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On containment measure and the mixed isoperimetric inequality

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Abstract

We first investigate whether for given convex domains K_0, K_1 in the Euclidean plane, for any rotation α, there is a translation x so that $x + \alpha K_1 \subset K_0$ or $x + \alpha K_1 \supset K_0$. Then, we estimate the mixed isoperimetric deficit $\Delta_2(K_0, K_1)$ of domains K_0 and K_1 via the known kinematic formulas of Poincaré and Blaschke in integral geometry. We obtain the sufficient condition for domain K_0 to contain, or to be contained in, convex domain $x + \alpha K_1$. Finally, we obtain the mixed isoperimetric inequality and some Bonnesen-style mixed inequalities. Those Bonnesen-style mixed inequalities obtained are the known Bonnesen-style inequalities if one of the domains is a disc. As a direct consequence, we obtain the strengthened Bonnesen isoperimetric inequality.
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1 Introductions and preliminaries

A set of points K in the Euclidean space R^n is convex if for all $x, y \in K$ and $0 \le \lambda \le 1$, $\lambda x + (1 - \lambda)y \in K$. The convex hull K^* of K is the intersection of all convex sets that contain K. The Minkowski sum of convex sets K and L is defined by

$$K + L = \{x + y : x \in K, y \in L\},$$

and the scalar product of convex set K for $\lambda \ge 0$ is defined by

$$\lambda K = \{\lambda x : x \in K\}.$$

A homothety of a convex set K is of the form $x + \lambda K$ for $x \in R^n$, $\lambda > 0$. A convex body is a compact convex set with nonempty interior. A domain is a set with nonempty interior.

One may be interested in the following strong containment problem: Whether for given convex domains K_0 and K_1, there exists a translation x so that $x + \alpha K_0 \subset K_1$ or $x + \alpha K_1 \subset K_0$ for any rotation α. It should be noted that this containment problem is much stronger than Hadwiger's one. Therefore, the strong containment problem could lead to general and fundamental geometric inequalities (*cf.* [1–9]).

The well-known classical isoperimetric problem says that the disc encloses the maximum area among all domains of fixed perimeters in the Euclidean plane R^2.

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Proposition 1 *Let Γ be a simple closed curve of length P in the Euclidean plane R^2, then the area A of the domain K enclosed by Γ satisfies*

$$P^2 - 4\pi A \geq 0. \tag{1}$$

The equality sign holds if and only if Γ is a circle.

Its analytic proofs root back to centuries ago. One can find some simplified and beautiful proofs that lead to generalizations of the discrete case, higher dimensions, the surface of constant curvature and applications to other branches of mathematics (*cf.* [1, 3–5, 10–53]).

The isoperimetric deficit

$$\Delta_2(K) = P^2 - 4\pi A \tag{2}$$

measures the difference between domain K of area A and perimeter P, and a disc of radius $P/2\pi$.

During the 1920s, Bonnesen proved a series of inequalities of the form

$$\Delta_2(K) = P^2 - 4\pi A \geq B_K, \tag{3}$$

where the quantity B_K is an invariant of geometric significance having the following basic properties:

1. B_K is nonnegative;
2. B_K is vanish only when K is a disc.

Many B_Ks are found during the past. The main interest is still focusing on those unknown invariants of geometric significance. See references [3–5, 12, 17, 23, 31, 32, 36] for more details. The following Bonnesen's isoperimetric inequality is well known.

Proposition 2 *Let K be a domain of area A, bounded by a simple closed curve of perimeter P in the Euclidean plane R^2. Let r and R be, respectively, the maximum inscribed radius and minimum circumscribed radius of K. Then we have the following Bonnesen's isoperimetric inequality*:

$$P^2 - 4\pi A \geq \pi^2 (R - r)^2, \tag{4}$$

where the equality holds if and only if K is a disc.

Since for any domain K in R^2, its convex hull K^* increases the area A^* and decreases the perimeter P^*, that is, $A^* \geq A$ and $P^* \leq P$, then we have $P^2 - 4\pi A \geq P^{*2} - 4\pi A^*$, that is, $\Delta_2(K) \geq \Delta_2(K^*)$. Therefore, the isoperimetric inequality and the Bonnesen-style inequality are valid for all domains in R^2 if these inequalities are valid for convex domains.

