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Approximating fixed points for a reversible semigroup of Lipschitzian mappings in a smooth Banach space

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Abstract

In this paper, we approximate a fixed point of the semigroup $\varphi = \{T_s : s \in S\}$ of Lipschitzian mappings from a nonempty compact convex subset C of a smooth Banach space E into C with a uniform Lipschitzian condition and with respect to a finite family of sequences $\{\mu_{i,n}\}_{i=1,n=1}^{m,\ \infty}$ of left strong regular invariant means defined on an appropriate invariant subspace of $l^{\infty}(S)$. Our result extends the main results announced by several others.

MSC: 47H09; 47H10; 47J25

Keywords: smooth Banach space; asymptotically nonexpansive semigroup; reversible semigroup; invariant mean

1 Introduction

Let E be a real Banach space with the topological dual E^*, and let C be a nonempty closed and convex subset of E. Recall that a mapping T of C into itself is said to be

(1) Lipschitzian with Lipschitz constant $l > 0$ if

$$\|Tx - Ty\| \le l\|x - y\|, \quad \forall x, y \in C.$$

(2) nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

(3) asymptotically nonexpansive if there exists a sequence $\{l_n\}$ of positive numbers such that $\lim_{n \to \infty} l_n = 1$ and

$$\|T^n x - T^n y\| \le l_n \|x - y\|, \quad \forall x, y \in C.$$

A semigroup S is called left reversible if any two closed right ideals of S have non-void intersection, *i.e.*, $aS \cap bS \ne \emptyset$ for $a, b \in S$. In this case, $(S, \preceq)$ is a directed set when the binary relation $\preceq$ on S is defined by $a \preceq b$ if and only if $aS \supset bS$ for $a, b \in S$.

Notation Throughout the rest of this paper, S will always denote a left reversible semigroup with an identity e.

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In [1], Lau *et al.* studied iterative schemes for approximating a fixed point of the semigroup $\varphi = \{T(s) : s \in S\}$ of nonexpansive mappings on a nonempty compact convex subset C of a smooth (and strictly convex) Banach space and introduced the following iteration process. Let $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_{\mu_n} x_n, \quad n \geq 1, \tag{1}$$

where $\{\mu_n\}_{n=1}^{\infty}$ is a sequence of left strong regular invariant means defined on an appropriate invariant subspace of $l^{\infty}(S)$.

$\varphi = \{T(s) : s \in S\}$ is called a representation of S as a Lipschitzian mapping on C with Lipschitz constant $\{l(s) : s \in S\}$ if $T(s)$ is a Lipschitzian with Lipschitz constant $l(s)$ for each $s \in S$, $T(st) = T(s)T(t)$ for each $t, s \in S$ and $T(e) = I$. φ is called an asymptotically nonexpansive semigroup on C if φ is a representation of S as a Lipschitzian mapping on C with Lipschitz constant $\{l(s); s \in s\}$ and $\lim_s l(s) \leq 1$.

In 2008, Saeidi proved the following theorem.

Theorem 1.1 [2] *Let S be a left reversible semigroup and $\varphi = \{T_s : s \in S\}$ be a representation of S as a Lipschitzian mapping from a nonempty compact convex subset C of a smooth Banach space E into C, with uniform Lipschitzian constant $\lim_s K(s) \leq 1$, and let f be an α-contraction on C for some $0 < \alpha < 1$. Let X be a left invariant φ-stable subspace of $L^{\infty}(\varphi)$ containing 1, let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of left strong regular invariant means defined on X such that $\lim_{n \to \infty} \|\mu_{n+1} - \mu_n\| = 0$, and let $\{c_n\}_{n=1}^{\infty}$ be a sequence defined by*

$$c_n = \sup_{x,y \in C} \left(\|T_{\mu_n} x - T_{\mu_n} y\| - \|x - y\| \right), \quad n \geq 1.$$

Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $(0, 1)$ such that

(C$_1$) $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$,
(C$_2$) $\lim_{n \to \infty} \alpha_n = 0$,
(C$_3$) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(C$_4$) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$,
(C$_5$) $\limsup_{n \to \infty} \frac{c_n}{\alpha_n} \leq 0$.

If $\{x_n\}_{n=1}^{\infty}$ is a sequence generated by $x_1 \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_{\mu_n} x_n, \quad n \geq 1,$$

then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to some $z \in \mathrm{Fix}(\varphi)$, the set of common fixed points of φ, which is the unique solution of the variational inequality

$$\langle (f - I)z, J(y - z) \rangle \leq 0, \quad \forall y \in \mathrm{Fix}(\varphi).$$

Equivalently, one has $z = Pfz$, where P is the unique sunny nonexpansive retraction of C onto $F(\varphi)$.

In 2007, Zhang *et al.* [3] introduced the following composite iteration scheme:

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) T t_n x_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n, \end{cases} \tag{2}$$

where $\{T(t) : t \geq 0\}$ is a nonexpansive semigroup from C to C, u is an arbitrary (but fixed) element in C, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset [0,1]$, $\{t_n\} \subset \mathbb{R}^+$, and proved some strong convergence theorems of an explicit composite iteration scheme for nonexpansive mappings in the framework of a reflexive Banach space with a uniformly Gâteaux differentiable norm, uniformly smooth Banach space and uniformly convex Banach space with a weakly continuous normalized duality mapping.

Motivated and inspired by Zhang *et al.* [3] and Saeidi [2], Katchang and Kumam proved the following theorem.

Theorem 1.2 [4] *Let S be a left reversible semigroup, and let $\varphi = \{T_s : s \in S\}$ be a representation of S as a Lipschitzian mapping from a nonempty compact convex subset C of a smooth Banach space E into C, with uniform Lipschitzian constant $\lim_s K(s) \leq 1$, and let f be an α-contraction on C for some $0 < \alpha < 1$. Let X be a left invariant φ-stable subspace of $L^\infty(\varphi)$ containing 1, let $\{\mu_n\}_{n=1}^\infty$ be a sequence of left strong regular invariant means defined on X such that $\lim_{n \to \infty} \|\mu_{n+1} - \mu_n\| = 0$, and let $\{c_n\}_{n=1}^\infty$ be a sequence defined by*

$$c_n = \sup_{x,y \in C} \left(\|T_{\mu_n}x - T_{\mu_n}y\| - \|x - y\| \right), \quad n \geq 1.$$

Let $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$, $\{\gamma_n\}_{n=1}^\infty$ and $\{\delta_n\}_{n=1}^\infty$ be sequences in $(0,1)$ such that

(C$_1$) $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$,
(C$_2$) $\lim_{n \to \infty} \alpha_n = 0$,
(C$_3$) $\sum_{n=1}^\infty \alpha_n = \infty$,
(C$_4$) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$,
(C$_5$) $\limsup_{n \to \infty} \frac{c_n}{\alpha_n} \leq 0$,
(C$_6$) $\lim_{n \to \infty} \delta_n = 0$.

