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Optimal bounds for the Neuman-Sándor mean in terms of the first Seiffert and quadratic means

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Abstract
In this paper, we find the least value α and the greatest value β such that the double inequality

$$P^{\alpha}(a,b)Q^{1-\alpha}(a,b) < M(a,b) < P^{\beta}(a,b)Q^{1-\beta}(a,b)$$

holds true for all $a,b>0$ with $a\neq b$, where $P(a,b)$, $M(a,b)$ and $Q(a,b)$ are the first Seiffert, Neuman-Sándor and quadratic means of a and b, respectively.

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1 Introduction

Let u, v and w be the bivariate means such that $u(a,b) < w(a,b) < v(a,b)$ for all $a,b>0$ with $a\neq b$. The problems of finding the best possible parameters α and β such that the inequalities $\alpha u(a,b)+(1-\alpha)v(a,b) < w(a,b) < \beta u(a,b)+(1-\beta)v(a,b)$ and $u^{\alpha}(a,b)v^{1-\alpha}(a,b) < w(a,b) < u^{\beta}(a,b)v^{1-\beta}(a,b)$ hold for all $a,b>0$ with $a\neq b$ have attracted the interest of many mathematicians.

For $a,b>0$ with $a\neq b$, the first Seiffert mean $P(a,b)$ [1], the Neuman-Sándor mean $M(a,b)$ [2], the quadratic mean $Q(a,b)$ are defined by

$$P(a,b)=\frac{a-b}{4\arctan(\sqrt{a/b})-\pi},\qquad M(a,b)=\frac{a-b}{2\sinh^{-1}(\frac{a-b}{a+b})},\qquad Q(a,b)=\sqrt{\frac{a^2+b^2}{2}}, \tag{1.1}$$

respectively. In here, $\sinh^{-1}(x)=\log(x+\sqrt{x^2+1})$ is the inverse hyperbolic sine function.

Recently, the means P, M and Q have been the subject of intensive research. In particular, many remarkable inequalities for these means can be found in the literature [3–14]. The first Seiffert mean $P(a,b)$ can be rewritten as (see [2, Eq. (2.4)])

$$P(a,b)=\frac{a-b}{2\arcsin[(a-b)/(a+b)]}. \tag{1.2}$$

Let $H(a,b) = 2ab/(a+b)$, $L(a,b) = (b-a)/(\log b - \log a)$, $A(a,b) = (a+b)/2$, $T(a,b) = (a-b)/[2\arctan((a-b)/(a+b))]$ and $C(a,b) = (a^2+b^2)/(a+b)$ be the harmonic, logarithmic, arithmetic, second Seiffert and contra-harmonic means of a and b, respectively. Then it is known that the inequalities

$$H(a,b) < L(a,b) < P(a,b) < A(a,b) < M(a,b) < T(a,b) < Q(a,b) < C(a,b)$$

hold for all $a, b > 0$ with $a \neq b$.

Neuman and Sándor [2, 15] proved that the inequalities

$$\begin{aligned}
&\frac{\pi}{4\log(1+\sqrt{2})} T(a,b) < M(a,b) < \frac{A(a,b)}{\log(1+\sqrt{2})},\\
&\sqrt{2T^2(a,b) - Q^2(a,b)} < M(a,b) < \frac{T^2(a,b)}{Q(a,b)},\\
&H\big(T(a,b), A(a,b)\big) < M(a,b) < L\big(A(a,b), Q(a,b)\big), \qquad T(a,b) > H\big(M(a,b), Q(a,b)\big),\\
&M(a,b) < \frac{A^2(a,b)}{P(a,b)}, \qquad A^{2/3}(a,b)Q^{1/3}(a,b) < M(a,b) < \frac{2A(a,b) + Q(a,b)}{3},\\
&\sqrt{A(a,b)T(a,b)} < M(a,b) < \sqrt{A^2(a,b) + T^2(a,b)},\\
&\frac{A(x,y)}{A(1-x,1-y)} < \frac{M(x,y)}{M(1-x,1-y)} < \frac{T(x,y)}{T(1-x,1-y)},\\
&\frac{1}{A(1-x,1-y)} - \frac{1}{A(x,y)} < \frac{1}{M(1-x,1-y)} - \frac{1}{M(x,y)} < \frac{1}{T(1-x,1-y)} - \frac{1}{T(x,y)},\\
&A(x,y)A(1-x,1-y) < M(x,y)M(1-x,1-y) < T(x,y)T(1-x,1-y)
\end{aligned}$$

hold for all $a, b > 0$ and $x, y \in (0, 1/2]$ with $a \neq b$ and $x \neq y$.