In this paper, we first investigate the stronger containment problem: Whether for given convex bodies K_0, K_1 in the Euclidean plane R^2, there is a translation x so that $x + \alpha K_0 \subset K_1$ or $x + \alpha K_1 \subset K_0$ for any rotation α. Then we investigate the mixed isoperimetric deficit $\Delta_2(K_0, K_1)$ of domains K_0 and K_1.

Definition 1 Let K_0 and K_1 be two domains of areas A_0 and A_1, and of perimeters P_0 and P_1, respectively. Then the mixed isoperimetric deficit of K_0 and K_1 is defined as

$$\Delta_2(K_0, K_1) = P_0^2 P_1^2 - 16\pi^2 A_0 A_1. \tag{5}$$

Since the convex hull K^* of a set K in the Euclidean plane R^2 decreases the circum perimeter and increases the area, we have

$$\Delta_2(K_0, K_1) = P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \geq P_0^{*2} P_1^{*2} - 16\pi^2 A_0^* A_1^* = \Delta_2\left(K_0^*, K_1^*\right).$$

Therefore, we can only consider the convex domains when we estimate the mixed isoperimetric deficit low bound.

Via the kinematic formulas of Poincaré and Blaschke in integral geometry, we obtain sufficient conditions for convex domain K_1 to contain, or to be contained in, another convex domain K_0 for a translation x and any rotation α (Theorem 1 and Theorem 2). We obtain the mixed isoperimetric inequality and some Bonnesen-style mixed inequalities (Theorem 3, Theorem 4, Corollary 2, Corollary 3, Corollary 4, Theorem 5 and Theorem 6). One immediate consequence of our results is the strengthening Bonnesen isoperimetric inequality (Corollary 3). These new Bonnesen-style mixed inequalities obtained are fundamental and generalize some known Bonnesen-style inequalities (Corollary 5).

2 The containment measure

Let K_k ($k = 0, 1$) be two domains of areas A_k with simple boundaries of perimeters P_k in the Euclidean plane R^2. Let dg denote the kinematic density of the group G_2 of rigid motions, that is, translations and rotations, in R^2. Let K_1 be convex, and let tK_1 ($t \in (0, +\infty)$) be a homothetic copy of K_1, then we have the known kinematic formula of Poincaré (*cf.* [3, 36])

$$\int_{\{g \in G_2 : \partial K_0 \cap t\partial(gK_1) \neq \emptyset\}} n\left\{\partial K_0 \cap t\partial(gK_1)\right\} dg = 4tP_0 P_1, \tag{6}$$

where $n\{\partial K_0 \cap t\partial(gK_1)\}$ denotes the number of points of intersection $\partial K_0 \cap t\partial(gK_1)$.

Let m_n be the kinematic measure of the set of positions g, for which $t\partial(gK_1)$ has exactly n intersection points with ∂K_0, *i.e.*, $m_n = m\{g \in G_2 : n\{\partial(K_0) \cap t\partial(gK_1)\} = n\}$. Notice that the measure $m_n = 0$ for the odd n, then the formula of Poincaré can be rewritten as

$$\sum_{n=1}^{\infty} (2n) m_{2n} = 4tP_0 P_1,$$

that is,

$$\sum_{n=1}^{\infty} n m_{2n} = 2tP_0 P_1. \tag{7}$$

We consider the homothetic copy tK_1 ($t \in (0, +\infty)$) of K_1.

Let $\chi(K_0 \cap t(gK_1))$ be the Euler-Poincaré characteristics of the intersection $K_0 \cap t(gK_1)$. From the Blaschke's kinematic formula (*cf.* [3, 36]):

$$\int_{\{g\in G_2:K_0\cap t(gK_1)\neq\emptyset\}} \chi\left(K_0 \cap t(gK_1)\right) dg = 2\pi\left(t^2A_1 + A_0\right) + tP_0P_1, \tag{8}$$

we have

$$\sum_{n=1}^{\infty} m_{2n} = 2\pi\left(t^2A_1 + A_0\right) + tP_0P_1. \tag{9}$$

