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Optimality and duality for nonsmooth multiobjective optimization problems

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Abstract

In this paper, we consider a nonsmooth multiobjective programming problems including support functions with inequality and equality constraints. Necessary and sufficient optimality conditions are obtained by using higher-order strong convexity for Lipschitz functions. Mond-Weir type dual problem and duality theorems for a strict minimizer of order m are given.

Keywords: nonsmooth multiobjective programming; strict minimizers; optimality conditions; duality

1 Introduction

Nonlinear analysis problems are a new and vital area of optimization theory, mathematical physics, economics, engineering and functional analysis. Moreover, nonsmooth problems occur naturally and frequently in optimization.

In 1970, Rockafellar wrote in his book that practical applications are not necessarily differentiable in applied mathematics (see [1]). So, dealing with nondifferentiable mathematical programming problems was very important. Vial [2] studied strongly and weakly convex sets and ρ-convex functions.

Auslender [3] introduced the notion of lower second-order directional derivative and obtained necessary and sufficient conditions for a strict local minimizer. Based on Auslender's results, Studniarski [4] proved necessary and sufficient conditions for the problem of the feasible set defined by an arbitrary set. Moreover, Ward [5] derived necessary and sufficient conditions for strict minimizer of order m in nondifferentiable scalar programs. Jimenez [6] introduced the notion of super-strict efficiency for vector problems and gave necessary conditions for strict minimality. Jimenez and Novo [7, 8] obtained first- and second-order optimality conditions for vector optimization problems. Bhatia [9] gave the higher-order strong convexity for Lipschitz functions and established optimality conditions for the new concept of strict minimizer of higher order for a multiobjective optimization problem.

Kim and Bae [10] formulated nondifferentiable multiobjective programs with the support functions. Also, Bae *et al.* [11] established duality theorems for nondifferentiable multiobjective programming problems under generalized convexity assumptions. Also, Kim and Lee [12] introduced the nonsmooth multiobjective programming problems involving locally Lipschitz functions and support functions. They introduced Karush-Kuhn-Tucker type optimality conditions and established duality theorems for (weak) Pareto-optimal solutions. Recently, Bae and Kim [10] established optimality conditions and duality theorems for a nondifferentiable multiobjective programming problem with support functions.

In this paper, we consider nonsmooth multiobjective programming with inequality and equality constraints. In Section 2, we introduce the concept of a strict minimizer of order m and higher-order strong convexity for this problem. In Section 3, necessary and sufficient optimality theorems are established for a strict minimizer of order m under generalized strong convexity assumptions. In Section 4, we formulate a Mond-Weir type dual problem and obtain weak and strong duality theorems.

2 Preliminaries

Let $x, y \in \mathbb{R}^n$. The following notation will be used for vectors in $\mathbb{R}^n$:

$$x < y \iff x_i < y_i, \quad i = 1, 2, \ldots, n;$$

$$x \leqq y \iff x_i \leqq y_i, \quad i = 1, 2, \ldots, n;$$

$$x \leq y \iff x_i \leqq y_i, \quad i = 1, 2, \ldots, n \text{ but } x \neq y;$$

$x \not< y$ is the negation of $x < y$;

$x \not\leq y$ is the negation of $x \leq y$.

For $x, u \in \mathbb{R}$, $x \leqq u$ and $x < u$ have the usual meaning. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, and let $\mathbb{R}^n_+$ be its nonnegative orthant.

Definition 2.1 [13] Let D be a compact convex set in $\mathbb{R}^n$. The support function $s(\cdot|D)$ is defined by

$$s(x|D) := \max\{x^T y : y \in D\}.$$

The support function $s(\cdot|D)$ has a subdifferential. The subdifferential of $s(\cdot|D)$ at x is given by

$$\partial s(x|D) := \{z \in D : z^T x = s(x|D)\}.$$

The support function $s(\cdot|D)$ is convex and everywhere finite, that is, there exists $z \in D$ such that

$$s(y|D) \geq s(x|D) + z^T(y - x) \quad \text{for all } y \in D.$$

Equivalently,

$$z^T x = s(x|D).$$

We consider the following multiobjective programming problem.

$$\text{(MOP)} \quad \text{Minimize} \quad f(x) + s(x|D) = \big(f_1(x) + s(x|D_1), \ldots, f_p(x) + s(x|D_p)\big)$$
$$\text{subject to} \quad g(x) \leqq 0, \qquad h(x) = 0, \quad x \in X,$$

where $f : X \to \mathbb{R}^p$, $g : X \to \mathbb{R}^q$ and $h : X \to \mathbb{R}^r$ are locally Lipschitz functions, respectively, and X is the convex set of $\mathbb{R}^n$. For each $i \in P = \{1, 2, \ldots, p\}$, D_i is a compact convex subset of $\mathbb{R}^n$.