If $\{x_n\}_{n=1}^\infty$ is a sequence generated by $x_1 \in C$ and

$$\begin{cases} y_n = \delta_n x_n + (1 - \delta_n) T_{\mu_n} x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, \quad n \geq 1, \end{cases} \qquad (3)$$

then the sequence $\{x_n\}_{n=1}^\infty$ converges strongly to some $z \in \mathrm{Fix}(\varphi)$, which is the unique solution of the variational inequality

$$\langle (f - I)z, J(y - z) \rangle \leq 0, \quad \forall y \in F(\varphi).$$

Equivalently, one has $z = Pfz$, where P is the unique sunny nonexpansive retraction of C onto $F(\varphi)$.

Recently, many authors studied fixed point results for a nonlinear semigroup mapping, for example, [5–8].

In this paper, motivated and inspired by Qianglian *et al.* [9], Lau *et al.* [1], Zhang *et al.* [3], Saeidi [2], Katchang and Kumam [4], Sunthrayuth and Kumam [10, 11] and Wattanawitoon and Kumam [12], we introduce the composite explicit viscosity iterative schemes as

follows:

$$x_{n+1} = \alpha_n f(y_{1,n}) + \beta_n x_n + \gamma_n T_{\mu_{1,n}} y_{1,n},$$

$$y_{i,n} = \delta_{i,n} y_{i+1,n} + (I - \delta_{i,n}) T_{\mu_{i,n}} y_{i+1,n}, \quad i = 1, 2, \ldots, m, \tag{4}$$

$$y_{m+1,n} = x_n$$

for an asymptotically nonexpansive semigroup $\varphi = \{T_s : s \in S\}$ on a compact convex subset C of a smooth Banach space E with respect to a finite family of left regular sequences $\{\mu_{i,n}\}_{i=1,n=1}^{m,\ \infty}$ of invariant means defined on an appropriate invariant subspace of $l^\infty(S)$. We prove, under certain appropriate assumptions on the sequences $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$, $\{\gamma_n\}_{n=1}^\infty$ and $\{\delta_n\}_{i=1,n=1}^{m,\ \infty}$, that $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{i=1,n=1}^{m,\ \infty}$ defined by (4) converge strongly to $z \in \mathrm{Fix}(\varphi)$, which is the unique solution of the variational inequality

$$\langle (f - I)z, J(y - z) \rangle \leq 0, \quad \forall y \in \mathrm{Fix}(\varphi).$$

Our result improves and extends many previous results (*e.g.*, [1, 2, 4, 13–15] and many others).

2 Preliminaries

Let E^* be the topological dual of a real Banach space E. The value of $j \in E^*$ at $x \in E$ will be denoted by $\langle x, j \rangle$ or $j(x)$. With each $x \in E$, we associate the set

$$J(x) = \big\{ j \in E^* : \langle x, j \rangle = \|x\|^2 = \|j\|^2 \big\}.$$

Using the Hahn-Banach theorem, it is immediately clear that $J(x) \neq \phi$ for each $x \in E$. The multi-valued mapping J from E into E^* is said to be the (normalized) duality mapping. A Banach space E is said to be smooth if the duality mapping J is single-valued. As is well known, the duality mapping is norm to weak-star continuous when E is smooth, see [16].

Let $B(S)$ be the Banach space of all bounded real-valued functions defined on S with supremum norm. For each $s \in S$, we define the left and right translation operators $l(s)f$ and $r(s)f$ on $B(S)$ by

$$\big(l(s)f\big)(t) = f(st), \quad \text{and} \quad \big(r(s)f\big)(t) = f(ts),$$

for each $s \in S$ and $f \in B(S)$, respectively. Let X be a subspace of $B(S)$ containing 1 and let X^* be its topological dual. An element μ of X^* is said to be a mean on X if $\|\mu\| = \mu(1) = 1$. Let $\mu \in X^*$. Then we define $r(s)^*\mu, l(s)^*\mu \in X^*$ by $(r(s)^*\mu)f = \mu(r(s)f)$, $(l(s)^*\mu)f = \mu(l(s)f)$ for each $f \in X$ and $s \in S$. It is easy to see that if μ is a mean on X, then $r(s)^*\mu$ and $l(s)^*\mu$ are also. We often write $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$. Let X be left invariant (resp. right invariant), *i.e.*, $l(s)(X) \subset X$ (resp. $r(s)(X) \subset X$) for each $s \in S$. A mean μ on X is said to be left invariant (resp. right invariant) if $l(s)^*\mu = \mu$ (resp. $r(s)^*\mu = \mu$) for each $s \in S$ and $f \in X$. X is said to be left (resp. right) amenable if X has a left (resp. right) invariant mean. The semigroup S is amenable (*i.e.*, S is both left and right amenable) when S is a commutative semigroup or a solvable group. However, the free group (or semigroup) on two generators is not left amenable. If a semigroup S is left amenable, then S is left

reversible, but the converse is not true see [17], [18, p.335]. A net $\{\mu_\alpha\}$ of means on X is said to be strongly left regular if

$$\lim_\alpha \|l(s)^* \mu_\alpha - \mu_\alpha\| = 0,$$

for each $s \in S$, where $l(s)^*$ is the adjoint operator of l_s.

Let $\varphi = \{T(s) : s \in S\}$ be a representation of S as a Lipschitzian mapping on C with Lipschitz constant $\{l(s) : s \in S\}$. By Fix(φ) we denote the set of common fixed points of φ, *i.e.*,

$$\text{Fix}(\varphi) = \bigcap_{s \in S} \{x \in C : T(s)x = x\}.$$

We denote by C_a the set of almost periodic elements in C, *i.e.*, all $x \in C$ such that $\{T(s)x : s \in S\}$ is relatively compact in the norm topology of E. Let X be a subspace of $B(S)$ such that the functions (i) $s \to \langle T(s)x, x^* \rangle$ and (ii) $s \to \|T(s)x - y\|$ on S are in X for all $x, y \in C$ and $x^* \in E^*$. We will call a subspace X of $B(S)$ satisfying (i) and (ii) φ-stable. We know that if X is a subspace of $B(S)$ containing 1 and the function $s \to \langle T(s)x, x^* \rangle$ on S is in X for all $x \in C$ and $x^* \in E^*$, then there exists a unique point $x_0 \in E$ such that $\mu \langle T(\cdot)x, x^* \rangle = \langle x_0, x^* \rangle$ for a mean μ on X, $x \in C$ and $x^* \in E$. We denote such a point $x_0 \in E$ by $T_\mu x$. See [19] for more details.