The formula of Poincaré (7) and the formula of Blaschke (9) give

$$\sum_{n=2}^{\infty} m_{2n}(n-1) = tP_0P_1 - 2\pi\left(t^2A_1 + A_0\right).$$

Since all m_k are non-negative, we have

$$tP_0P_1 - 2\pi\left(t^2A_1 + A_0\right) \geq 0; \quad t \in (0, +\infty). \tag{10}$$

On the other hand, since domains K_k $(k = 0, 1)$ are assumed to be simply connected and bounded by simple curves, we have $\chi(K_0 \cap t(gK_1)) = n(g)$ = the number of connected components of the intersection $K_0 \cap g(tK_1)$. The fundamental kinematic formula of Blaschke (8) can be rewritten as

$$\int_{\{g\in G_2:K_0\cap t(gK_1)\neq\emptyset\}} n(g)\, dg = 2\pi\left(t^2A_1 + A_0\right) + tP_0P_1. \tag{11}$$

If μ denotes set of all positions of K_1, in which either $t(gK_1) \subset K_0$ or $t(gK_1) \supset K_0$, then the above formula of Blaschke can be rewritten as

$$\int_{\mu} dg + \int_{\{g\in G_2:\partial K_0\cap t\partial(gK_1)\neq\emptyset\}} n(g)\, dg = 2\pi\left(t^2A_1 + A_0\right) + tP_0P_1. \tag{12}$$

When $\partial K_0 \cap t\partial(gK_1) \neq \emptyset$, each component of $K_0 \cap t(gK_1)$ is bounded by at least an arc of ∂K_0 and an arc of $t\partial(gK_1)$. Therefore, $n(g) \leq n\{\partial K_0 \cap t\partial(gK_1)\}/2$. Then by formulas of Poincaré and Blaschke, we obtain

$$\int_{\mu} dg \geq 2\pi\left(t^2A_1 + A_0\right) - tP_0P_1. \tag{13}$$

Therefore, this inequality immediately gives the following answer for the strong containment problem (*cf.* [1–9, 17, 36, 50, 54–60]).

Theorem 1 *Let K_k $(k = 0, 1)$ be two domains of areas A_k with simple boundaries of perimeters P_k in R^2. Let K_1 be convex. A sufficient condition for tK_1 to contain, or to be contained in, another domain K_0 for a translation and any rotation, is*

$$2\pi A_1t^2 - P_0P_1t + 2\pi A_0 > 0. \tag{14}$$

Moreover, if $t^2A_1 \geq A_0$, then tK_1 contains K_0.

As a direct consequence of Theorem 1, we have the following analog of Ren's theorem (*cf.* [36, 50, 58–60]).

Theorem 2 *Let K_k $(k = 0, 1)$ be two convex domains with areas A_k and perimeters P_k. Denote by $\Delta_2(K_k) = P_k^2 - 4\pi A_k$ the isoperimetric deficit of K_k. Then a sufficient condition for tK_1, a homothetic copy of the convex domain K_1, to contain domain K_0 for a translation and any rotation, is*

$$tP_1 - P_0 > \sqrt{t^2\Delta_2(K_1) + \Delta_2(K_0)}. \tag{15}$$

Proof Condition (15) means that $tP_1 > P_0$ and

$$2\pi A_1 t^2 - P_0 P_1 t + 2\pi A_0 > 0. \tag{16}$$

By Theorem 1, we conclude that tK_1 either contains K_0 or is contained in K_0. This inequality also leads to

$$2\pi\left(t^2 A_1 - A_0\right) > tP_0P_1 - 4\pi A_0 > P_0^2 - 4\pi A_0 = \Delta_2(K_0).$$

The isoperimetric inequality guarantees that $t^2 A_1 > A_0$. We complete the proof of the theorem. □

3 Bonnesen-style mixed inequalities

Let $r_{01} = \max\{t : t(gK_1) \subseteq K_0, g \in G_2\}$, the maximum inscribed radius of K_0 with respect to K_1, and $R_{01} = \min\{t : t(gK_1) \supseteq K_0, g \in G_2\}$, the minimum circum scribed radius of K_0 with respect to K_1. Note that r_{01}, R_{01} are, respectively, the maximum inscribed radius, the minimum circum radius of K_0 when K_1 is the unit disc. It is obvious that $r_{01} \le R_{01}$. Therefore, for $t \in [r_{01}, R_{01}]$ neither tK_1 contains K_0 nor it is contained in K_0. Then by Theorem 1, we have the following.