Further, let $S := \{x \in X \mid g_j(x) \leqq 0, j = 1, \ldots, q, h_l(x) = 0, l = 1, \ldots, r\}$ be the feasible set of (MOP), $B(x^0, \epsilon) = \{x \in \mathbb{R}^n \mid \|x - x^0\| < \epsilon\}$ be an open ball with center x^0 and radius ϵ and $I(x^0) := \{j \in \{1, \ldots, q\} \mid g_j(x^0) = 0\}$ be the index set of active constraints at x^0.

We introduce the following definitions due to Jimenez [6].

Definition 2.2 A point $x^0 \in X$ is called a strict local minimizer for (MOP) if there exists $\epsilon > 0$ such that

$$f(x) + s(x|D) \not< f(x^0) + s(x^0|D), \quad \forall x \in B(x^0, \epsilon) \cap X.$$

Definition 2.3 Let $m \geqq 1$ be an integer. A point $x^0 \in X$ is called a strict local minimizer of order m for (MOP) if there exist $\epsilon > 0$ and $c \in \operatorname{int} \mathbb{R}^p_+$ such that

$$f(x) + s(x|D) \not< f(x^0) + s(x^0|D) + \|x - x^0\|^m c, \quad \forall x \in B(x^0, \epsilon) \cap X.$$

Definition 2.4 Let $m \geqq 1$ be an integer. A point $x^0 \in X$ is called a strict minimizer of order m for (MOP) if there exists $c \in \operatorname{int} \mathbb{R}^p_+$ such that

$$f(x) + s(x|D) \not< f(x^0) + s(x^0|D) + \|x - x^0\|^m c, \quad \forall x \in X.$$

Definition 2.5 [14] Suppose that $f : X \to \mathbb{R}$ is Lipschitz on X. Clarke's generalized directional derivative of f at $x \in X$ in the direction $d \in \mathbb{R}^n$, denoted by $f^0(x, d)$, is defined as

$$f^0(x, d) = \limsup_{y \to x\ t \downarrow 0} \frac{f(y + td) - f(y)}{t}.$$

Definition 2.6 [14] Clarke's generalized gradient of f at $x \in X$, denoted by $\partial f(x)$, is defined as

$$\partial f(x) = \{\xi \in \mathbb{R}^n : f^0(x, d) \geq \langle \xi, d \rangle\ \forall d \in \mathbb{R}^n\}.$$

Definition 2.7 For a nonempty subset X of $\mathbb{R}^n$, we denote X^*, the dual cone of X, defined by

$$X^* = \{u \in \mathbb{R}^n \mid u^T x \geqq 0, \forall x \in X\}.$$

Further, for $x^0 \in X$, $N_X(x^0)$ denotes the normal cone to X at x^0 defined by

$$N_X(x^0) = \{d \in \mathbb{R}^n \mid \langle d, x - x^0 \rangle \leqq 0, \forall x \in X\}.$$

It is clear that $(X - x^0)^* = -N_X(x^0)$.

We recall the notion of strong convexity of order m introduced by Lin and Fukushima in [15].

Definition 2.8 A function $f : X \to \mathbb{R}$ is said to be strongly convex of order m on a convex set X if there exists $c > 0$ such that for $x_1, x_2 \in X$ and $t \in [0, 1]$,

$$f\big(tx_1 + (1-t)x_2\big) \leqq tf(x_1) + (1-t)f(x_2) - ct(1-t)\|x_1 - x_2\|^m.$$

Proposition 2.1 [15] *If f_i, $i = 1, \dots, p$, are strongly convex of order m on a convex set X, then $\sum_{i=1}^p t_i f_i$ and $\max_{1 \leq i \leq p} f_i$ are also strongly convex of order m on X, where $t_i \geq 0$, $i = 1, \dots, p$.*

Definition 2.9 A locally Lipschitz function f is said to be strongly quasiconvex of order m on X if there exists a constant $c > 0$ such that for $x_1, x_2 \in X$,

$$f(x_1) \leqq f(x_2) \quad \Longrightarrow \quad \langle \xi, x_1 - x_2 \rangle + \|x_1 - x_2\|^m c \leqq 0, \quad \forall \xi \in \partial f(x_2).$$

For each $k \in \{1, \dots, p\}$ and $x \in X$, we consider the following scalarizing problem of (MOP) due to the one in [16].

$$\begin{aligned} (P_k(x^0)) \quad & \text{Minimize} \quad f_k(x) + s(x|D_k) \\ & \text{subject to} \quad f_i(x) + s(x|D_i) \leqq f_i\big(x^0\big) + s\big(x^0|D_i\big), \quad k \in P, i \neq k, \\ & \qquad g_j(x) \leqq 0, \quad j = 1, \dots, q, \qquad h_l(x) = 0, \quad l = 1, \dots, r. \end{aligned}$$

The following definition is due to the one in [17].