Lemma 2.1 [20] *Let S be a semigroup and C be a nonempty closed convex subset of a reflexive Banach space E. Let $\varphi = \{T_t : t \in S\}$ be a nonexpansive semigroup on H such that $\{T_t x : t \in S\}$ is bounded for some $x \in C$, let X be a subspace of $B(S)$ such that $1 \in X$ and the mapping $t \to \langle T_t x, y^* \rangle$ is an element of X for each $x \in C$ and $y^* \in E^*$, and μ is a mean on X. If we write $T_\mu x$ instead of $\int T_t x \, d\mu(t)$, then the following hold:*
    (i) *T_μ is a nonexpansive mapping from C into C.*
    (ii) *$T_\mu x = x$ for each $x \in \text{Fix}(\varphi)$.*
    (iii) *$T_\mu x \in \overline{co}\{T_t x : t \in S\}$ for each $x \in C$.*

Lemma 2.2 [21] *Let $\varphi = \{T(s) : s \in S\}$ be a representation of S as a Lipschitzian mapping from a nonempty weakly compact convex subset C of a Banach space E into C, with uniform Lipschitzian constant $\lim_s l(s) \leq 1$ on the Lipschitz constant of mappings. Let X be a left invariant and φ-stable subspace of $B(S)$, and let $\{\mu_n\}_{n=1}^\infty$ be an asymptotically left invariant sequence of means on X. If $z \in C_a$ and $\liminf_{n \to \infty} \|T_{\mu_n} z - z\| = 0$, then z is a common fixed point of φ.*

Lemma 2.3 [2] *Let $\varphi = \{T(s) : s \in S\}$ be a representation of S as a Lipschitzian mapping from a nonempty weakly compact convex subset C of a Banach space E into C, with uniform Lipschitzian constant $\lim_s l(s) \leq 1$ on the Lipschitz constant of mappings. Let X be a left invariant subspace of $B(S)$ containing 1 such that the mapping $s \to \langle T(s)x, x^* \rangle$ on S is in X for all $x \in C$ and $x^* \in E^*$, and $\{\mu_n\}_{n=1}^\infty$ is an asymptotically left invariant sequence of means on X. Then*

$$\limsup_{n \to \infty} \left(\sup_{x,y \in C} \left(\|T_{\mu_n} x - T_{\mu_n} y\| - \|x - y\| \right) \right) \leq 0.$$

Let D be a subset of B, where B is a subset of a Banach space E, and let P be a retraction of B onto D, that is, $Px = x$ for each $x \in D$. Then P is said to be sunny [22] if for each $x \in B$ and $t \geq 0$ with $Px + t(x - Px) \in B$, $P(Px + t(x - Px)) = Px$. A subset D of B is said to be a sunny nonexpansive retract of B if there exists a sunny nonexpansive retraction P of B into D.

Lemma 2.4 [1] *Let $\varphi = \{T(s) : s \in S\}$ be a representation of S as a Lipschitzian mapping from a nonempty compact convex subset C of a smooth Banach space E into C, with uniform Lipschitzian constant $\lim_s l(s) \leq 1$ on the Lipschitz constant of mappings. Let X be a left invariant and φ-stable subspace of $B(S)$ containing 1 and μ be a left invariant mean on X. Then $\mathrm{Fix}(\varphi)$ is a sunny nonexpansive retract of C, and the sunny nonexpansive retraction of C onto $\mathrm{Fix}(\varphi)$ is unique.*

Lemma 2.5 [16] *Let C be a nonempty convex subset of a smooth Banach space E, let D be a nonempty subset of C, and let $P \colon C \to D$ be a retraction. Then the following are equivalent:*
  (a) *P is sunny nonexpansive.*
  (b) *$\langle x - Px, J(y - Px) \rangle \leq 0$ for all $x \in C$ and $y \in D$.*
  (c) *$\langle x - y, J(Px - Py) \rangle \geq \|Px - Py\|^2$ for all $x, y \in C$.*

Lemma 2.6 [3] *Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be bounded sequences in a Banach space X, and let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $[0,1]$ such that $0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1$. Suppose $x_{n+1} = \alpha_n x_n + (1 - \alpha_n) y_n$ for all integers $n \geq 0$ and*

$$\limsup_{n \to \infty} \left(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \right) \leq 0.$$

Then $\lim_{n \to \infty} \|y_n - x_n\| = 0$.

Lemma 2.7 [19] *Let E be a real smooth Banach space and J be the duality mapping. Then*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y) \rangle, \quad \forall x, y \in E.$$

Lemma 2.8 [23] *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - b_n) a_n + b_n c_n, \quad n \geq 0,$$

where $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ are sequences of real numbers satisfying the following conditions:
  (i) *$\{b_n\}_{n=1}^{\infty} \subset (0,1)$, $\sum_{n=0}^{\infty} b_n = \infty$,*
  (ii) *either $\limsup_{n \to \infty} c_n \leq 0$ or $\sum_{n=0}^{\infty} |b_n c_n| < \infty$.*
Then $\lim_{n \to \infty} a_n = 0$.

Lemma 2.9 [16] *Let (X, d) be a metric space. A subset C of X is compact if and only if every sequence in C contains a convergent subsequence with limit in C.*

3 The main result

In this section, we establish a strong convergence theorem for finding a common fixed point of an asymptotically nonexpansive semigroup in a smooth Banach space.

Theorem 3.1 *Let $\varphi = \{T(s) : s \in S\}$ be a representation of S as a Lipschitzian mapping from a nonempty compact convex subset C of a smooth Banach space E into C, with uniform Lipschitzian constant $\lim_s l(s) \le 1$ on the Lipschitz constant of mappings, such that $\mathrm{Fix}(\varphi) \ne \emptyset$, and let f be a contraction of C into itself with constant $\alpha \in (0,1)$. Let X be a left invariant and φ-stable subspace of B(S) containing 1 and the function $t \to \langle T_t x, y \rangle$ is an element of X for each $x \in C$ and $y \in H$, and let $\{\mu_{i,n}\}_{i=1,n=1}^{m,\ \infty}$ be a finite family of left regular sequences of invariant means on X such that for $i = 1, 2, \dots, m$, $\lim_{n \to \infty} \|\mu_{i,n+1} - \mu_{i,n}\| = 0$. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $(0,1)$ satisfying conditions (C_1)-(C_4), and let $\{\delta_n\}_{i=1,n=1}^{m,\ \infty}$ be a sequence in $(0,1)$ satisfying the condition*

(C_5') $\lim_{n \to \infty} \delta_{i,n} = 1$, $i = 1, 2, \dots, m$.

If $\{x_n\}_{n=1}^{\infty}$ and $\{y_{i,n}\}_{i=1,n=1}^{m,\ \infty}$ are sequences generated by $x_1 \in C$ and

$$x_{n+1} = \alpha_n f(y_{1,n}) + \beta_n x_n + \gamma_n T_{\mu_{1,n}} y_{1,n},$$

$$y_{i,n} = \delta_{i,n} y_{i+1,n} + (I - \delta_{i,n}) T_{\mu_{i,n}} y_{i+1,n}, \quad i = 1, 2, \dots, m, \tag{5}$$

$$y_{m+1,n} = x_n,$$

then $\{x_n\}_{n=1}^{\infty}$ and $\{y_{i,n}\}_{i=1,n=1}^{m,\ \infty}$ converge strongly to $z \in \mathrm{Fix}(\varphi)$, which is the unique solution of the variational inequality

$$\langle (f - I)z, J(y - z) \rangle \le 0, \quad \forall y \in \mathrm{Fix}(\varphi). \tag{6}$$

Equivalently, $z = Pf(z)$, where P denotes the unique sunny nonexpansive retraction of C onto $\mathrm{Fix}(\varphi)$.