Li *et al.* [16] proved that the double inequality $L_{p_0}(a,b) < M(a,b) < L_2(a,b)$ holds for all $a, b > 0$ with $a \neq b$, where $L_p(a,b) = [(b^{p+1} - a^{p+1})/((p+1)(b-a))]^{1/p}$ $(p \neq -1, 0)$, $L_0(a,b) = 1/e(b^b/a^a)^{1/(b-a)}$ and $L_{-1}(a,b) = (b-a)/(\log b - \log a)$ is the pth generalized logarithmic mean of a and b, and $p_0 = 1.843\ldots$ is the unique solution of the equation $(p+1)^{1/p} = 2\log(1+\sqrt{2})$.

In [13], Neuman proved that the double inequalities

$$Q^{\alpha}(a,b)A^{1-\alpha}(a,b) < M(a,b) < Q^{\beta}(a,b)A^{1-\beta}(a,b) \tag{1.3}$$

and

$$C^{\lambda}(a,b)A^{1-\lambda}(a,b) < M(a,b) < C^{\mu}(a,b)A^{1-\mu}(a,b) \tag{1.4}$$

hold for all $a, b > 0$ with $a \neq b$ if $\alpha \leq 1/3$, $\beta \geq 2[\log(2+\sqrt{2}) - \log 3]/\log 2$, $\lambda \leq 1/6$ and $\mu \geq [\log(2+\sqrt{2}) - \log 3]/\log 2$.

Jiang and Qi [17, 18] gave the best possible parameters α, β, t_1 and t_2 in $(0, 1/2)$ such that the inequalities

$$\begin{aligned}
&Q\big(\alpha a + (1-\alpha)b, \alpha b + (1-\alpha)a\big) < M(a,b) < Q\big(\beta a + (1-\beta)b, \beta b + (1-\beta)a\big),\\
&Q_{t_1,p}(a,b) < M(a,b) < Q_{t_2,p}(a,b)
\end{aligned}$$

hold for all $a,b>0$ with $a\neq b$ and $p\geq 1/2$, where $Q_{t,p}(a,b)=C^{p}(ta+(1-t)b,tb+(1-t)a)A^{1-p}(a,b)$.

Inspired by inequalities (1.3) and (1.4), in this paper, we present the optimal upper and lower bounds for the Neuman-Sándor mean $M(a,b)$ in terms of the geometric convex combinations of the first Seiffert mean $P(a,b)$ and the quadratic mean $Q(a,b)$. All numerical computations are carried out using Mathematica software.

2 Lemmas

In order to establish our main result, we need several lemmas, which we present in this section.

Lemma 2.1 *The double inequality*

$$x+\frac{x^3}{3}-\frac{2x^5}{15}<\sqrt{1+x^2}\sinh^{-1}(x)<x+\frac{x^3}{3}-\frac{2x^5}{15}+\frac{8x^7}{105} \tag{2.1}$$

holds for $x\in(0,1)$.

Proof To show inequality (2.1), it suffices to prove that

$$\omega_1(x)=\sqrt{1+x^2}\sinh^{-1}(x)-\left(x+\frac{x^3}{3}-\frac{2x^5}{15}\right)>0 \tag{2.2}$$

and

$$\omega_2(x)=\sqrt{1+x^2}\sinh^{-1}(x)-\left(x+\frac{x^3}{3}-\frac{2x^5}{15}+\frac{8x^7}{105}\right)<0 \tag{2.3}$$

for $x\in(0,1)$.

From the expressions of $\omega_1(x)$ and $\omega_2(x)$, we get

$$\omega_1(0)=\omega_2(0)=0, \tag{2.4}$$

$$\omega_1'(x)=\frac{x\omega_1^*(x)}{\sqrt{1+x^2}},\qquad \omega_2'(x)=\frac{x\omega_2^*(x)}{\sqrt{1+x^2}}, \tag{2.5}$$

where

$$\omega_1^*(x)=\sinh^{-1}(x)-\left(x-\frac{2x^3}{3}\right)\sqrt{1+x^2},$$

$$\omega_2^*(x)=\sinh^{-1}(x)-\left(x-\frac{2x^3}{3}+\frac{8x^5}{15}\right)\sqrt{1+x^2},$$

$$\omega_1^*(0)=\omega_2^*(0)=0, \tag{2.6}$$

$$\omega_1^{*\prime}(x)=\frac{8x^4}{3\sqrt{1+x^2}}>0 \tag{2.7}$$

and

$$\omega_2^{*\prime}(x)=-\frac{16x^6}{5\sqrt{1+x^2}}<0 \tag{2.8}$$

for $x\in(0,1)$.