Definition 2.10 Let x^0 be a feasible solution for (MOP). We say that the basic regularity condition (BRC) is satisfied at x^0 if there exist no non-zero scalars $\lambda_i^0 \geqq 0$, $w_i \in D_i$, $i = 1, \dots, p$, $i \neq k$, $k \in P$, $\mu_j^0 \geqq 0$, $j \in I(x^0)$, $\mu_j^0 = 0$, $j \notin I(x^0)$, and ν_l^0, $l = 1, \dots, r$, such that

$$0 \in \sum_{i=1, i \neq k}^{p} \lambda_i^0 \big(\partial f_i\big(x^0\big) + w_i\big) + \sum_{j=1}^{q} \mu_j^0 \partial g_j(x_0) + \sum_{l=1}^{r} \nu_l^0 \partial h_l\big(x^0\big) + N_X\big(x^0\big).$$

3 Optimality conditions

In this section, we establish Fritz John necessary optimality conditions, Karush-Kuhn-Tucker necessary optimality conditions and Karush-Kuhn-Tucker sufficient optimality condition for a strict minimizer of (MOP).

Theorem 3.1 (Fritz John necessary optimality conditions) *Suppose that x^0 is a strict minimizer of order m for* (MOP) *and f_i, $i = 1, \dots, p$, g_j, $j = 1, \dots, q$, and h_l, $l = 1, \dots, r$, are locally Lipschitz functions at x^0. Then there exist $\lambda_i^0 \geqq 0$, $w_i^0 \in D_i$, $i = 1, \dots, p$, $\mu_j^0 \geqq 0$, $j = 1, \dots, q$, and ν_l^0, $l = 1, \dots, r$, not all zero such that*

$$\begin{aligned} & 0 \in \sum_{i=1}^{p} \lambda_i^0 \big(\partial f_i\big(x^0\big) + w_i^0\big) + \sum_{j=1}^{q} \mu_j^0 \partial g_j\big(x^0\big) + \sum_{l=1}^{r} \nu_l^0 \partial h_l\big(x^0\big) + N_X\big(x^0\big), \\ & \big\langle w_i^0, x^0 \big\rangle = s\big(x^0|D_i\big), \quad i = 1, \dots, p, \\ & \mu_j^0 g_j\big(x^0\big) = 0, \quad j = 1, \dots, q. \end{aligned}$$

Proof Since x^0 is a strict minimizer of order m for (MOP), it is a strict minimizer for (MOP). It can be shown that x^0 solves the following problem:

$$\begin{aligned} &\text{minimize} \quad F(x) \\ &\text{subject to} \quad g(x) \leqq 0, \qquad h(x) = 0, \end{aligned}$$

where

$$\begin{aligned} F(x) = \max\bigl\{ &\bigl(f_1(x) + s(x|D_1)\bigr) - \bigl(f_1\bigl(x^0\bigr) + s\bigl(x^0|D_1\bigr)\bigr), \dots, \\ &\bigl(f_p(x) + s(x|D_p)\bigr) - \bigl(f_p\bigl(x^0\bigr) + s\bigl(x^0|D_p\bigr)\bigr)\bigr\}. \end{aligned}$$

If it is not so, then there exits $x^1 \in \mathbb{R}^n$ such that $F(x^1) < F(x^0)$, $g(x^1) \leqq 0$, $h(x^1) = 0$. Since $F(x^0) = 0$, we have $F(x^1) < 0$. This contradicts the fact that x^0 is a strict minimizer for (MOP). Since x^0 minimizes $F(x)$, from Theorem 6.1.1 in Clarke [14], there exists $(\lambda, \mu, \nu) \in (\mathbb{R}^p, \mathbb{R}^q, \mathbb{R}^r)$ not all zero such that

$$0 \in \sum_{i=1}^{p} \lambda_i \partial F\bigl(x^0\bigr) + \sum_{j \in I(x^0)} \mu_j \partial g_j\bigl(x^0\bigr) + \sum_{l=1}^{r} \nu_l \partial h_l\bigl(x^0\bigr) + N_X\bigl(x^0\bigr).$$

