) + d(x_{n+j}, T_{n+j}x_{n+j}) \to 0 \quad (\text{as } n \to \infty). \end{aligned}$$

This implies that the sequence

$$\bigcup_{j=1}^{N} \big\{d(x_n, T_{n+j}x_n)\big\}_{n=1}^{\infty} \to 0 \quad (\text{as } n \to \infty).$$

Since for each $i = 1,2,\dots,N$, $\{d(x_n, T_i x_n)\}_{n=1}^{\infty}$ is a subsequence of $\bigcup_{j=1}^{N}\{d(x_n, T_{n+j}x_n)\}_{n=1}^{\infty}$, therefore we have

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0, \quad \forall i = 1,2,\dots,N.$$

Conclusion (3.13) is proved.

(III) Now we show that $\{x_n\}$ $\triangle$-converges to a point in $\mathcal{F}$.

Let $W_\omega(x_n) := \bigcup_{\{u_n\}\subset\{x_n\}} A(\{u_n\})$. We first prove that $W_\omega(x_n) \subset \mathcal{F}$.

In fact, let $u \in W_\omega(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.7, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle\text{-}\lim_{n\to\infty} v_n = v \in C$. In view of (3.13), $\lim_{n\to\infty} d(v_n, T_i v_n) = 0$. It follows from Lemma 2.10 that $v \in \mathcal{F}$; so, by (3.9), the limit $\lim_{n\to\infty} d(x_n, v)$ exists. By Lemma 2.14, $u = v$. This implies that $W_\omega(x_n) \subset \mathcal{F}$.

Next let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$, and let $A(\{x_n\}) = \{x\}$. Since $u \in W_\omega(x_n) \subset \mathcal{F}$, from (3.9) the limit $\lim_{n\to\infty} d(x_n, u)$ exists. In view of Lemma 2.14, $x = u$. This implies that $W_\omega(x_n)$ consists of exactly one point. We know that $\{x_n\}$ $\triangle$-converges to some point $q^* \in \mathcal{F}$.

The conclusion of Theorem 3.5 is proved. □

Now we prove the strong convergence results for the following iterative scheme:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = p((1-\alpha_n)x_n \oplus \alpha_n T_{i(n)}(PT_{i(n)})^{k(n)-1} y_n), \quad n \geq 1, \\ y_n = p((1-\beta_n)x_n \oplus \beta_n T_{i(n)}(PT_{i(n)})^{k(n)-1} x_n), \end{cases} \tag{3.22}$$

where C is a nonempty closed and convex subset of a complete CAT(0) space X for each $i = 1,2,\dots$, $T_i : C \to X$ is a uniformly L_i-Lipschitzian and $(\{\mu_n^{(i)}\}, \{\nu_n^{(i)}\}, \zeta^{(i)})$-total asymptotically nonexpansive nonself mapping defined by (2.4), and for each positive integer $n \geq 1$, $i(n)$ and $k(n)$ are the unique solutions to the following positive integer equation:

$$n = i(n) + \frac{(k(n)-1)k(n)}{2}, \quad k(n) \geq i(n). \tag{3.23}$$

Lemma 3.6 [23] (1) *The unique solutions to the positive integer equation* (3.23) *are*

$$\begin{aligned} &i(n) = n - \frac{(k(n)-1)k(n)}{2}, \\ &k(n) = \left[\frac{1}{2} + \sqrt[2]{2n - \frac{7}{4}}\right], \qquad k(n) \geq i(n) \quad \textit{and} \quad k(n) \to \infty \quad (\textit{as } n \to \infty), \end{aligned}$$

where $[x]$ *denotes the maximal integer that is not larger than* x.

(2) *For each* $i \geq 1$, *denote*

$$\begin{aligned} &\Gamma_i := \left\{ n \in \mathcal{N} : n = i + \frac{(k(n)-1)k(n)}{2}, k(n) \geq i \right\}, \quad \textit{and} \\ &K_i := \left\{ k(n) : n \in \Gamma_i, n = i + \frac{(k(n)-1)k(n)}{2}, k(n) \geq i \right\}, \end{aligned}$$

then $k(n) + 1 = k(n+1)$, $\forall n \in \Gamma_i$.