Therefore, inequality (2.2) follows from (2.4)-(2.7), and inequality (2.3) follows from (2.4)-(2.6) and (2.8). □

Lemma 2.2 *The inequality*

$$\frac{x^3}{\sqrt{1+x^2}} > \left[\sinh^{-1}(x)\right]^3$$

holds for $x \in (0,1)$.

Proof Let $x \in (0,1)$, then from (1.3) we have

$$M(1+x,1-x) > A^{2/3}(1+x,1-x)Q^{1/3}(1+x,1-x). \tag{2.9}$$

Therefore, Lemma 2.2 follows from (2.9). □

Lemma 2.3 *The inequality*

$$\sqrt{1-x^2}\arcsin(x) > x - \frac{x^3}{3} - \frac{x^5}{3} \tag{2.10}$$

holds for $x \in (0,0.7)$, *and the inequality*

$$\sqrt{1-x^2}\arcsin(x) < x - \frac{x^3}{3} - \frac{2x^5}{15} \tag{2.11}$$

holds for $x \in (0,1)$, *where* $\arcsin(x)$ *is the inverse sine function*.

Proof Let

$$\varphi_1(x) = \sqrt{1-x^2}\arcsin(x) - x + \frac{x^3}{3} + \frac{x^5}{3}, \tag{2.12}$$

$$\varphi_2(x) = \sqrt{1-x^2}\arcsin(x) - x + \frac{x^3}{3} + \frac{2x^5}{15}. \tag{2.13}$$

Then simple computations lead to

$$\varphi_1(0) = \varphi_2(0) = 0, \tag{2.14}$$

$$\varphi_1'(x) = \frac{x\varphi_1^*(x)}{\sqrt{1-x^2}}, \qquad \varphi_2'(x) = \frac{x\varphi_2^*(x)}{\sqrt{1-x^2}}, \tag{2.15}$$

where

$$\varphi_1^*(x) = \left(x + \frac{5x^3}{3}\right)\sqrt{1-x^2} - \arcsin(x),$$

$$\varphi_2^*(x) = \left(x + \frac{2x^3}{3}\right)\sqrt{1-x^2} - \arcsin(x).$$

Note that

$$\varphi_1^*(0) = \varphi_2^*(0) = 0, \qquad \varphi_1^*(0.7) = 0.1327\ldots, \tag{2.16}$$

$$\varphi_1^{*\prime}(x) = \frac{x^2(9-20x^2)}{3\sqrt{1-x^2}}, \tag{2.17}$$

$$\varphi_2^{*\prime}(x) = -\frac{8x^4}{3\sqrt{1-x^2}} < 0 \tag{2.18}$$

for $x \in (0,1)$.

From (2.17) we clearly see that $\varphi_1^*(x)$ is strictly increasing on $(0, 3\sqrt{5}/10]$ and strictly decreasing on $[3\sqrt{5}/10, 0.7)$. This in conjunction with (2.16) implies that

$$\varphi_1^*(x) > 0 \tag{2.19}$$

for $x \in (0, 0.7)$.

Therefore, inequality (2.10) follows from (2.12), (2.14), (2.15) and (2.19), and inequality (2.11) follows from (2.12) and (2.14)-(2.16) together with (2.18). □

Lemma 2.4 *Let*

$$\Phi(x) = \frac{1}{\sqrt{1+x^2}\sinh^{-1}(x)} - \frac{1}{x(1+x^2)}.$$

Then the inequality

$$\Phi(x) > \frac{2x}{3} - \frac{34x^3}{45} + \frac{754x^5}{945} - x^7 \tag{2.20}$$

holds for $x \in (0, 0.7)$, *and*

$$\Phi(x) < \frac{2x}{3} - \frac{34x^3}{45} + \frac{4x^5}{5} \tag{2.21}$$

holds for $x \in (0,1)$.

Proof To show inequalities (2.20) and (2.21), it suffices to prove that

$$\begin{aligned}\phi_1(x) &:= x(1+x^2)\sinh^{-1}(x)\left[\Phi(x) - \left(\frac{2x}{3} - \frac{34x^3}{45} + \frac{754x^5}{945} - x^7\right)\right] \\ &= x\sqrt{1+x^2} - \sinh^{-1}(x) \\ &\quad - x(1+x^2)\sinh^{-1}(x)\left(\frac{2x}{3} - \frac{34x^3}{45} + \frac{754x^5}{945} - x^7\right) > 0\end{aligned} \tag{2.22}$$

for $x \in (0, 0.7)$, and

$$\begin{aligned}\phi_2(x) &:= x(1+x^2)\sinh^{-1}(x)\left[\Phi(x) - \left(\frac{2x}{3} - \frac{34x^3}{45} + \frac{4x^5}{5}\right)\right] \\ &= x\sqrt{1+x^2} - \sinh^{-1}(x) \\ &\quad - x(1+x^2)\sinh^{-1}(x)\left(\frac{2x}{3} - \frac{34x^3}{45} + \frac{4x^5}{5}\right) < 0\end{aligned} \tag{2.23}$$

for $x \in (0,1)$.