Letting $\mu_j = 0$, for $j \notin I(x^0)$, we have

$$0 \in \sum_{i=1}^{p} \lambda_i \partial F\bigl(x^0\bigr) + \sum_{j=1}^{q} \mu_j \partial g_j\bigl(x^0\bigr) + \sum_{l=1}^{r} \nu_l \partial h_l\bigl(x^0\bigr) + N_X\bigl(x^0\bigr).$$

Since $F(x) = \max\{(f(x) + s(x|D)) - (f(x^0) + s(x^0|D))\}$ for any $x \in X$ and $s(x^0|D_i) = (x^0)^T w_i$, $i = 1, \dots, p$, we have

$$\begin{aligned} \partial F(x^0) &\subset co\bigl\{\partial\bigl(f_i\bigl(x^0\bigr) + s\bigl(x^0|D_i\bigr)\bigr)\bigr\} \\ &= co\bigl\{\bigl(\partial f_i\bigl(x^0\bigr) + w_i\bigr)\bigr\}, \end{aligned}$$

where $co\{\partial(f_i(x^0) + s(x^0|D_i))\}$ denotes the convex hull of $\{\partial(f_i(x^0) + s(x^0|D_i))\}$. Hence, there exist $\lambda_i^0 \geqq 0$, $w_i^0 \in D_i$, $i = 1, \dots, p$, $\mu_j^0 \geqq 0$, $j = 1, \dots, q$, and ν_l, $l = 1, \dots, r$, not all zero such that

$$\begin{aligned} &0 \in \sum_{i=1}^{p} \lambda_i\bigl(\partial f_i\bigl(x^0\bigr) + w_i^0\bigr) + \sum_{j=1}^{q} \mu_j \partial g_j\bigl(x^0\bigr) + \sum_{l=1}^{r} \nu_l \partial h_l\bigl(x^0\bigr) + N_X\bigl(x^0\bigr), \\ &\bigl\langle w_i^0, x^0\bigr\rangle = s\bigl(x^0|D_i\bigr), \quad i = 1, \dots, p, \\ &\mu_j^0 g_j\bigl(x^0\bigr) = 0, \quad j = 1, \dots, q. \end{aligned}$$

□

Theorem 3.2 (Karush-Kuhn-Tucker necessary optimality conditions) *Suppose that x^0 is a strict minimizer of order m for* (MOP) *and f_i, $i = 1, \dots, p$, g_j, $j = 1, \dots, q$, and h_l, $l = 1, \dots, r$, are locally Lipschitz functions at x^0. If the basic regularity condition* (*BRC*) *holds at x^0, then there exist $\lambda_i^0 \geqq 0$, $w_i^0 \in D_i$, $i = 1, \dots, p$, $\mu_j^0 \geqq 0$, $j = 1, \dots, q$, and ν_l^0, $l = 1, \dots, r$, such that*

$$0 \in \sum_{i=1}^{p} \lambda_i^0 \left(\partial f_i\left(x^0\right) + w_i^0\right) + \sum_{j=1}^{q} \mu_j^0 \partial g_j\left(x^0\right) + \sum_{l=1}^{r} \nu_l^0 \partial h_l\left(x^0\right) + N_X\left(x^0\right),$$
$$\left\langle w_i^0, x^0\right\rangle = s\left(x^0|D_i\right), \quad i = 1, \ldots, p,$$
$$\mu_j^0 g_j\left(x^0\right) = 0, \quad j = 1, \ldots, q,$$
$$\left(\lambda_1^0, \ldots, \lambda_p^0\right) \neq (0, \ldots, 0).$$

Proof Since x^0 is a strict minimizer of order m for (MOP), by Theorem 3.1, there exist $w_i^0 \in D_i$, $\lambda_i^0 \geqq 0$, $i = 1, \ldots, p$, $\mu_j^0 \geqq 0$, $j = 1, \ldots, q$, and ν_l, $l = 1, \ldots, r$, not all zero such that

$$0 \in \sum_{i=1}^{p} \lambda_i^0 \left(\partial f_i\left(x^0\right) + w_i^0\right) + \sum_{j=1}^{q} \mu_j^0 \partial g_j\left(x^0\right) + \sum_{l=1}^{r} \nu_l^0 \partial h_l\left(x^0\right) + N_X\left(x^0\right),$$
$$\left\langle w_i^0, x^0\right\rangle = s\left(x^0|D_i\right), \quad i = 1, \ldots, p,$$
$$\mu_j^0 g_j\left(x^0\right) = 0, \quad j = 1, \ldots, q.$$