Proof From Lemma 2.1 and the definition of $\{y_{i,n}\}_{i=1,n=1}^{m,\ \infty}$, for every $z \in \mathrm{Fix}(\varphi)$, we have

$$
\begin{aligned}
\|y_{i,n} - z\| &= \left\| \delta_{i,n} y_{i+1,n} + (1 - \delta_{i,n}) T_{\mu_{i,n}} y_{i+1,n} - z \right\| \\
&\le \delta_{i,n} \|y_{i+1,n} - z\| + (1 - \delta_{i,n}) \| T_{\mu_{i,n}} y_{i+1,n} - T_{\mu_{i,n}} z\| \\
&= \delta_{i,n} \|y_{i+1,n} - z\| + (1 - \delta_{i,n}) \|y_{i+1,n} - z\| \\
&= \|y_{i+1,n} - z\|.
\end{aligned}
$$

Therefore, we have

$$\|y_{1,n} - z\| \le \|y_{2,n} - z\| \le \cdots \le \|y_{m,n} - z\| \le \|x_n - z\|. \tag{7}$$

We shall divide the proof into several steps.

Step 1. Let $\{t_n\}_{n=1}^{\infty}$ be a sequence in C. Then

$$\lim_{n \to \infty} \| T_{\mu_{i,n+1}} t_n - T_{\mu_{i,n}} t_n \| = 0, \quad i = 1, 2, \dots, m.$$

Proof of Step 1. This assertion is proved in [24, 25].

Step 2. $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$.

Proof of Step 2. From the definition of $\{y_{i,n}\}_{i=1,n=1}^{m,\ \infty}$, we have

$$\|y_{i,n+1} - y_{i,n}\|$$
$$= \left\| \delta_{i,n+1} y_{i+1,n+1} + (1 - \delta_{i,n+1}) T_{\mu_{i,n+1}} y_{i+1,n+1} \right.$$
$$\left. - \delta_{i,n} y_{i+1,n} - (1 - \delta_{i,n}) T_{\mu_{i,n}} y_{i+1,n} \right\|$$
$$= \left\| \delta_{i,n+1} y_{i+1,n+1} - \delta_{i,n+1} y_{i+1,n} + \delta_{i,n+1} y_{i+1,n} \right.$$
$$\left. + (1 - \delta_{i,n+1}) T_{\mu_{i,n+1}} y_{i+1,n+1} - \delta_{i,n} y_{i+1,n} - (1 - \delta_{i,n}) T_{\mu_{i,n}} y_{i+1,n} \right\|$$
$$\leq \delta_{i,n+1} \|y_{i+1,n+1} - y_{i+1,n}\| + |\delta_{i,n+1} - \delta_{i,n}| \|y_{i+1,n}\|$$
$$+ (1 - \delta_{i,n+1}) \|T_{\mu_{i,n+1}} y_{i+1,n+1}\| + (1 - \delta_{i,n}) \|T_{\mu_{i,n}} y_{i+1,n}\|$$
$$\leq \|y_{i+1,n+1} - y_{i+1,n}\| + |\delta_{i,n+1} - \delta_{i+1,n}| \|y_{i+1,n}\|$$
$$+ (1 - \delta_{i,n+1}) \|T_{\mu_{i,n+1}} y_{i+1,n+1}\| + (1 - \delta_{i,n}) \|T_{\mu_{i,n}} y_{i+1,n}\|,$$

which implies that

$$\|y_{i,n+1} - y_{i,n}\|$$
$$\leq \|x_{n+1} - x_n\| + \sum_{j=i}^{m} \left(|\delta_{j,n+1} - \delta_{j,n}| \|y_{j+1,n}\| \right.$$
$$\left. + (1 - \delta_{i,n+1}) \|T_{\mu_{i,n+1}} y_{i+1,n+1}\| + (1 - \delta_{i,n}) \|T_{\mu_{i,n}} y_{i+1,n}\| \right). \tag{8}$$

Define

$$x_{n+1} = (1 - \beta_n) z_n + \beta_n x_n, \quad n \geq 1. \tag{9}$$

Observe that from the definition of z_n, we obtain

$$z_{n+1} - z_n = \frac{x_{n+2} - \beta_{n+1} x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n}$$
$$= \frac{\alpha_{n+1} f(y_{1,n+1}) + \gamma_{n+1} T_{\mu_{1,n+1}} y_{1,n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n f(y_{1,n}) + \gamma_n T_{\mu_{1,n}} y_{1,n}}{1 - \beta_n}$$
$$= \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \left(f(y_{1,n+1}) - T_{\mu_{1,n+1}} y_{1,n+1} \right) - \frac{\alpha_n}{1 - \beta_n} \left(f(y_{1,n}) - T_{\mu_{1,n}} y_{1,n} \right)$$
$$+ T_{\mu_{1,n+1}} y_{1,n+1} - T_{\mu_{1,n}} y_{1,n}.$$

It follows that

$$\|z_{n+1} - z_n\|$$
$$\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \left\| f(y_{1,n+1}) - T_{\mu_{1,n+1}} y_{1,n+1}) \right\|$$
$$+ \frac{\alpha_n}{1 - \beta_n} \left\| f(y_{1,n}) - T_{\mu_{1,n}} y_{1,n} \right\|$$
$$+ \|y_{1,n+1} - y_{1,n}\| + \|T_{\mu_{1,n+1}} y_{1,n} - T_{\mu_{1,n}} y_{1,n}\|. \tag{10}$$

Substituting (8) into (10), we obtain

$$\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|$$

$$\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \left\| f(y_{1,n+1}) - T_{\mu_{1,n+1}} y_{1,n+1} \right\|$$

$$+ \frac{\alpha_n}{1 - \beta_n} \left\| f(y_{1,n}) - T_{\mu_{1,n}} y_{1,n} \right\|$$

$$+ \left\| T_{\mu_{1,n+1}} y_{1,n} - T_{\mu_{1,n}} y_{1,n} \right\| + \sum_{j=1}^{m} \left(|\delta_{j,n+1} - \delta_{j,n}| \|y_{j+1,n}\| \right)$$

$$+ (1 - \delta_{i,n+1}) \| T_{\mu_{i,n+1}} y_{i+1,n+1} \| + (1 - \delta_{i,n}) \| T_{\mu_{i,n}} y_{i+1,n} \|). \tag{11}$$