From the expressions of $\phi_1(x)$ and $\phi_2(x)$, one has

$$\phi_1(0) = \phi_2(0) = 0, \tag{2.24}$$

$$\phi_1'(x) = \frac{x}{945\sqrt{1+x^2}}\phi_1^*(x), \qquad \phi_2'(x) = -\frac{2x}{45\sqrt{1+x^2}}\phi_2^*(x), \tag{2.25}$$

where

$$\begin{aligned}\phi_1^*(x) &= x(1{,}260 + 84x^2 - 40x^4 + 191x^6 + 945x^8) \\ &\quad - 2(630 - 168x^2 + 120x^4 - 764x^6 - 4{,}725x^8)\sqrt{1+x^2}\sinh^{-1}(x),\end{aligned} \tag{2.26}$$

$$\begin{aligned}\phi_2^*(x) &= x(18x^6 + x^4 - 2x^2 - 30) \\ &\quad + 2(15 - 4x^2 + 3x^4 + 72x^6)\sqrt{1+x^2}\sinh^{-1}(x).\end{aligned} \tag{2.27}$$

Note that

$$\begin{aligned}&630 - 168x^2 + 120x^4 - 764x^6 - 4{,}725x^8 \\ &\quad > 630 - 168 \times (0.7)^2 - 764 \times (0.7)^6 - 4{,}725 \times (0.7)^8 = 185.4\ldots > 0\end{aligned} \tag{2.28}$$

for $x \in (0, 0.7)$.

Lemma 2.1 and equations (2.26)-(2.28) lead to

$$\begin{aligned}\phi_1^*(x) &> x(1{,}260 + 84x^2 - 40x^4 + 191x^6 + 945x^8) \\ &\quad - 2(630 - 168x^2 + 120x^4 - 764x^6 - 4{,}725x^8)\left(x + \frac{x^3}{3} - \frac{2x^5}{15} + \frac{8x^7}{105}\right) \\ &= \frac{x^7}{105}(157{,}311 + 1{,}151{,}003x^2 + 307{,}438x^4 - 120{,}076x^6 + 75{,}600x^8) > 0\end{aligned} \tag{2.29}$$

for $x \in (0, 0.7)$, and

$$\begin{aligned}\phi_2^*(x) &> x(18x^6 + x^4 - 2x^2 - 30) \\ &\quad + 2(15 - 4x^2 + 3x^4 + 72x^6)\left(x + \frac{x^3}{3} - \frac{2x^5}{15}\right) \\ &= \frac{x^5}{15}(5 + 2{,}476x^2 + 708x^4 - 288x^6) > 0\end{aligned} \tag{2.30}$$

for $x \in (0, 1)$.

Therefore, inequality (2.22) follows from (2.24), (2.25) and (2.29), and inequality (2.23) follows from (2.24), (2.25) and (2.30). □

Lemma 2.5 *Let*

$$\Upsilon(x) = \frac{1}{x(1+x^2)} - \frac{1}{\sqrt{1-x^2}\arcsin(x)}.$$

Then the inequality

$$\Upsilon(x) > -\frac{4x}{3} + \frac{34x^3}{45} - \frac{3x^5}{2} \tag{2.31}$$

holds for $x \in (0, 0.7)$, *and*

$$\Upsilon(x) < -\frac{4x}{3} + \frac{34x^3}{45} - \frac{8x^5}{9} \tag{2.32}$$

holds for $x \in (0, 1)$.