It can be shown that $(\lambda_1^0, \ldots, \lambda_p^0) \neq (0, \ldots, 0)$. If $\lambda_i^0 = 0$, $i = 1, \ldots, p$, then we have

$$0 \in \sum_{i=1, i\neq k}^{p} \lambda_i^0 \left(\partial f_i\left(x^0\right) + w_i\right) + \sum_{j=1}^{q} \mu_j^0 \partial g_j(x_0) + \sum_{l=1}^{r} \nu_l^0 \partial h_l\left(x^0\right) + N_X\left(x^0\right)$$

for each $k \in P = \{1, \ldots, p\}$. Since the basic regularity condition (BRC) holds at x^0, we have $\lambda_k = 0$, $k \in P$, $k \neq i = \{1, \ldots, p\}$, $\mu_j = 0$, $j \in I(x^0)$, and $\nu_l = 0$, $l = 1, \ldots, r$. This contradicts the fact that λ_i, λ_k, $k \in P$, $k \neq i$, μ_j, $j \in I(x^0)$ and ν_l, $l = 1, \ldots, r$, are not all simultaneously zero. Hence, $(\lambda_1, \ldots, \lambda_p) \neq (0, \ldots, 0)$. □

Theorem 3.3 (Karush-Kuhn-Tucker sufficient optimality conditions) *Assume that there exist* $\lambda_i^0 \geqq 0$, $w_i^0 \in D_i$, $i = 1, \ldots, p$, $\mu_j^0 \geqq 0$, $j = 1, \ldots, q$, *and* ν_l^0, $l = 1, \ldots, r$, *such that for* $x^0 \in X$,

$$0 \in \sum_{i=1}^{p} \lambda_i^0 \left(\partial f_i\left(x^0\right) + w_i^0\right) + \sum_{j=1}^{q} \mu_j^0 \partial g_j\left(x^0\right) + \sum_{l=1}^{r} \nu_l^0 \partial h_l\left(x^0\right) + N_X\left(x^0\right),$$
$$\left\langle w_i^0, x^0\right\rangle = s\left(x^0|D_i\right), \quad i = 1, \ldots, p,$$
$$\mu_j^0 g_j\left(x^0\right) = 0, \quad j = 1, \ldots, q,$$
$$\left(\lambda_1^0, \ldots, \lambda_p^0\right) \neq (0, \ldots, 0).$$

Assume further that f_i, $i = 1, \ldots, p$, *are strongly convex of order* m *on* X, g_j, $j \in I(x^0)$ *are strongly quasiconvex of order* m *on* X *and* $\nu^T h$ *is strongly quasiconvex of order* m *on* X. *Then* x^0 *is a strict minimizer of order* m *for* (MOP).

Proof Since f_i, $i = 1, \ldots, p$, are strongly convex of order m on X and $(\cdot)^T w_i$, $i = 1, \ldots, p$, are convex, there exists $c_i > 0$, $i = 1, \ldots, p$, such that for all $x \in X$, $\xi_i \in \partial f_i(x^0)$ and $w_i \in D_i$, $i = 1, \ldots, p$,

$$f_i(x) - f_i\left(x^0\right) \geqq \left\langle \xi_i, x - x^0\right\rangle + \left\|x - x^0\right\|^m c_i, \qquad x^T w_i - \left(x^0\right)^T w_i \geqq \left\langle x_i, x - x^0\right\rangle.$$

So, we obtain

$$\left(f_i(x) + x^T w_i\right) - \left(f_i(x^0) + (x^0)^T x_i\right) \geqq \langle \xi_i + w_i, x - x^0 \rangle + \|x - x^0\|^m c_i. \tag{3.1}$$

For $\lambda_i^0 \geqq 0$, $i = 1, \ldots, p$, (3.1) implies

$$\begin{aligned}&\sum_{i=1}^{p} \lambda_i^0 \left(f_i(x) + x^T w_i\right) - \sum_{i=1}^{p} \lambda_i^0 \left(f_i(x^0) + (x^0)^T w_i\right) \\ &\quad \geqq \sum_{i=1}^{p} \lambda_i^0 \langle \xi_i + w_i, x - x^0 \rangle + \sum_{i=1}^{p} \lambda_i^0 \|x - x^0\|^m c_i.\end{aligned} \tag{3.2}$$

For $x \in X$, we have

$$\begin{aligned}&g_j(x) \leqq g_j(x^0), \quad j \in I(x^0), \\ &\nu^T h(x) = \nu^T h(x^0).\end{aligned}$$