It follows from Step 1, conditions (C_2) and (C_5') that

$$\limsup_{n \to \infty} \left(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \right) \leq 0.$$

Applying Lemma 2.6 to (9), we get

$$\lim_{n \to \infty} \|x_n - z_n\| = 0.$$

Consequently,

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} (1 - \beta_n) \|x_n - z_n\| = 0.$$

Step 3. We claim that $\omega(\{x_n\}_{n=1}^{\infty}) \subset \text{Fix}(\varphi)$, where

$$\omega\left(\{x_n\}_{n=1}^{\infty}\right) := \left\{ x \in C : \{x_{n_j}\}_{j=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}, \lim_{j \to \infty} \|x_{n_j} - x\| = 0 \right\}.$$

Proof of Step 3. From Lemma 2.9, we get $\omega(\{x_n\}_{n=1}^{\infty}) \neq \emptyset$.
Let $x \in \omega(\{x_n\}_{n=1}^{\infty})$. Then there exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that

$$\lim_{j \to \infty} \|x_{n_j} - x\| = 0. \tag{12}$$

Observe that

$$\|x_n - T_{\mu_{1,n}} x_n\|$$

$$\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_{\mu_{1,n}} x_n\|$$

$$= \|x_n - x_{n+1}\| + \left\| \alpha_n \left(f(y_{1,n}) - T_{\mu_{1,n}} x_n \right) + \beta_n (x_n - T_{\mu_{1,n}} x_n) \right.$$

$$+ \left. \gamma_n (T_{\mu_{1,n}} y_{1,n} - T_{\mu_{1,n}} x_n) \right\|$$

$$\leq \|x_n - x_{n+1}\| + \alpha_n \left\| f(y_{1,n}) - T_{\mu_{1,n}} x_n \right\| + \beta_n \|x_n - T_{\mu_{1,n}} x_n\|$$

$$+ \gamma_n \| T_{\mu_{1,n}} y_{1,n} - T_{\mu_{1,n}} x_n \|$$

$$\leq \|x_n - x_{n+1}\| + \alpha_n \left\| f(y_{1,n}) - T_{\mu_{1,n}} x_n \right\| + \beta_n \|x_n - T_{\mu_{1,n}} x_n\|$$

$$+ \gamma_n \|y_{1,n} - x_n\|$$

$$\leq \|x_n - x_{n+1}\| + \alpha_n \left\| f(y_{1,n}) - T_{\mu_{1,n}} x_n \right\| + \beta_n \|x_n - T_{\mu_{1,n}} x_n\|$$

$$+ \gamma_n \sum_{i=1}^{m} \|y_{i,n} - y_{i+1,n}\|$$

$$= \|x_n - x_{n+1}\| + \alpha_n \left\| f(y_{1,n}) - T_{\mu_{1,n}} x_n \right\| + \beta_n \|x_n - T_{\mu_{1,n}} x_n\|$$

$$+ \gamma_n \sum_{i=1}^{m} (1 - \delta_{i,n}) \|y_{i+1,n} - T_{\mu_{i,n}} y_{i+1,n}\|.$$

Therefore, we have

$$(1 - \beta_n) \|x_n - T_{\mu_{1,n}} x_n\|$$

$$\leq \|x_n - x_{n+1}\| + \alpha_n \left\| f(y_{1,n}) - T_{\mu_{1,n}} x_n \right\|$$

$$+ \gamma_n \sum_{i=1}^{m} (1 - \delta_{i,n}) \|y_{i+1,n} - T_{\mu_{i,n}} y_{i+1,n}\|. \tag{13}$$

From condition (C_4), it follows that

$$\liminf_{n \to \infty} (1 - \beta_n) > 0. \tag{14}$$

By conditions (C_2) and (C_5'), Step 2, (13) and (14), we have

$$\lim_{n \to \infty} \|x_n - T_{\mu_{1,n}} x_n\| = 0. \tag{15}$$

Indeed, observe that

$$\limsup_{j \to \infty} \|x - T_{\mu_{1,n_j}} x\| \leq \limsup_{j \to \infty} \left(\|x - x_{n_j}\| + \|x_{n_j} - T_{\mu_{1,n_j}} x_{n_j}\| \right.$$

$$\left. + \|T_{\mu_{1,n_j}} x_{n_j} - T_{\mu_{1,n_j}} x\| \right).$$

Thus, due to (15), Lemma 2.2 and Lemma 2.3, we get $x \in \text{Fix}(\varphi)$.

Step 4. $\{x_n\}_{n=1}^{\infty}$ converges strongly to $z = Pf(z)$.

Proof of Step 4. From Lemma 2.4 there exists a unique sunny nonexpansive retraction P of C onto $\text{Fix}(\varphi)$. Since f is a contraction of C into itself, therefore Pf is a contraction. Then the Banach contraction guarantees that Pf has a unique fixed point z. By Lemma 2.5, z is the unique solution of the variational inequality

$$\langle (f - I)z, J(y - z) \rangle \leq 0, \quad \forall y \in \text{Fix}(\varphi). \tag{16}$$

Let us show that

$$\limsup_{n \to \infty} \langle (f - I)z, J(x_n - z) \rangle \leq 0.$$

Indeed, we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n \to \infty} \langle (f - I)z, J(x_n - z) \rangle = \lim_{k \to \infty} \langle (f - I)z, J(x_{n_k} - z) \rangle. \tag{17}$$

Since C is compact, we may assume, with no loss of generality, that $\{x_{n_k}\}$ converges strongly to some $y \in C$. By Step 3, $y \in \mathrm{Fix}(\varphi)$. Because the duality mapping J is norm to weak-star continuous from (16) and (17), we have

$$\limsup_{n \to \infty} \langle (f - I)z, J(x_n - z) \rangle \leq 0. \tag{18}$$

Using Lemma 2.1, Lemma 2.7 and relation (7), we have

$$\|x_{n+1} - z\|^2$$
$$= \left\| \alpha_n \big(f(y_{1,n}) - z \big) + \beta_n (x_n - z) + \gamma_n (T_{\mu_{1,n}} y_{1,n} - z) \right\|^2$$
$$\leq \left\| \beta_n (x_n - z) + \gamma_n (T_{\mu_{1,n}} y_{1,n} - z) \right\|^2 + 2\alpha_n \big\langle f(y_{1,n}) - z, J(x_{n+1} - z) \big\rangle$$
$$\leq \big[\beta_n \|x_n - z\| + \gamma_n \|T_{\mu_{1,n}} y_{1,n} - z\| \big]^2 + 2\alpha_n \big\langle f(y_{1,n}) - f(z), J(x_{n+1} - z) \big\rangle$$
$$\quad + 2\alpha_n \big\langle f(z) - z, J(x_{n+1} - z) \big\rangle$$
$$\leq \big[\beta_n \|x_n - z\| + \gamma_n \|y_{1,n} - z\| \big]^2 + 2\alpha_n \alpha \|y_{1,n} - z\| \, \|J(x_{n+1} - z)\|$$
$$\quad + 2\alpha_n \big\langle f(z) - z, J(x_{n+1} - z) \big\rangle$$
$$\leq \big[\beta_n \|x_n - z\| + \gamma_n \|x_n - z\| \big]^2 + 2\alpha_n \alpha \|x_n - z\| \, \|x_{n+1} - z\|$$
$$\quad + 2\alpha_n \big\langle f(z) - z, J(x_{n+1} - z) \big\rangle$$
$$\leq (1 - \alpha_n)^2 \|x_n - z\|^2 + \alpha_n \alpha \big(\|x_n - z\|^2 + \|x_{n+1} - z\|^2 \big)$$
$$\quad + 2\alpha_n \big\langle f(z) - z, J(x_{n+1} - z) \big\rangle,$$