Theorem 3 *Let K_k $(k = 0, 1)$ be two convex domains with areas A_k and perimeters P_k. Then*

$$2\pi A_1 t^2 - P_0 P_1 t + 2\pi A_0 \le 0; \quad r_{01} \le t \le R_{01}. \tag{17}$$

When K_1 is the unit disc, this reduces to the following known Bonnesen inequality (*cf.* [3, 9, 31, 36, 61]).

Corollary 1 *Let K be a convex domain with a simple boundary ∂K of length P and area A. Denote by R and r, respectively, the radius of the minimum circumscribed disc and radius of the maximum inscribed disc of K. Then*

$$\pi t^2 - Pt + A \le 0; \quad r \le t \le R. \tag{18}$$

By the two special cases of inequality (17):

$$2\pi A_1 r_{01}^2 - P_0 P_1 r_{01} + 2\pi A_0 \le 0; \qquad 2\pi A_1 R_{01}^2 - P_0 P_1 R_{01} + 2\pi A_0 \le 0,$$

we obtain the following.

Theorem 4 *Let K_k $(k = 0,1)$ be two convex domains in the Euclidean plane R^2 with areas A_k and perimeters P_k. If K_1 is convex, then*

$$P_0^2P_1^2 - 16\pi^2 A_0A_1 \geq 4\pi^2 A_1^2(R_{01} - r_{01})^2 + \left[2\pi A_1(R_{01} + r_{01}) - P_0P_1\right]^2,$$

where the equality holds if and only if $r_{01} = R_{01}$, that is, K_0 and K_1 are discs.

Proof By inequalities (19), we have

$$-8\pi^2 A_0A_1 \geq 8\pi^2 A_1^2 r_{01}^2 - 4\pi A_1 r_{01} P_0P_1, \qquad -8\pi^2 A_0A_1 \geq 8\pi^2 A_1^2 R_{01}^2 - 4\pi A_1 R_{01} P_0P_1,$$
$$P_0^2P_1^2 - 16\pi^2 A_0A_1 \geq P_0^2P_1^2 + 8\pi^2 A_1^2 r_{01}^2 + 8\pi^2 A_1^2 R_{01}^2 - 4\pi A_1 r_{01} P_0P_1 - 4\pi A_1 R_{01} P_0P_1.$$

Since

$$\begin{aligned}
&P_0^2P_1^2 + 8\pi^2 A_1^2 r_{01}^2 + 8\pi^2 A_1^2 R_{01}^2 - 4\pi A_1 r_{01} P_0P_1 - 4\pi A_1 R_{01} P_0P_1 \\
&\quad = 4\pi^2 A_1^2 r_{01}^2 + 4\pi^2 A_1^2 R_{01}^2 - 8\pi^2 A_1^2 r_{01} R_{01} \\
&\qquad + P_0^2P_1^2 + 4\pi^2 A_1^2 r_{01}^2 + 4\pi^2 A_1^2 R_{01}^2 + 8\pi^2 A_1^2 r_{01} R_{01} - 4\pi A_1 r_{01} P_0P_1 - 4\pi A_1 R_{01} P_iP_j \\
&\quad = 4\pi^2 A_1^2 (R_{01} - r_{01})^2 + (2\pi A_1 r_{01} + 2\pi A_1 R_{01} - P_0P_1)^2,
\end{aligned}$$

therefore,

$$P_0^2P_1^2 - 16\pi^2 A_0A_1 \geq 4\pi^2 A_1^2(R_{01} - r_{01})^2 + \left[2\pi A_1(r_{01} + R_{01}) - P_0P_1\right]^2.$$

We complete the proof of Theorem 4. □

The following Kotlyar's inequality (*cf.* [3, 24]) is an immediate consequence of Theorem 4.

Corollary 2 (Kotlyar) *Let K_k $(k = 0,1)$ be two domains in R^2 with areas A_k and perimeters P_k. If K_1 is convex, then*

$$P_0^2P_1^2 - 16\pi^2 A_0A_1 \geq 4\pi^2 A_1^2(R_{01} - r_{01})^2, \tag{19}$$

where the equality holds if and only if both K_0 and K_1 are discs.