Since g_j, $j \in I(x^0)$ are strongly quasiconvex of order m on X and $\nu^T h$ is strongly quasiconvex of order m on X, it follows that there exist $c_j > 0$, $\eta_j \in \partial g_j(x^0)$, $j \in I(x^0)$, $c > 0$, and $\zeta \in \partial \nu^T h(x^0)$ such that

$$\begin{aligned}&\langle \eta_j, x - x^0 \rangle + \|x - x^0\|^m c_j \leqq 0, \\ &\langle \zeta, x - x^0 \rangle + \|x - x^0\|^m c \leqq 0.\end{aligned} \tag{3.3}$$

For $\mu_j^0 \geqq 0$, $j \in I(x^0)$, we obtain

$$\left\langle \sum_{j \in I(x^0)} \mu_j^0 \eta_j, x - x^0 \right\rangle + \sum_{j \in I(x^0)} \mu_j^0 \|x - x^0\|^m c_j \leqq 0. \tag{3.4}$$

Since $\mu_j^0 = 0$ for $j \notin I(x^0)$, (3.4) implies

$$\left\langle \sum_{j=1}^{q} \mu_j^0 \eta_j, x - x^0 \right\rangle + \sum_{j=1}^{q} \mu_j^0 \|x - x^0\|^m c_j \leqq 0. \tag{3.5}$$

By (3.2), (3.3) and (3.5), we get

$$\sum_{i=1}^{p} \lambda_i^0 \left(f_i(x) + x^T w_i\right) - \sum_{i=1}^{p} \lambda_i^0 \left(f_i(x^0) + (x^0)^T w_i\right) \geqq \|x - x^0\|^m a,$$

where $a = \sum_{i=1}^{p} \lambda_i^0 c_i + \sum_{j=1}^{q} \mu_j^0 c_j + \sum_{l=1}^{m} \nu_l^0 c_l$. This implies that

$$\sum_{i=1}^{p} \lambda_i^0 \left[\left(f_i(x) + x^T w_i\right) - \left(f_i(x^0) + (x^0)^T w_i\right) - \|x - x^0\|^m d_i\right] \geqq 0, \tag{3.6}$$

where $d = ae$.

Suppose that x^0 is not a strict minimizer of order m for (MOP). Then there exist $x, x^0 \in X$ and $c \in \mathbb{R}^p_+$ such that

$$f(x) + s(x|D) < f(x^0) + s(x^0|D) + \|x - x^0\|^m c, \quad \forall x \in X.$$

Since $x^T w \leqq s(x|D)$ and $(x^0)^T w = s(x^0|D)$, we have

$$\begin{aligned} f(x) + x^T w &\leqq f(x) + s(x|D) \\ &< f(x^0) + s(x^0|D) + \|x - x^0\|^m c \\ &= f(x^0) + (x^0)^T w + \|x - x^0\|^m c. \end{aligned}$$

For $\lambda_i^0 \geqq 0$, we obtain

$$\sum_{i=1}^p \lambda_i^0 \big(f_i(x) + x^T w_i\big) < \sum_{i=1}^p \lambda_i^0 \big(f_i(x^0) + (x^0)^T w_i\big) + \sum_{i=1}^p \lambda_i^0 \|x - x^0\|^m c_i.$$

This is a contradiction to (3.6). □

Remark 3.1 Suppose that g_j, $j \in I(x^0)$ are strongly convex of order m on X and that $\nu^T h$ is strongly convex of order m on X. Then the conclusion of Theorem 3.3 also holds.

Proof It follows on the lines of Theorem 3.3. □

4 Duality theorems

Now we propose the following Mond-Weir type dual (MOD) to (MOP):

$$\text{(MOD)} \quad \text{Maximize} \quad f(u) + u^T w$$

$$\text{subject to} \quad 0 \in \sum_{i=1}^p \lambda_i \big(\partial f_i(u) + w_i\big) + \sum_{j=1}^q \mu_j \partial g_j(u) + \sum_{l=1}^r \nu_l \partial h_l(u) + N_X(u), \tag{4.1}$$

$$\sum_{j=1}^q \mu_j g_j(u) + \sum_{l=1}^r \nu_l h_l(u) \geqq 0, \tag{4.2}$$

$$\lambda_i \geqq 0, \qquad w_i \in D_i, \quad i = 1, \ldots, p, \lambda^T e = 1, \tag{4.3}$$

$$\mu_j \geqq 0, \quad j = 1, \ldots, q, \qquad \nu_l, \quad l = 1, \ldots, r, u \in X. \tag{4.4}$$