and consequently,

$$\|x_{n+1} - z\|^2$$
$$\leq \frac{(1 - \alpha_n)^2 + \alpha_n \alpha}{1 - \alpha_n \alpha} \|x_n - z\|^2 + \frac{2\alpha_n}{1 - \alpha_n \alpha} \big\langle f(z) - z, J(x_{n+1} - z) \big\rangle$$
$$\leq \left(1 - \frac{2\alpha_n(1 - \alpha_n)}{1 - \alpha_n \alpha} \right) \|x_n - z\|^2$$
$$\quad + \frac{2\alpha_n(1 - \alpha_n)}{1 - \alpha_n \alpha} \left(\frac{1}{1 - \alpha} \big\langle f(z) - z, J(x_{n+1} - z) \big\rangle + \frac{\alpha_n}{2(1 - \alpha)} \|x_n - z\|^2 \right).$$

Then we have

$$\|x_{n+1} - z\|^2 \leq (1 - b_n) \|x_n - z\|^2 + b_n c_n, \tag{19}$$

where $b_n = \frac{2\alpha_n(1 - \alpha_n)}{1 - \alpha_n \alpha}$ and

$$c_n = \frac{1}{1 - \alpha} \big\langle f(z) - z, J(x_{n+1} - z) \big\rangle + \frac{\alpha_n}{2(1 - \alpha)} \|x_n - z\|^2.$$

It follows from conditions (C_2), (C_3) and (18) that

$$\sum_{n=1}^{\infty} b_n = \infty, \qquad \limsup_{n \to \infty} c_n \leq 0.$$

Therefore, applying Lemma 2.8 to (19), we have that $\{x_n\}_{n=1}^{\infty}$ converges strongly to $z = Pf(z)$ and since for $i = 1, 2, \ldots, m$, $\|y_{i,n} - z\| \leq \|x_n - z\|$, therefore $\{y_n\}_{i=1,n=1}^{m,\,\infty}$ converges strongly to $z = Pf(z)$. This completes the proof. □

4 Applications

Let $\{g_{n,j}\}_{n,j=0}^{\infty}$ be a family of real numbers. Then $\{g_{n,j}\}$ is said to be the strongly regular summation method [26, 27] if $\{g_{n,j}\}$ satisfies the following conditions:

(S$_1$) $g_{n,j} \geq 0$,

(S$_2$) $\sum_{j=0}^{\infty} g_{n,j} = 1$ for every n,

(S$_3$) $\lim_{n \to \infty} g_{n,j} = 0$ for every j,

(S$_4$) $\lim_{n \to \infty} \sum_{j=0}^{\infty} |g_{n,j+1} - g_{n,j}| = 0$.

Corollary 4.1 *Let C be a compact convex subset of a smooth Banach space E, and let f be a contraction of C into itself with constant $\alpha \in (0, 1)$. Let T be an asymptotically nonexpansive mapping of C into itself with Lipschitz constants $\{k(j)\}$, and for $i = 1, 2, \ldots, m$, let $\{g_{(i,n),j}\}_{n,j=0}^{\infty}$ be a finite family of strongly regular summation methods such that*

$$\lim_{n \to \infty} \sum_{j=0}^{\infty} |g_{(i,n),j+1} - g_{(i,n),j}| = 0 \quad and \quad \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} g_{(i,n),j}\big(k(j) - 1\big) < \infty.$$

Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $(0, 1)$ satisfying conditions (C$_1$)-(C$_4$), and let $\{\delta_{i,n}\}_{i=1,n=1}^{m,\,\infty}$ be a sequence in $(0, 1)$ satisfying condition (C$_5'$). If $\{x_n\}_{n=1}^{\infty}$ and $\{y_{i,n}\}_{i=1,n=1}^{m,\,\infty}$ are sequences generated by $x_1 \in C$ and

$$x_{n+1} = \alpha_n f(y_{1,n}) + \beta_n x_n + \gamma_n \sum_{j=0}^{\infty} g_{(1,n),j} T(j) y_{1,n},$$

$$y_{i,n} = \delta_{i,n} y_{i+1,n} + (I - \delta_{i,n}) \sum_{j=0}^{\infty} g_{(i,n),j} T(j) y_{i+1,n}, \quad i = 1, 2, \ldots, m,$$

$$y_{m+1,n} = x_n,$$

then $\{x_n\}_{n=1}^{\infty}$ and $\{y_{i,n}\}_{i=1,n=1}^{m,\,\infty}$ converge strongly to $z \in \text{Fix}(T)$, which is the unique solution of the variational inequality

$$\big\langle (f - I)z, J(y - z) \big\rangle \leq 0, \quad \forall y \in \text{Fix}(T).$$

Equivalently, $z = Pf(z)$, where P denotes the unique sunny nonexpansive retraction of C onto $\text{Fix}(T)$.

Proof Denote by $\mathbb{Z}_+ = (\mathbb{Z}, +)$ the semigroup of nonnegative integers. It is obvious that $\varphi = \{T^j : j \in \mathbb{Z}\}$ is an asymptotically nonexpansive semigroup on C. For every $n \in \mathbb{Z}_+$ and $f \in l^{\infty}(\mathbb{Z}_+)$, define

$$\mu_{i,n} f = \sum_{j=0}^{\infty} g_{(i,n),j} f(j).$$

Hence $\{\mu_{i,n}\}_{i=1,n=1}^{m,\ \infty}$ is a strongly regular sequence of means on $l^\infty(\mathbb{Z}_+)$ and $\lim_{n\to\infty}\|\mu_{i,n+1} - \mu_{i,n}\| = 0$ [28]. Further, for each $y \in C$, we have

$$T_{\mu_{i,n}} = \sum_{j=0}^{\infty} g_{(i,n),j} T(j)y.$$

By Theorem 3.1, $\{x_n\}_{n=1}^{\infty}$ and $\{y_{i,n}\}_{i=1,n=1}^{m,\ \infty}$ converge strongly to $z \in \mathrm{Fix}(T)$. This completes the proof. □