Theorem 3.7 *Let X be a complete* CAT(0) *space, C be a nonempty bounded closed and convex subset of X, and for each $i \geq 1$, let $T_i : C \to X$ be a uniformly L_i-Lipschitzian and $(\{\mu_n^{(i)}\}, \{\nu_n^{(i)}\}, \zeta^{(i)})$-total asymptotically nonexpansive nonself mapping defined by* (2.4), *satisfying the following conditions*:

(i) $\sum_{i=1}^{\infty}\sum_{n=1}^{\infty} \nu_n^{(i)} < \infty$, $\sum_{i=1}^{\infty}\sum_{n=1}^{\infty} \mu_n^{(i)} < \infty$;

(ii) *there exists a constant $M^* > 0$ such that $\zeta^{(i)}(r) \leq M^* r$, $\forall r \geq 0$, $i = 1, 2, \ldots$;*

(iii) *there exist constants $a, b \in (0,1)$ with $0 < b(1-a) \leq \frac{1}{2}$ such that $\{\alpha_n\}, \{\beta_n\} \subset [a, b]$.*

If $\mathcal{F} := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and there exist a mapping $T_k \in \{T_i\}$ and a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$, $\forall r > 0$, such that

$$f\big(d(x_n, \mathcal{F})\big) \leq d(x_n, T_k x_n), \quad \forall n \geq 1, \tag{3.24}$$

then the sequence $\{x_n\}$ defined by (3.22) *converges strongly* (*i.e., in metric topology*) *to some point $q^* \in \mathcal{F}$.*

Proof We observe that for each $i \geq 1$, $T_i : C \to X$ is a $(\{\mu_n^{(i)}\}, \{\nu_n^{(i)}\}, \zeta^{(i)})$-total asymptotically nonexpansive nonself mapping. By condition (ii), for each $n \geq 1$ and any $x, y \in C$, we have

$$\begin{aligned} d\big(T_i(PT_i)^{n-1}x, T_i(PT_i)^{n-1}y\big) &\leq d(x,y) + \nu_n^{(i)}\zeta^{(i)}\big(d(x,y)\big) + \mu_n^{(i)} \\ &\leq \big(1 + \nu_n^{(i)} M^*\big) d(x,y) + \mu_n^{(i)}, \quad \forall n \geq 1. \end{aligned} \tag{3.25}$$

(I) We first prove that the following limits exist:

$$\lim_{n\to\infty} d(x_n, q) \quad \text{for each } q \in \mathcal{F} \quad \text{and} \quad \lim_{n\to\infty} d(x_n, \mathcal{F}). \tag{3.26}$$

In fact, since $q \in \mathcal{F}$ and $P : X \to C$ is nonexpansive, it follows from Lemma 2.4, (3.25) that

$$\begin{aligned} d(y_n, q) &= d\big(p\big((1-\beta_n)x_n \oplus \beta_n T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n\big), q\big) \\ &\leq d\big((1-\beta_n)x_n \oplus \beta_n T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n, q\big) \\ &\leq (1-\beta_n)d(x_n,q) + \beta_n d\big(T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n, q\big) \\ &\leq (1-\beta_n)d(x_n,q) + \beta_n\big(\big(1 + \nu_{k(n)}^{(i(n))}M^*\big)d(x_n,q) + \mu_{k(n)}^{(i(n))}\big) \\ &\leq \big(1 + \nu_{k(n)}^{(i(n))}M^*\big)d(x_n,q) + \mu_{k(n)}^{(i(n))} \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} d(x_{n+1}, q) &= d\big(p\big((1-\alpha_n)x_n \oplus \alpha_n T_{i(n)}(PT_{i(n)})^{k(n)-1}y_n\big), q\big) \\ &\leq d\big((1-\alpha_n)x_n \oplus \alpha_n T_{i(n)}(PT_{i(n)})^{k(n)-1}y_n, q\big) \\ &\leq (1-\alpha_n)d(x_n,q) + \alpha_n d\big(T_{i(n)}(PT_{i(n)})^{k(n)-1}y_n, q\big) \\ &\leq (1-\alpha_n)d(x_n,q) + \alpha_n\big(\big(1 + \nu_{k(n)}^{(i(n))}M^*\big)d(y_n,q) + \mu_{k(n)}^{(i(n))}\big). \end{aligned} \tag{3.28}$$