Proof Let

$$\begin{aligned}\epsilon_1(x) &:= x(1+x^2)\sqrt{1-x^2}\arcsin(x)\left[\Upsilon(x) + \left(\frac{4x}{3} - \frac{34x^3}{45} + \frac{3x^5}{2}\right)\right] \\ &= \sqrt{1-x^2}\arcsin(x) - x(1+x^2) \\ &\quad + x(1+x^2)\sqrt{1-x^2}\arcsin(x)\left(\frac{4x}{3} - \frac{34x^3}{45} + \frac{3x^5}{2}\right)\end{aligned} \tag{2.33}$$

and

$$\begin{aligned}\epsilon_2(x) &:= x(1+x^2)\sqrt{1-x^2}\arcsin(x)\left[\Upsilon(x) + \left(\frac{4x}{3} - \frac{34x^3}{45} + \frac{8x^5}{9}\right)\right] \\ &= \sqrt{1-x^2}\arcsin(x) - x(1+x^2) \\ &\quad + x(1+x^2)\sqrt{1-x^2}\arcsin(x)\left(\frac{4x}{3} - \frac{34x^3}{45} + \frac{8x^5}{9}\right).\end{aligned} \tag{2.34}$$

An easy calculation gives rise to

$$\epsilon_1(0) = \epsilon_2(0) = 0, \tag{2.35}$$

$$\epsilon_1'(x) = \frac{x}{90(1-x^2)}\epsilon_1^*(x), \qquad \epsilon_2'(x) = \frac{x}{45(1-x^2)}\epsilon_2^*(x), \tag{2.36}$$

where

$$\begin{aligned}\epsilon_1^*(x) &= -x(150 - 202x^2 - 15x^4 - 68x^6 + 135x^8) \\ &\quad + (150 - 152x^2 + 142x^4 + 611x^6 - 1{,}215x^8)\sqrt{1-x^2}\arcsin(x),\end{aligned} \tag{2.37}$$

$$\begin{aligned}\epsilon_2^*(x) &= -x(1-x^2)(75 - 26x^2 - 6x^4 - 40x^6) \\ &\quad + (75 - 76x^2 - 94x^4 + 278x^6 - 360x^8)\sqrt{1-x^2}\arcsin(x).\end{aligned} \tag{2.38}$$

Note that

$$\begin{aligned}&150 - 152x^2 + 142x^4 + 611x^6 - 1{,}215x^8 \\ &\quad > 150 - 152 \times (0.7)^2 - 1{,}215 \times (0.7)^8 = 5.477\ldots > 0\end{aligned} \tag{2.39}$$

for $x \in (0, 0.7)$.

It follows from (2.10), (2.37) and (2.39) that

$$\begin{aligned}\epsilon_1^*(x) &> -x\left(150-202x^2-15x^4-68x^6+135x^8\right)\\ &\quad+\left(150-152x^2+142x^4+611x^6-1{,}215x^8\right)\left(x-\frac{x^3}{3}-\frac{x^5}{3}\right)\\ &= \frac{x^5}{3}\left[\frac{1{,}183}{4}+709\left(\frac{1}{4}-x^4\right)+2{,}047x^2\left(1-2x^2\right)+604x^6+1{,}215x^8\right]>0 \end{aligned} \tag{2.40}$$

for $x \in (0, 0.7)$.

We claim that

$$\epsilon_2^*(x) < 0 \tag{2.41}$$

for $x \in (0,1)$. Indeed, let $q(x) = 75 - 76x^2 - 94x^4 + 278x^6 - 360x^8$, then $q(0.8009) = 0.000171\ldots$, $q(0.80091) = -0.00356\ldots$ and

$$q'(x) = -4x\left[38+\frac{10{,}759x^2}{320}+720x^2\left(x^2-\frac{139}{480}\right)\right]<0$$

for $x \in (0,1)$. Therefore, there exists unique $x_0 = 0.80090\ldots \in (0,1)$ such that $q(x) > 0$ for $x \in (0, x_0)$ and $q(x) \le 0$ for $[x_0, 1)$. This in conjunction with (2.11) and (2.38) leads to

$$\begin{aligned}\epsilon_2^*(x) &< -x\left(1-x^2\right)\left(75-26x^2-6x^4-40x^6\right)\\ &\quad+\left(75-76x^2-94x^4+278x^6-360x^8\right)\left(x-\frac{x^3}{3}-\frac{2x^5}{15}\right)\\ &= -\frac{2x^5}{15}\left[\frac{1{,}897{,}305{,}741}{27{,}436{,}644}+2{,}619\left(x^2-\frac{2{,}651}{5{,}238}\right)^2+2x^4\left(1-x^2\right)\left(491+180x^2\right)\right]<0\end{aligned}$$

for $x \in (0, x_0)$ and $\epsilon_2^*(x) \le -x(1-x^2)(75-26x^2-6x^4-40x^6) < 0$ for $x \in [x_0, 1)$.

Therefore, inequality (2.31) follows from (2.33), (2.35), (2.36) and (2.40), and inequality (2.32) follows from (2.33)-(2.36) and (2.41). □

Lemma 2.6 *Let*

$$\mu(x) = \frac{1+3x^2}{(x+x^3)^2} - \frac{1}{(1+x^2)[\sinh^{-1}(x)]^2} - \frac{x}{(1+x^2)^{3/2}\sinh^{-1}(x)}.$$

Then $\mu(x) < 0.2$ *for* $x \in [0.7, 1)$.