Theorem 4.1 (Weak duality) *Let x and $(u, w, \lambda, \mu, \nu)$ be feasible solutions of* (MOP) *and* (MOD), *respectively. If f_i, $i = 1, \ldots, p$, are strongly convex of order m at u and $\sum_{j=1}^q \mu_j g_j(\cdot) + \sum_{l=1}^r \nu_l h_l(\cdot)$ is strongly quasiconvex of order m at u, then the following cannot hold*:

$$f_i(x) + s(x|D_i) < f_i(u) + u^T w_i, \quad i = 1, \ldots, p. \tag{4.5}$$

Proof Since x is a feasible solution of (MOP) and $(u, w, \lambda, \mu, \nu)$ is a feasible solution of (MOD), we have

$$\sum_{j=1}^{q} \mu_j g_j(x) + \sum_{l=1}^{r} \mu_l h_l(x) \leqq \sum_{j=1}^{q} \mu_j g_j(u) + \sum_{l=1}^{r} \mu_l h_l(u).$$

Since $\sum_{j=1}^{q} \mu_j g_j(\cdot) + \sum_{l=1}^{r} \nu_l h_l(\cdot)$ is strongly quasiconvex of order m at u, it follows that there exist $c_j > 0$, $\eta_j \in \partial g_j(u)$, $j = 1, \ldots, q$, $c_l > 0$, and $\zeta_l \in \partial h_l(u)$ such that

$$\left\langle \sum_{j=1}^{q} \mu_j \eta_j + \sum_{l=1}^{r} \nu_l \zeta_l, x - u \right\rangle + \sum_{j=1}^{q} \|x - u\|^m c_j + \sum_{l=1}^{r} \|x - u\|^m c_l \leqq 0. \tag{4.6}$$

Now, suppose contrary to the result that (4.5) holds. Since $x^T w_i \leqq s(x|D_i)$, $i = 1, \ldots, p$, we obtain

$$f_i(x) + x^T w_i < f_i(u) + u^T w_i, \quad i = 1, \ldots, p.$$

For $c \in \operatorname{int} \mathbb{R}^p_+$, we obtain

$$f_i(x) + x^T w_i < f_i(u) + u^T w_i + \|x - u\|^m c_i, \quad i = 1, \ldots, p. \tag{4.7}$$

For $\lambda_i \geqq 0$, we obtain

$$\sum_{i=1}^{p} \lambda_i \big(f_i(x) + x^T w_i\big) < \sum_{i=1}^{p} \lambda_i \big(f_i(u) + u^T w_i\big) + \sum_{i=1}^{p} \lambda_i \|x - u\|^m c_i. \tag{4.8}$$

Since f_i, $i = 1, \ldots, p$, are strongly convex of order m at u and $(\cdot)^T w_i$, $i = 1, \ldots, p$, are convex at u, there exists $c_i > 0$, $i = 1, \ldots, p$, such that for all $x \in X$, $\xi_i \in \partial f_i(x^0)$ and $w_i \in D_i$, $i = 1, \ldots, p$,

$$\begin{aligned} &f_i(x) - f_i(u) \geqq \langle \xi_i, x - u \rangle + \|x - u\|^m c_i, \\ &x^T w_i - u^T w_i \geqq \langle w_i, x - u \rangle. \end{aligned}$$

So, we obtain

$$\big(f_i(x) + x^T w_i\big) - \big(f_i(u) + u^T w_i\big) \geqq \langle \xi_i + w_i, x - u \rangle + \|x - u\|^m c_i. \tag{4.9}$$

For $\lambda_i \geqq 0$, $i = 1, \ldots, p$, we obtain

$$\begin{aligned} &\sum_{i=1}^{p} \lambda_i \big(f_i(x) + x^T w_i\big) - \sum_{i=1}^{p} \lambda_i \big(f_i(u) + u^T w_i\big) \\ &\quad \geqq \left\langle \sum_{i=1}^{p} \lambda_i (\xi_i + w_i), x - u \right\rangle + \sum_{i=1}^{p} \lambda_i \|x - u\|^m c_i. \end{aligned} \tag{4.10}$$

By (4.6) and (4.10), we get

$$\sum_{i=1}^{p} \lambda_i \big(f_i(x) + x^T w_i\big) - \sum_{i=1}^{p} \lambda_i \big(f_i(u) + u^T w_i\big) \geqq \|x - u\|^m a, \tag{4.11}$$

where $a = \sum_{i=1}^{p} \lambda_i c_i + \sum_{j=1}^{q} \mu_j c_j + \sum_{l=1}^{r} \nu_l c_l$. This implies that

$$\sum_{i=1}^{p} \lambda_i \big[\big(f_i(x) + x^T w_i \big) - \big(f_i(u) + u^T w_i \big) - \|x-u\|^m d_i \big] \geqq 0, \tag{4.12}$$

where $d = ae$, since $\lambda^T e = 1$. This is a contradiction to (4.8). □

Lemma 4.1 *If $g_j(\cdot)$, $j = 1, \ldots, m$, are strongly convex of order m on X and $\nu^T h$ is strongly convex of order m on X, then the same conclusion of Theorem* 4.1 *also holds.*