Substituting (3.27) into (3.28) and simplifying it, we have

$$d(x_{n+1}, q) \leq (1+\sigma_n)d(x_n,q) + \xi_n, \quad \forall n \geq 1, q \in \mathcal{F}, \tag{3.29}$$

and so

$$d(x_{n+1},\mathcal{F}) \leq (1+\sigma_n)d(x_n,\mathcal{F}) + \xi_n, \quad \forall n \geq 1, \tag{3.30}$$

where $\sigma_n = b\nu_{k(n)}^{(i(n))}M^*(2+\nu_{k(n)}^{(i(n))}M^*)$, $\xi_n = b(2+\nu_{k(n)}^{(i(n))}M^*)\mu_{k(n)}^{(i(n))}$. By condition (i), $\sum_{n=1}^{\infty}\sigma_n < \infty$ and $\sum_{n=1}^{\infty}\xi_n < \infty$. By Lemma 2.13, the limits $\lim_{n\to\infty} d(x_n,\mathcal{F})$ and $\lim_{n\to\infty} d(x_n,q)$ exist for each $q \in \mathcal{F}$.

(II) Next we prove that for each $i \geq 1$, there exists a subsequence $\{x_m\} \subset \{x_n\}$ such that

$$\lim_{m\to\infty} d(x_m, T_i x_m) = 0. \tag{3.31}$$

In fact, for each given $q \in \mathcal{F}$, from the proof of (I), we know that $\lim_{n\to\infty} d(x_n,q)$ exists. Without loss of generality, we may assume that

$$\lim_{n\to\infty} d(x_n,q) = r \geq 0. \tag{3.32}$$

From (3.27) one gets

$$\limsup_{n\to\infty} d(y_n,q) \leq \lim_{n\to\infty}\big(\big(1+\nu_{k(n)}^{(i(n))}M^*\big)d(x_n,q) + \mu_{k(n)}^{(i(n))}\big) = r. \tag{3.33}$$

Since

$$\begin{aligned} d\big(T_{i(n)}(PT_{i(n)})^{k(n)-1}y_n,q\big) &\leq d(y_n,q) + \nu_{k(n)}^{(i(n))}\zeta^{(i(n))}\big(d(y_n,q)\big) + \mu_{k(n)}^{(i(n))} \\ &\leq \big(1+\nu_{k(n)}^{(i(n))}M^*\big)d(y_n,q) + \mu_{k(n)}^{(i(n))}, \quad \forall n \geq 1, \end{aligned}$$

we have

$$\limsup_{n\to\infty} d\big(T_{i(n)}(PT_{i(n)})^{k(n)-1}y_n,q\big) \leq r. \tag{3.34}$$

In addition, it follows from (3.29) that

$$\begin{aligned} d(x_{n+1},q) &\leq d\big((1-\alpha_n)x_n \oplus \alpha_n T_{i(n)}(PT_{i(n)})^{k(n)-1}y_n,q\big) \\ &\leq (1+\sigma_n)d(x_n,q) + \xi_n, \quad \forall n \geq 1, q \in \mathcal{F}, \end{aligned}$$

which implies that

$$\lim_{n\to\infty} d\big((1-\alpha_n)x_n \oplus \alpha_n T_{i(n)}(PT_{i(n)})^{k(n)-1}y_n,q\big) = r. \tag{3.35}$$

From (3.32)-(3.35) and Lemma 2.12, one has

$$\lim_{n\to\infty} d\big(x_n, T_{i(n)}(PT_{i(n)})^{k(n)-1}y_n\big) = 0. \tag{3.36}$$

Since

$$\begin{aligned} d(x_n,q) &\leq d\big(x_n, T_{i(n)}(PT_{i(n)})^{k(n)-1}y_n\big) + d\big(T_{i(n)}(PT_{i(n)})^{k(n)-1}y_n,q\big) \\ &\leq d\big(x_n, T_{i(n)}(PT_{i(n)})^{k(n)-1}y_n\big) + \big(1+\nu_{k(n)}^{(i(n))}M^*\big)d(y_n,q) + \mu_{k(n)}^{(i(n))}, \quad \forall n \geq 1. \end{aligned}$$