Proof Let

$$\mu_1(x) = \frac{1}{x^2} - \frac{1}{[\sinh^{-1}(x)]^2}, \qquad \mu_2(x) = \frac{2}{\sqrt{1+x^2}} - \frac{x}{\sinh^{-1}(x)}.$$

Then

$$\mu(x) = \frac{\mu_1(x)}{1+x^2} + \frac{\mu_2(x)}{(1+x^2)^{3/2}}. \tag{2.42}$$

Lemma 2.2 together with $x > \sinh^{-1}(x)$ gives $\mu_1(x) < 0$ and

$$\mu_1'(x) = \frac{2}{x^3[\sinh^{-1}(x)]^3}\left[\frac{x^3}{\sqrt{1+x^2}} - \left(\sinh^{-1}(x)\right)^3\right] > 0$$

for $x \in (0,1)$. This in turn implies that

$$\left[\frac{\mu_1(x)}{1+x^2}\right]' = \frac{\mu_1'(x)(1+x^2) - 2x\mu_1(x)}{(1+x^2)^2} > 0 \tag{2.43}$$

for $x \in (0,1)$.

On the other hand, from the expression of $\mu_2(x)$, we get

$$\mu_2(1) = 0.2796\ldots > 0, \tag{2.44}$$

$$\mu_2'(x) = -\frac{2x}{(1+x^2)^{3/2}} + \frac{\mu_2^*(x)}{[\sinh^{-1}(x)]^2}, \tag{2.45}$$

where

$$\mu_2^*(x) = \frac{x}{\sqrt{1+x^2}} - \sinh^{-1}(x), \tag{2.46}$$

$$\mu_2^*(0) = 0, \tag{2.47}$$

$$\mu_2^{*\prime}(x) = -\frac{x^2}{(1+x^2)^{3/2}} < 0 \tag{2.48}$$

for $x \in (0,1)$.

From (2.44)-(2.48) we clearly see that $\mu_2'(x) < 0$ and $\mu_2(x) > 0$ for $x \in (0,1)$. This in turn implies that

$$\left[\frac{\mu_2(x)}{(1+x^2)^{3/2}}\right]' = \frac{\mu_2'(x)(1+x^2)^{3/2} - 3x\sqrt{1+x^2}\mu_2(x)}{(1+x^2)^3} < 0 \tag{2.49}$$

for $x \in (0,1)$.

Equation (2.42) and inequalities (2.43) and (2.49) lead to the conclusion that

$$\mu(x) \le \frac{\mu_1(1)}{2} + \frac{\mu_2(0.7)}{[1+(0.7)^2]^{3/2}} = 0.167\ldots < 0.2$$

for $x \in [0.7,1)$. □

Lemma 2.7 *Let*

$$\nu(x) = -\frac{1+3x^2}{(x+x^3)^2} + \frac{1}{(1-x^2)\arcsin^2(x)} - \frac{x}{(1-x^2)^{3/2}\arcsin(x)}.$$

Then $\nu(x) < -1.48$ *for* $x \in [0.7,1)$.

Proof Differentiating $\nu(x)$ yields

$$\nu'(x) = \frac{(x+x^3)^3\arcsin(x)\nu_1(x) + (1-x^2)\nu_2(x)}{x^3(1-x^2)^{5/2}(1+x^2)^3\arcsin^3(x)}, \tag{2.50}$$

where

$$\nu_1(x) = 3x\sqrt{1-x^2} - (1+2x^2)\arcsin(x), \tag{2.51}$$

$$\nu_2(x) = 2(1+3x^2+6x^4)\left[\sqrt{1-x^2}\arcsin(x)\right]^3 - 2(x+x^3)^3. \tag{2.52}$$

Equation (2.51) leads to

$$\nu_1(0.7) = -0.03558\ldots, \tag{2.53}$$

$$\nu_1'(x) = \frac{2-8x^2-4x\sqrt{1-x^2}\arcsin(x)}{\sqrt{1-x^2}} < 0 \tag{2.54}$$

for $x \in [0.7, 1)$.

Therefore,

$$\nu_1(x) < 0 \tag{2.55}$$

for $x \in [0.7, 1)$ follows from (2.53) and (2.54).