Let K_1 be the unit disc, then Theorem 4 immediately leads to the following inequality that strengthens the Bonnesen isoperimetric inequality (4).

Corollary 3 *Let K be a domain of area A, bounded by a simple closed curve of length P in the Euclidean plane R^2. Let r and R be, respectively, the inscribed radius and circumscribed radius of K, then*

$$P^2 - 4\pi A \geq \pi^2(R - r)^2 + \left[\pi(R + r) - P\right]^2, \tag{20}$$

where the equality holds if and only if K is a disc.

One immediate consequence of Theorem 4 is the following mixed isoperimetric inequality:

$$P_0^2P_1^2 - 16\pi^2 A_0A_1 \geq 0,$$

where the equality holds if and only if K_0 and K_1 are discs.

One may wish to consider the following Bonnesen-style mixed inequality:

$$P_0^2P_1^2 - 16\pi^2 A_0A_1 \geq B_{K_0,K_1},$$

where B_{K_0,K_1} is an invariant of K_0 and K_1. B_{K_0,K_1} is, of course, assumed to be nonnegative and vanishes only when both K_0 and K_1 are discs.

The inequality (17) can be rewritten as the following several inequalities:

$$\begin{aligned} &P_0^2P_1^2 - 16\pi^2 A_0A_1 \geq (P_0P_1 - 4\pi A_1 t)^2; \\ &P_0^2P_1^2 - 16\pi^2 A_0A_1 \geq (P_0P_1 - \frac{4\pi A_0}{t})^2; \quad r_{01} \leq t \leq R_{01}, \\ &P_0^2P_1^2 - 16\pi^2 A_0A_1 \geq 4\pi^2(\frac{A_0}{t} - A_1 t)^2, \end{aligned} \tag{21}$$

Therefore, we obtain the following Bonnesen-style mixed inequalities.

Corollary 4 *Let K_k $(k = 0,1)$ be two convex domains in the Euclidean plane R^2 with areas A_k and perimeters P_k. Then for $r_{01} \leq t \leq R_{01}$, we have*

$$\begin{aligned} &P_0^2P_1^2 - 16\pi^2 A_0A_1 \geq 4\pi^2 A_1^2 (R_{01} - t)^2 + \left[2\pi A_1(t + R_{01}) - P_0P_1\right]^2; \\ &P_0^2P_1^2 - 16\pi^2 A_0A_1 \geq 4\pi^2 A_1^2 (t - r_{01})^2 + \left[2\pi A_1(r_{01} + t) - P_0P_1\right]^2; \\ &P_0^2P_1^2 - 16\pi^2 A_0A_1 \geq (P_0P_1 - 4\pi A_1 r_{01})^2; \\ &P_0^2P_1^2 - 16\pi^2 A_0A_1 \geq \left(\frac{4\pi A_0}{r_{01}} - P_0P_1\right)^2; \\ &P_0^2P_1^2 - 16\pi^2 A_0A_1 \geq 4\pi^2\left(\frac{A_0}{r_{01}} - A_1 r_{01}\right)^2; \\ &P_0^2P_1^2 - 16\pi^2 A_0A_1 \geq (P_0P_1 - 4\pi A_1 t)^2; \\ &P_0^2P_1^2 - 16\pi^2 A_0A_1 \geq \left(P_0P_1 - \frac{4\pi A_0}{t}\right)^2; \\ &P_0^2P_1^2 - 16\pi^2 A_0A_1 \geq 4\pi^2\left(\frac{A_0}{t} - A_1 t\right)^2; \\ &P_0^2P_1^2 - 16\pi^2 A_0A_1 \geq (4\pi A_1 R_{01} - P_0P_1)^2; \\ &P_0^2P_1^2 - 16\pi^2 A_0A_1 \geq \left(P_0P_1 - \frac{4\pi A_0}{R_{01}}\right)^2; \\ &P_0^2P_1^2 - 16\pi^2 A_0A_1 \geq 4\pi^2\left(A_1 R_{01} - \frac{A_0}{R_{01}}\right)^2. \end{aligned} \tag{22}$$

Each inequality holds as an equality if and only if both K_0 and K_1 are discs.