Taking lim inf as $n \to \infty$ on both sides in the inequality above, from (3.36) we have

$$\liminf_{n\to\infty} d(y_n, q) \geq r,$$

which combined with (3.33) implies that

$$\liminf_{n\to\infty} d(y_n, q) = r. \tag{3.37}$$

Using (3.27) we have

$$\begin{aligned} r = \lim_{n\to\infty} d(y_n, q) &\leq \lim_{n\to\infty} d\big((1-\beta_n)x_n \oplus \beta_n T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n, q\big) \\ &\leq \lim_{n\to\infty}\big(1 + \nu_{k(n)}^{(i(n))}M^*\big)d(x_n, q) + \mu_{k(n)}^{(i(n))} = r. \end{aligned} \tag{3.38}$$

This implies that

$$\lim_{n\to\infty} d\big((1-\beta_n)x_n \oplus \beta_n T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n, q\big) = r. \tag{3.39}$$

Similarly, we have that

$$\limsup_{n\to\infty} d\big(T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n, q\big) \leq \limsup_{n\to\infty}\big(1 + \nu_{k(n)}^{(i(n))}M^*\big)d(x_n, q) + \mu_{k(n)}^{(i(n))} \leq r.$$

This together with (3.32) and (3.39) and Lemma 2.12 yields that

$$\lim_{n\to\infty} d\big(x_n, T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n\big) = 0. \tag{3.40}$$

Therefore we obtain

$$\begin{aligned} d(x_n, y_n) &= d\big(x_n, (1-\beta_n)x_n \oplus \beta_n T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n\big) \\ &\leq \beta_n d\big(x_n, T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n\big) \to 0 \quad (\text{as } n \to \infty). \end{aligned} \tag{3.41}$$

Furthermore, it follows from (3.36) that

$$\begin{aligned} d(x_{n+1}, x_n) &= d\big((1-\alpha_n)x_n \oplus \alpha_n T_{i(n)}(PT_{i(n)})^{k(n)-1}y_n, x_n\big) \\ &\leq \alpha_n d\big(T_{i(n)}(PT_{i(n)})^{k(n)-1}y_n, x_n\big) \to 0 \quad (\text{as } n \to \infty). \end{aligned} \tag{3.42}$$

Now combined with (3.41) this shows that

$$d(x_{n+1}, y_n) \leq d(x_{n+1}, x_n) + d(x_n, y_n) \to 0 \quad (\text{as } n \to \infty). \tag{3.43}$$

From Lemma 3.6, (3.37), (3.40), (3.42) and (3.43), we have that for each given positive integer $i \geq 1$, there exist subsequences $\{x_m\}_{m\in\Gamma_i}$, $\{y_m\}_{m\in\Gamma_i}$ and $\{k_m\}_{m\in\Gamma_i} \subset K_i := \{k(m) : m \in \Gamma_i, m = i + \frac{(k(m)-1)k(m)}{2}, k(m) \geq i\}$ such that

$$\begin{aligned} d(x_m, T_i x_m) &\leq d\big(x_m, T_i(PT_i)^{k(m)-1}x_m\big) + d\big(T_i(PT_i)^{k(m)-1}x_m, T_i(PT_i)^{k(m)-1}y_{m-1}\big) \\ &\quad + d\big(T_i(PT_i)^{k(m)-1}y_{m-1}, T_i x_m\big) \end{aligned}$$

$$
\begin{aligned}
&\le d\big(x_m, T_i(PT_i)^{k(m)-1}x_m\big) + d(x_m, y_{m-1}) + \nu_{k(m)}^{(i)}\zeta^{(i)}\big(d(x_m, y_{m-1})\big) \\
&\quad + \mu_{k(m)}^{(i)} + L_i d\big((PT_i)^{k(m)-1}y_{m-1}, x_m\big) \\
&= d\big(x_m, T_i(PT_i)^{k(m)-1}x_m\big) + d(x_m, y_{m-1}) + \nu_{k(m)}^{(i)}\zeta^{(i)}\big(d(x_m, y_{m-1})\big) \\
&\quad + \mu_{k(m)}^{(i)} + L_i d\big(T_i(PT_i)^{k(m-1)-1}y_{m-1}, x_m\big) \quad \big(\text{by Lemma 3.6(2)}\big) \\
&\le d\big(x_m, T_i(PT_i)^{k(m)-1}x_m\big) + d(x_m, y_{m-1}) + \nu_{k(m)}^{(i)}\zeta^{(i)}\big(d(x_m, y_{m-1})\big) \\
&\quad + \mu_{k(m)}^{(i)} + L_i d\big(T_i(PT_i)^{k(m-1)-1}y_{m-1}, x_{m-1}\big) \\
&\quad + L_i d(x_{m-1}, x_m) \to 0 \quad (\text{as } m \to \infty).
\end{aligned}
$$

Conclusion (3.31) is proved.