It follows from (2.52) and (2.11) that

$$\begin{aligned}\nu_2(x) &< 2(1+3x^2+6x^4)\left(x-\frac{x^3}{3}\right)^3 - 2(x+x^3)^3 \\ &= -\frac{2x^5}{27}(27-9x^2+163x^4-51x^6+6x^8) < 0 \end{aligned} \tag{2.56}$$

for $x \in [0.7, 1)$.

Equation (2.50) together with inequalities (2.55) and (2.56) leads to the conclusion that $\nu(x)$ is strictly decreasing on $[0.7, 1)$. This in turn implies that

$$\nu(x) \le \nu(0.7) = -1.48798\ldots < -1.48$$

for $x \in [0.7, 1)$. □

Lemma 2.8 *Let* $\lambda_0 = [2\log(\log(1+\sqrt{2})) + \log 2]/[2\log\pi - \log 2] = 0.2760\ldots$, *and* $\Theta(x) = \Phi(x) + \lambda_0\Upsilon(x)$, *where* $\Phi(x)$ *and* $\Upsilon(x)$ *are defined as in Lemmas* 2.4 *and* 2.5, *respectively. Then the function* $\Theta(x)$ *is strictly decreasing on* $[0.7, 1)$.

Proof Let $\mu(x)$ and $\nu(x)$ be defined as in Lemmas 2.6 and 2.7, respectively. Then differentiating $\Theta(x)$ yields

$$\Theta'(x) = \Phi'(x) + \lambda_0\Upsilon'(x) = \mu(x) + \lambda_0\nu(x) < 0.2 - 1.48\lambda_0 = -0.208\ldots < 0$$

for $x \in [0.7, 1)$. This in turn implies that $\Theta(x)$ is strictly decreasing on $[0.7, 1)$. □

3 Main result

Theorem 3.1 *The double inequality*

$$P^{\alpha}(a,b)Q^{1-\alpha}(a,b) < M(a,b) < P^{\beta}(a,b)Q^{1-\beta}(a,b)$$

holds for all $a,b>0$ *with* $a\neq b$ *if and only if* $\alpha\geq 1/2$ *and* $\beta\leq[2\log(\log(1+\sqrt{2}))+\log 2]/[2\log\pi-\log 2]=0.2760\ldots$.

Proof Since $P(a,b)$, $M(a,b)$ and $Q(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a>b$. Let $p\in(0,1)$, $\lambda_0=[2\log(\log(1+\sqrt{2}))+\log 2]/[2\log\pi-\log 2]$ and $x=(a-b)/(a+b)$. Then $x\in(0,1)$,

$$\frac{P(a,b)}{A(a,b)}=\frac{x}{\arcsin(x)},\qquad \frac{M(a,b)}{A(a,b)}=\frac{x}{\sinh^{-1}(x)},\qquad \frac{Q(a,b)}{A(a,b)}=\sqrt{1+x^2},$$

$$\frac{\log[Q(a,b)]-\log[M(a,b)]}{\log[Q(a,b)]-\log[P(a,b)]}=\frac{\log(1+x^2)-2\log x+2\log[\sinh^{-1}(x)]}{\log(1+x^2)-2\log x+2\log[\arcsin(x)]}\tag{3.1}$$

and

$$\lim_{x\to 0^+}\frac{\log(1+x^2)-2\log x+2\log[\sinh^{-1}(x)]}{\log(1+x^2)-2\log x+2\log[\arcsin(x)]}=\frac{1}{2},\tag{3.2}$$

$$\lim_{x\to 1^-}\frac{\log(1+x^2)-2\log x+2\log[\sinh^{-1}(x)]}{\log(1+x^2)-2\log x+2\log[\arcsin(x)]}=\lambda_0.\tag{3.3}$$

The difference between the convex combination of $\log[P(a,b)]$, $\log[Q(a,b)]$ and $\log[M(a,b)]$ is given by

$$\begin{aligned}&p\log\bigl[P(a,b)\bigr]+(1-p)\log\bigl[Q(a,b)\bigr]-\log\bigl[M(a,b)\bigr]\\&\quad=p\log\biggl[\frac{x}{\arcsin(x)}\biggr]+\frac{1-p}{2}\log\bigl(1+x^2\bigr)-\log\biggl[\frac{x}{\sinh^{-1}(x)}\biggr]:=D_p(x).\end{aligned}\tag{3.4}$$

Equation (3.4) leads to

$$D_p\bigl(0^+\bigr)=0,\qquad D_p\bigl(1^-\bigr)=\log\bigl[\sqrt{2}\log(1+\sqrt{2})\bigr]-p\log\biggl(\frac{\pi}{\sqrt{2}}\biggr),\qquad D_{\lambda_0}\bigl(1^-\bigr)=0,\tag{3.5}$$

$$D_p'(x)=-\frac{p}{\sqrt{1-x^2}\arcsin(x)}-\frac{(1-p)}{x(1+x^2)}+\frac{1}{\sqrt{1+x^2}\sinh^{-1}(x)}=\Phi(x)+p\Upsilon(x),\tag{3.6}$$

where $\Phi(x)$ and $\Upsilon(x)$ are defined as in Lemmas 2.4 and 2.5, respectively.