Example 4.2 Let C be a compact convex subset of a smooth Banach space E such that $0 \in C$, and let f be a contraction of C into itself with constant $\alpha \in (0,1)$. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $(0,1)$ satisfying conditions (C$_1$)-(C$_4$), and let $\{\delta_n\}_{n=1}^{\infty}$ be a sequence in $(0,1)$ satisfying condition (C$_5'$). Let $\{t_n\}_{n=1}^{\infty}$ be sequences in $(0,1)$ with $t_1 > t_2 > \cdots$, $\lim_{n\to\infty} t_n = 0$, $\lim_{n\to\infty} \frac{t_{n+1}}{t_n} = 1$ and $\sum_{n=0}^{\infty} t_n < \infty$. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences generated by $x_1 \in C$ and

$$x_{n+1} = \alpha_n f(y_n) + \beta_n x_n + \gamma_n t_n \sum_{j=0}^{\infty} (1-t_n)^j \sqrt[n]{\left(1 + \frac{1}{n^2}\right)^j} y_n,$$

$$y_n = \delta_n x_n + (I - \delta_n) t_n \sum_{j=0}^{\infty} (1-t_n)^j \sqrt[n]{\left(1 + \frac{1}{n^2}\right)^j} x_n.$$

Then $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ converge strongly to $z \in C$.

Proof We define

$$T: C \to C,$$

$$Tx = \sqrt[n]{\left(1 + \frac{1}{n^2}\right)}x.$$

Obviously, T is an asymptotically nonexpansive mapping with Lipschitz constants $l_n = (1 + \frac{1}{n^2})$. Define $g_{n,j} = t_n(1 - t_n)^j$. Then it follows that $\{q_{n,j}\}$ is a strongly regular summation method [28]. We also have

$$\sum_{n=0}^{\infty} \sum_{j=0}^{\infty} g_{n,j}(l_n - 1) = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} t_n(1 - t_n)^j \left(1 + \frac{1}{n^2} - 1\right)$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} t_n(1 - t_n)^j \frac{1}{n^2}$$

$$\leq \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} t_n \frac{1}{n^2}$$

$$= \sum_{n=0}^{\infty} t_n \sum_{j=0}^{\infty} \frac{1}{n^2} < \infty.$$

Therefore, by taking $y_{1,n} = y_n$ and $\delta_{1,n} = \delta_n$ in Corollary 4.1, we complete the proof. □

Let $Q = \{q_{n,m}\}_{n,m=0}^{\infty}$ be a matrix satisfying the following conditions:

(a) $\sup_{n \geq 0} \sum_{m=0}^{\infty} |q_{n,m}| < \infty$,

(b) $\sum_{m=0}^{\infty} q_{n,m} = 1$ for every $n \in \mathbb{N}$,

(c) $\lim_{n \to \infty} \sum_{m=0}^{\infty} |q_{n,m+1} - q_{n,m}| = 0$.

Such a matrix Q is called strongly regular in the sense of Lorentz [29]. If Q is a strongly regular matrix, then for each $m \in \mathbb{N}$, we have $\lim_{n \to \infty} q_{n,m} = 0$, see [30]. Strongly regular matrices were used in the context of nonlinear ergodic theory in [31] and [32].

Corollary 4.3 *Let C be a compact convex subset of a smooth Banach space E. Let T be an asymptotically nonexpansive mapping of C into itself, and let $Q = \{q_{n,m}\}_{n,m=0}^{\infty}$ be a strongly regular matrix. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $(0,1)$ satisfying conditions (C_1)-(C_4), and let $\{\delta_n\}_{i=1,n=1}^{m,\ \infty}$ be a sequence in $(0,1)$ satisfying condition (C_5'). If $\{x_n\}_{n=1}^{\infty}$ and $\{y_{i,n}\}_{i=1,n=1}^{m,\ \infty}$ are sequences generated by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n f(y_{1,n}) + \beta_n x_n + \gamma_n \sum_{m=0}^{\infty} q_{(1,n),m} T^m y_{1,n},$$

$$y_{i,n} = \delta_{i,n} y_{i+1,n} + (I - \delta_{i,n}) \sum_{m=0}^{\infty} q_{(i,n),m} T^m y_{i+1,n}, \quad i = 1, 2, \ldots, m,$$

$$y_{m+1,n} = x_n,$$

then $\{x_n\}_{n=1}^{\infty}$ and $\{y_{i,n}\}_{i=1,n=1}^{m,\ \infty}$ converge strongly to $z \in \mathrm{Fix}(T)$, which is the unique solution of the variational inequality

$$\langle (f - I)z, J(y - z) \rangle \leq 0, \quad \forall y \in \mathrm{Fix}(T).$$

Equivalently, $z = Pf(z)$, where P denotes the unique sunny nonexpansive retraction of C onto $\mathrm{Fix}(T)$.

Proof Let $x \in C$. For each $n \in \mathbb{N}$, define

$$\mu_{i,n} f = \sum_{m=0}^{\infty} q_{(i,n),m} f(m)$$

for each $f \in L^{\infty}(\mathbb{N})$. Hence $\{\mu_{i,n}\}_{i=1,n=1}^{m,\ \infty}$ is a strongly regular sequence of means on $l^{\infty}(\mathbb{Z}_+)$ and $\lim_{n \to \infty} \|\mu_{i,n+1} - \mu_{i,n}\| = 0$ [33]. Further, for each $y \in C$, we have

$$T_{\mu_{i,n}} = \sum_{m=0}^{\infty} q_{(i,n),m} T^m y.$$

By Theorem 3.1, $\{x_n\}_{n=1}^{\infty}$ and $\{y_{i,n}\}_{i=1,n=1}^{m,\ \infty}$ converge strongly to $z \in \mathrm{Fix}(T)$. This completes the proof. □

Example 4.4 Let C be a compact convex subset of a smooth Banach space E such that $0 \in C$, and let f be a contraction of C into itself with constant $\alpha \in (0,1)$. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$

and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $(0, 1)$ satisfying conditions (C_1)-(C_4), and let $\{\delta_n\}_{n=1}^{\infty}$ be a sequence in $(0, 1)$ satisfying condition (C_5'). Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences generated by $x_1 \in C$ and

$$x_{n+1} = \alpha_n f(y_n) + \beta_n x_n + \gamma_n \sum_{m=0}^{n} \frac{2m}{n(n-1)} \sqrt[n]{\left(1 - \frac{1}{n^2}\right)^m} y_n,$$

$$y_n = \delta_n x_n + (I - \delta_n) \sum_{m=0}^{n} \frac{2m}{n(n-1)} \sqrt[n]{\left(1 - \frac{1}{n^2}\right)^m} x_n.$$

Then $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ converge strongly to $z \in C$.