(III) Now we prove that $\{x_n\}$ converges strongly (*i.e.*, in metric topology) to some point $q^* \in \mathcal{F}$.

In fact, it follows from (3.31) and (3.24) that for a given mapping T_k, there exists a subsequence $\{x_m\}_{m\in\Gamma_i}$ of $\{x_n\}$ such that

$$\lim_{m\to\infty} d(x_m, T_k x_m) = 0$$

and

$$f\big(d(x_m, \mathcal{F})\big) \le d(x_m, T_k x_m), \quad \forall m \ge 1.$$

Taking lim sup on both sides of the above inequality, one has

$$\limsup_{n\to\infty} f\big(d(x_m, \mathcal{F})\big) = 0.$$

By the property of f, this implies that

$$\lim_{m\to\infty} d(x_m, \mathcal{F}) = 0. \tag{3.44}$$

Next we prove that $\{x_m\}$ is a Cauchy sequence in C.

In fact, it follows from (3.29) that for any $q \in \mathcal{F}$,

$$d(x_{m+1}, q) \le (1 + \sigma_m) d(x_m, q) + \xi_m, \quad \forall m \ge 1,$$

where $\sum_{m=1}^{\infty} \sigma_m < \infty$ and $\sum_{m=1}^{\infty} \xi_m < \infty$. Hence, for any positive integers m, k, we have

$$d(x_{m+k}, x_m) \le d(x_{m+K}, q) + d(x_m, q) \le (1 + \sigma_{m+k-1}) d(x_{m+k-1}, q) + \xi_{m+k-1} + d(x_m, q).$$

Since for each $x \ge 0$, $1 + x \le e^x$, one gets

$$
\begin{aligned}
d(x_{m+k}, x_m) &\le e^{\sigma_{m+k-1}} d(x_{m+k-1}, q) + \xi_{m+k-1} + d(x_m, q) \\
&\le e^{\sigma_{m+k-1}+\sigma_{m+k-2}} d(x_{m+k-2}, q) + e^{\sigma_{m+k-1}} \xi_{m+k-2} + \xi_{m+k-1} + d(x_m, q) \\
&\le \cdots
\end{aligned}
$$

$$\begin{aligned}
&\le e^{\sum_{i=m}^{m+k-1}\sigma_i} d(x_m, q) + e^{\sum_{i=m+1}^{m+k-1}\sigma_i}\xi_m + e^{\sum_{i=m+2}^{m+k-1}\sigma_i}\xi_{m+1} + \cdots \\
&\quad + e^{\sigma_{m+k-1}}\xi_{m+k-2} + \xi_{m+k-1} + d(x_m, q) \\
&\le (1+M)d(x_m, q) + M\sum_{i=m}^{m+k-1}\xi_i \quad \text{for each } q \in \mathcal{F}
\end{aligned}$$

and

$$d(x_{m+k}, x_m) \le (1+M)d(x_m, \mathcal{F}) + M\sum_{i=m}^{m+k-1}\xi_i,$$

where $M = e^{\sum_{i=1}^{\infty}\sigma_i} < \infty$. By (3.44) we have

$$d(x_{m+k}, x_m) \le (1+M)d(x_m, \mathcal{F}) + M\sum_{i=m}^{m+k-1}\xi_i \to 0 \quad (\text{as } m, k \to \infty).$$

This shows that the subsequence $\{x_m\}$ is a Cauchy sequence in C. Since C is a closed subset in a CAT(0) space X, it is complete. Without loss of generality, we can assume that the subsequence $\{x_m\}$ converges strongly (*i.e.*, in metric topology in X) to some point $q^* \in C$. By Theorem 3.3, we know that $\mathcal{F}$ is a closed subset in C. Since $\lim_{m\to\infty} d(x_m, \mathcal{F}) = 0$, $q^* \in \mathcal{F}$. By using (3.26), it yields that the whole sequence $\{x_n\}$ converges in the metric topology to some point $q^* \in \mathcal{F}$.