From Lemmas 2.4 and 2.5, we clearly see that

$$\begin{aligned}D_{1/2}'(x)&=\Phi(x)+\frac{1}{2}\Upsilon(x)\\&<\frac{2x}{3}-\frac{34x^3}{45}+\frac{4x^5}{5}-\frac{1}{2}\biggl(\frac{4x}{3}-\frac{34x^3}{45}+\frac{8x^5}{9}\biggr)\\&=-\frac{16x^3}{45}\biggl(\frac{17}{16}-x^2\biggr)<0\end{aligned}\tag{3.7}$$

for $x\in(0,1)$, and

$$\begin{aligned}D_{\lambda_0}'(x)&=\Phi(x)+\lambda_0\Upsilon(x)\\&>\frac{2x}{3}-\frac{34x^3}{45}+\frac{754x^5}{945}-x^7-\lambda_0\biggl(\frac{4x}{3}-\frac{34x^3}{45}+\frac{3x^5}{2}\biggr)\end{aligned}$$

$$= x\left[\frac{2(1-2\lambda_0)}{3} - \frac{34(1-\lambda_0)}{45}x^2 + \left(\frac{754}{945} - \frac{3\lambda_0}{2}\right)x^4 - x^6\right]$$
$$:= xF_{\lambda_0}(x) > 0 \tag{3.8}$$

for $x \in (0, 0.7)$.

Note that

$$F_{\lambda_0}(0) = 2(1-2\lambda_0)/3 > 0, \qquad F_{\lambda_0}(0.7) = 0.00513\ldots > 0 \tag{3.9}$$

and

$$F''_{\lambda_0}(x) = -30\left[\left(x^2 - \frac{1{,}508 - 2{,}835\lambda_0}{9{,}450}\right)^2 + \frac{2{,}224{,}136 + 4{,}052{,}160\lambda_0 - 8{,}037{,}225\lambda_0^2}{89{,}302{,}500}\right]$$
$$< 0 \tag{3.10}$$

for $x \in (0, 0.7)$.

Inequalities (3.8)-(3.10) lead to the conclusion that

$$D'_{\lambda_0}(x) > 0 \tag{3.11}$$

for $x \in (0, 0.7)$.

It follows from Lemma 2.8 and (3.6) that $D'_{\lambda_0}(x)$ is strictly decreasing in $[0.7, 1)$. Then from (3.11) and $D'_{\lambda_0}(0.7) = 0.0626\ldots$ together with $D'_{\lambda_0}(1^-) = -\infty$, we know that there exists $x^* \in (0.7, 1)$ such that $D_{\lambda_0}(x)$ is strictly increasing on $(0, x^*]$ and strictly decreasing on $[x^*, 1)$. This in conjunction with (3.5) implies that

$$D_{\lambda_0}(x) > 0 \tag{3.12}$$

for $x \in (0, 1)$.

Equations (3.4), (3.5), (3.7) and (3.12) lead to the conclusion that

$$M(a,b) < P^{\lambda_0}(a,b)Q^{1-\lambda_0}(a,b) \tag{3.13}$$

and

$$M(a,b) > P^{1/2}(a,b)Q^{1/2}(a,b). \tag{3.14}$$

Therefore, Theorem 3.1 follows from (3.13) and (3.14) together with the following statements:

- If $\alpha < 1/2$, then (3.1) and (3.2) imply that there exists $\delta_1 \in (0,1)$ such that $M(a,b) < P^{\alpha}(a,b)Q^{1-\alpha}(a,b)$ for all $a, b > 0$ with $(a-b)/(a+b) \in (0, \delta_1)$.
- If $\beta > \lambda_0$, then (3.1) and (3.3) imply that there exists $\delta_2 \in (0,1)$ such that $M(a,b) > P^{\beta}(a,b)Q^{1-\beta}(a,b)$ for all $a, b > 0$ with $(a-b)/(a+b) \in (1-\delta_2, 1)$. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions
W-MG provided the main idea and carried out the proof of Theorem 3.1. X-HS carried out the proof of Lemmas 2.1-2.5. Y-MC carried out the proof of Lemmas 2.6-2.8. All authors read and approved the final manuscript.

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