Proof We define

$$T : C \to C,$$

$$Tx = \sqrt[n]{\left(1 - \frac{1}{n^2}\right)} x.$$

Obviously, T is an asymptotically nonexpansive mapping with Lipschitz constants $l_n = (1 - \frac{1}{n^2})$. Define

$$q_{n,m} = \begin{cases} \frac{2m}{n(n-1)}, & 0 \le m \le n, \\ 0, & m > n. \end{cases}$$

Then it follows that $\{q_{n,m}\}$ is a strongly regular matrix. Further, we have

$$\sum_{m=0}^{\infty} q_{n,m} = \sum_{m=0}^{n} \frac{2m}{n(n-1)} = \frac{2}{n(n-1)} \frac{n(n-1)}{2} = 1.$$

Therefore

$$\sup_{n \ge 0} \sum_{m=0}^{\infty} q_{n,m} < \infty.$$

On the other hand,

$$\sum_{m=0}^{\infty} |q_{n,m+1} - q_{n,m}| = \sum_{m=0}^{n} |q_{n,m+1} - q_{n,m}|$$

$$= \sum_{m=0}^{n-1} |q_{n,m+1} - q_{n,m}| + |q_{n,n+1} - q_{n,n}|$$

$$= \sum_{m=0}^{n-1} \left| \frac{2(m+1)}{n(n-1)} - \frac{2m}{n(n-1)} \right| + \left| 0 - \frac{2n}{n(n-1)} \right|$$

$$= \sum_{m=0}^{n-1} \left| \frac{2(m+1)}{n(n-1)} - \frac{2m}{n(n-1)} \right| + \frac{2n}{n(n-1)}$$

$$= \sum_{m=0}^{n-1} \frac{2}{n(n-1)} + \frac{2n}{n(n-1)}$$

$$= \frac{4n}{n(n-1)} \to 0$$

as $n \to \infty$. By taking $y_{1,n} = y_n$ and $\delta_{1,n} = \delta_n$ in Corollary 4.3, we complete the proof. $\qquad\square$

Corollary 4.5 *Let C be a compact convex subset of a smooth Banach space E such that $0 \in C$, and let T be an asymptotically nonexpansive mapping of C into itself with Lipschitz constants $\{k(j)\}$ satisfying $\sum_{j=0}^{\infty}(k(j) - 1) < \infty$. Let f be a contraction of C into itself with constant $\alpha \in (0,1)$, let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $(0,1)$ satisfying conditions (C_1)-(C_4), and let $\{\delta_n\}_{n=1}^{\infty}$ be a sequence in $(0,1)$ satisfying condition (C_5'). If $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequences generated by $x_1 \in C$ and*

$$y_n = \delta_n x_n + \frac{2(1 - \delta_n)}{(n+1)^2} \sum_{j=0}^{(n+1)^2} T^j x_n,$$

$$x_{n+1} = \alpha_n f(y_n) + \beta_n x_n + \frac{2\gamma_n}{(n+1)^2} \sum_{j=0}^{(n+1)^2} T^j y_n,$$

then $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ converge strongly to $z \in \text{Fix}(T)$, which is the unique solution of the variational inequality

$$\langle (f - I)z, J(y - z) \rangle \le 0, \quad \forall y \in \text{Fix}(T). \tag{20}$$

Equivalently, $z = Pf(z)$, where P denotes the unique sunny nonexpansive retraction of C onto $\text{Fix}(T)$.

Proof Denote by $\mathbb{Z}_+ = (\mathbb{Z}, +)$ the semigroup of nonnegative integers. It is obvious that $\varphi = \{T^j : j \in \mathbb{Z}\}$ is an asymptotically nonexpansive semigroup on C. For every $n \in \mathbb{Z}_+$ and $f \in l^{\infty}(\mathbb{Z}_+)$, define

$$\mu_n f = \frac{1}{(n+1)^2} \sum_{j=0}^{\infty} f(j).$$

Hence $\{\mu_n\}_{n=1}^{\infty}$ is a strongly regular sequence of means on $l^{\infty}(\mathbb{Z}_+)$ and $\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0$ [28]. Further, for each $y \in C$, we have

$$T_{\mu_n} = \frac{1}{(n+1)^2} \sum_{j=0}^{\infty} T^j y.$$

By Theorem 3.1, $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ converge strongly to $z \in \text{Fix}(T)$. This completes the proof. $\qquad\square$

Example 4.6 Let C be a compact convex subset of a smooth Banach space E such that $0 \in C$. Let f be a contraction of C into itself with constant $\alpha \in (0,1)$, let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $(0,1)$ satisfying conditions (C_1)-(C_4), and let $\{\delta_n\}_{n=1}^{\infty}$ be a sequence

in $(0,1)$ satisfying condition (C_5'). If $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequences generated by $x_1 \in C$ and

$$y_n = \delta_n x_n + \frac{2(1-\delta_n)}{(n+1)^2} \sum_{j=0}^{(n+1)^2} \sqrt[n]{\left(1 + \frac{1}{n\ln^2 n}\right)^j} x_n,$$

$$x_{n+1} = \alpha_n f(y_n) + \beta_n x_n + \frac{2\gamma_n}{(n+1)^2} \sum_{j=0}^{(n+1)^2} \sqrt[n]{\left(1 + \frac{1}{n\ln^2 n}\right)^j} y_n,$$

then $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ converge strongly to $z \in C$.

Proof We define

$$T : C \to C,$$

$$T(x) = \sqrt[n]{\left(1 + \frac{1}{n\ln^2 n}\right)} x.$$

Obviously, T is an asymptotically nonexpansive mapping with Lipschitz constants $l_n(T) = (1 + \frac{1}{n\ln^2 n})$. Moreover,

$$\sum_{n=0}^{\infty}(l_n - 1) = \sum_{n=0}^{\infty}\left(1 + \frac{1}{n\ln^2 n} - 1\right) = \sum_{n=0}^{\infty}\frac{1}{n\ln^2 n} < \infty.$$

Therefore, applying Corollary 4.5, the result follows. □

Remark 4.7 For deducing some more applications, we refer to [13, 19, 24, 25, 28, 30].

Remark 4.8 Theorem 3.1 improves and extends [4, Theorem 3.1] and [2, Theorem 3.1] in the following aspects.

(1) Theorem 3.1 extends [4, Theorem 3.1] and [2, Theorem 3.1] from one sequence of means to a finite family of sequences of means.

(2) In Theorem 3.1, by taking $T_{\mu_{i,n}} = I$ for $i = 1, 2, \ldots, m-1$, $T_{\mu_{m,n}} = T_{\mu_n}$ and $y_{m,n} = y_n$, one can see that [4, Theorem 3.1] is a special case of Theorem 3.1.

(3) In Theorem 3.1, by taking $T_{\mu_{i,n}} = I$ for $i = 1, 2, \ldots, m$, one can see that [2, Theorem 3.1] is a special case of Theorem 3.1.

(4) Theorem 3.1 gives all consequences of [4, Theorem 3.1] and [2, Theorem 3.1] without assumption C_5 used in [4, Theorem 3.1] and [2, Theorem 3.1].

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