The proof is completed. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors read and approved the final manuscript.

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References

1. Espinola, R, Nicolae, A: Geodesic Ptolemy spaces and fixed points. Nonlinear Anal. **74**(1), 27-34 (2011)
2. Kirk, WA: Geodesic geometry and fixed point theory. In: Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003). Colecc. Abierta, vol. 64, pp. 195-225. Seville University Publications, Seville (2003)
3. Kirk, WA: Geodesic geometry and fixed point theory II. In: International Conference on Fixed Point Theory and Applications, pp. 113-142. Yokohama Publications, Yokohama (2004)
4. Dhompongsa, S, Kirk, WA, Panyanak, B: Nonexpansive set-valued mappings in metric and Banach spaces. J. Nonlinear Convex Anal. **8**(1), 35-45 (2007)
5. Dhompongsa, S, Kirk, WA, Sims, B: Fixed points of uniformly Lipschitzian mappings. Nonlinear Anal. **65**(4), 762-772 (2006)
6. Dhompongsa, S, Panyanak, B: On Δ-convergence theorem in CAT(0) spaces. Comput. Math. Appl. **56**(10), 2572-2579 (2008)
7. Hussain, N, Khamsi, MA: On asymptotic pointwise contractions in metric spaces. Nonlinear Anal. **71**(10), 4423-4429 (2009)
8. Kirk, WA, Panyanak, B: A concept of convergence in geodesic spaces. Nonlinear Anal. **68**(12), 3689-3696 (2008)
9. Lim, TC: Remarks on some fixed point theorems. Proc. Am. Math. Soc. **60**, 179-182 (1976)
10. Abbas, M, Thakur, BS, Thakur, D: Fixed points of asymptotically nonexpansive mappings in the intermediate sense in CAT(0) spaces. Commun. Korean Math. Soc. **28**(1), 107-121 (2013)
11. Bridson, MR, Haefliger, A: Metric Spaces of Non-Positive Curvature. Grundlehren der Mathematischen Wissenschaften, vol. 319. Springer, Berlin (1999)
12. Brown, KS: Buildings. Springer, New York (1989)

13. Goebel, K, Reich, S: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings. Monographs and Textbooks in Pure and Applied Mathematics, vol. 83. Marcel Dekker, New York (1984)
14. Burago, D, Burago, Y, Ivanov, S: A Course in Metric Geometry. Graduate Studies in Mathematics, vol. 33. Am. Math. Soc., Providence (2001)
15. Gromov, M: Metric Structures for Riemannian and Non-Riemannian Spaces. Progress in Mathematics, vol. 152. Birkhäuser, Boston (1999)
16. Wang, L, Chang, SS, Ma, Z: Convergence theorems for total asymptotically nonexpansive nonself mappings in CAT(0) spaces. J. Inequal. Appl. **2013**, 135 (2013). doi:10.1186/1029-242X-2013-135
17. Dhompongsa, S, Panyanak, B: On Δ-convergence theorems in CAT(0) spaces. Comput. Math. Appl. **56**(10), 2572-2579 (2008)
18. Dhompongsa, S, Kirk, WA, Sims, B: Fixed points of uniformly Lipschitzian mappings. Nonlinear Anal. **65**(4), 762-772 (2006)
19. Lim, TC: Remarks on some fixed point theorems. Proc. Am. Math. Soc. **60**, 179-182 (1976)
20. Dhompongsa, S, Kirk, WA, Panyanak, B: Nonexpansive set-valued mappings in metric and Banach spaces. J. Nonlinear Convex Anal. **8**(1), 35-45 (2007)
21. Nanjaras, B, Panyanak, B: Demiclosed principle for asymptotically nonexpansive mappings in CAT(0) spaces. Fixed Point Theory Appl. **2010**, Article ID 268780 (2010)
22. Chang, SS, Wang, L, Lee, HWJ, Chan, CK, Yang, L: Total asymptotically nonexpansive mappings in CAT(0) space demiclosed principle and Δ-convergence theorems for total asymptotically nonexpansive mappings in CAT(0) spaces. Appl. Math. Comput. **219**, 2611-2617 (2012)
23. Deng, WQ, Bai, P: An implicit iterative process for common fixed points of two infinite families of asymptotically nonexpansive mappings in Banach spaces. J. Appl. Math. **2013**, Article ID 602582 (2013)

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