Proof It follows on the lines of Theorem 4.1. □

Definition 4.1 Let $m \geqq 1$ be an integer. A point $x^0 \in X$ is called a strict maximizer of order m for (MOD) if there exists $c \in \operatorname{int} \mathbb{R}_+^p$ such that

$$f\big(x^0\big) + \big(x^0\big)^T w + \big\|x - x^0\big\|^m c \nless f(x) + x^T w, \quad \forall x \in X.$$

Theorem 4.2 (Strong duality) *If x^0 is a strict minimizer of order m for* (MOP) *and the basic regularity condition* (*BRC*) *holds at x^0, then there exist $\lambda_i^0 \geqq 0$, $w_i^0 \in D_i$, $i = 1, \ldots, p$, $\mu_j^0 \geqq 0$, $j = 1, \ldots, q$, and ν_l^0, $l = 1, \ldots, r$, such that $(x^0, w^0, \lambda^0, \mu^0, \nu^0)$ is a feasible solution of* (MOD) *and $(x^0)^T w_i^0 = s(x^0|D_i)$, $i = 1, \ldots, p$. Moreover, if the assumptions of Theorem* 4.1 *are satisfied, then $(x^0, w^0, \lambda^0, \mu^0, \nu^0)$ is a strict maximizer of order m for* (MOD).

Proof By Theorem 3.3, there exists $\lambda_i^0 \geqq 0$, $w_i^0 \in D_i$, $i = 1, \ldots, p$, $\mu_j^0 \geqq 0$, $j = 1, \ldots, q$, and ν_l^0, $l = 1, \ldots, r$, such that

$$\begin{aligned}
&0 \in \sum_{i=1}^{p} \lambda_i^0 \big(\partial f_i\big(x^0\big) + w_i^0\big) + \sum_{j=1}^{q} \mu_j^0 \partial g_j\big(x^0\big) + \sum_{l=1}^{r} \nu_l^0 \partial h_l\big(x^0\big) + N_X\big(x^0\big), \\
&\big\langle w_i^0, x^0 \big\rangle = s\big(x^0|D_i\big), \quad i = 1, \ldots, p, \\
&\mu_j^0 g_j\big(x^0\big) = 0, \quad j = 1, \ldots, q, \\
&\big(\lambda_1^0, \ldots, \lambda_p^0\big) \neq (0, \ldots, 0).
\end{aligned}$$

Thus $(x^0, w^0, \lambda^0, \mu^0, \nu^0)$ is a feasible solution of (MOD) and $(x^0)^T w_i^0 = s(x^0|D_i)$, $i = 1, \ldots, p$. By Theorem 4.1, we obtain that the following holds:

$$\begin{aligned}
f_i\big(x^0\big) + \big(x^0\big)^T w_i^0 &= f_i\big(x^0\big) + s\big(x^0|D_i\big) \\
&\nless f_i(u) + u^T w_i, \quad i = 1, \ldots, p,
\end{aligned}$$

for a given feasible solution $(u, w, \lambda, \mu, \nu)$ of (MOD). For $x^0, u \in X$ and $c \in \operatorname{int} \mathbb{R}^p$, we have

$$\begin{aligned}
&f\big(x^0\big) + \big(x^0\big)^T w^0 + \big\|u - x^0\big\|^m c \\
&\quad \nless f(u) + u^T w.
\end{aligned}$$

Thus, $(x^0, w^0, \lambda^0, \mu^0, \nu^0)$ is a strict maximizer of order m for (MOD). □

Remark 4.1 Theorem 4.1 and Theorem 4.2 reduce to [13, Theorem 4.1 and Theorem 4.2] in an inequality constraint case. More exactly, $f_i(\cdot) + (\cdot)^T w_i$, $i = 1, \ldots, p$, and $g_j(\cdot)$, $j \in I(u)$ at the considered point in the framework of [13, Theorem 4.1 and Theorem 4.2] are strongly convex of order m and strongly quasiconvex of order m, respectively.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
DSK obtained necessary and sufficient optimality conditions by using higher-order strong convexity of Lipschitz functions, formulated a Mond-Weir type dual problem and established weak and strong duality theorems for a strict minimizer of order *m*. KDB carried out the duality studies and participated in the sequence alignment and drafted the manuscript. All authors read and approved the final manuscript.

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