On the other hand, let us consider the following Bonnesen quadratic polynomial

$$B_{K_0,K_1}(t) = 2\pi A_1 t^2 - P_0 P_1 t + 2\pi A_0.$$

It is clear that $B_{K_0,K_1}(0) > 0$ and $B_{K_0,K_1}(+\infty) > 0$. If K_1 is convex, then the mixed isoperimetric inequality guarantees that two roots $\frac{P_0P_1 \pm \sqrt{\Delta_2(K_0,K_1)}}{4\pi A_1}$ of $B_{K_0,K_1}(t) = 0$ exist and satisfy

$$0 < \frac{P_0P_1 - \sqrt{\Delta_2(K_0,K_1)}}{4\pi A_1} \le r_{01} \le R_{01} \le \frac{P_0P_1 + \sqrt{\Delta_2(K_0,K_1)}}{4\pi A_1} < +\infty. \tag{23}$$

The condition for existence of root(s) of the Bonnesen quadratic equation $B_{K_0,K_1}(t) = 0$ is the following symmetric mixed isoperimetric inequality:

$$\Delta_2(K_0,K_1) = P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \ge 0. \tag{24}$$

The Bonnesen function $B_{K_0,K_1}(t) = 2\pi A_1 t^2 - P_0 P_1 t + 2\pi A_0$ attains minimum value $-\frac{\Delta_2(K_0,K_1)}{8\pi A_1}$ at $t = \frac{P_0P_1}{4\pi A_1}$. The Bonnesen quadratic trinomial has only one root when $\Delta_2(K_0,K_1) = 0$. This means that both K_0 and K_1 are discs. This immediately leads to the following results.

Theorem 5 *Let K_k $(k = 0,1)$ be two convex domains of areas A_k and perimeters P_k in R^2. Then*

$$2\pi A_1 t^2 - P_0 P_1 t + 2\pi A_0 \ge -\frac{\Delta_2(K_0,K_1)}{8\pi A_1}. \tag{25}$$

Theorem 6 *Let K_k $(k = 0,1)$ be two convex domains of areas A_k and perimeters P_k in the Euclidean plane R^2. Then we have*

$$\frac{P_0P_1 - \sqrt{\Delta_2(K_0,K_1)}}{4\pi A_1} \le r_{01} \le \frac{P_0P_1}{4\pi A_1} \le R_{01} \le \frac{P_0P_1 + \sqrt{\Delta_2(K_0,K_1)}}{4\pi A_1}. \tag{26}$$

Each equality holds if and only if K_0 and K_1 are discs.

The following known Bonnesen-style inequalities are immediate consequences of Corollary 4, Theorem 5 and Theorem 6 when letting K_1 be the unit disc (*cf.* [3, 9, 12, 23, 31, 32, 36, 58, 62]).

Corollary 5 *Let K be a plane domain of area A, bounded by a simple closed curve of length P. Let r and R be, respectively, the in-radius and out-radius of K. Then for any disc of radius t, $r \le t \le R$, we have the following Bonnesen-style inequalities*:

$$P^2 - 4\pi A \ge (P - 2\pi t)^2;$$

$$P^2 - 4\pi A \ge \pi^2(t - r)^2 + \bigl[\pi(t + r) - P\bigr]^2;$$

$$P^2 - 4\pi A \ge \pi^2(R - t)^2 + \bigl[\pi(R + t) - P\bigr]^2;$$

$$P^2 - 4\pi A \ge \left(P - \frac{2A}{t}\right)^2; \qquad P^2 - 4\pi A \ge \left(\frac{A}{t} - \pi t\right)^2; \tag{27}$$

$$P^2 - 4\pi A \geq A^2\left(\frac{1}{r} - \frac{1}{R}\right)^2; \qquad P^2 - 4\pi A \geq P^2\left(\frac{R-r}{R+r}\right)^2;$$
$$P^2 - 4\pi A \geq A^2\left(\frac{1}{r} - \frac{1}{t}\right)^2; \qquad P^2 - 4\pi A \geq P^2\left(\frac{t-r}{t+r}\right)^2;$$
$$P^2 - 4\pi A \geq A^2\left(\frac{1}{t} - \frac{1}{R}\right)^2; \qquad P^2 - 4\pi A \geq P^2\left(\frac{R-t}{R+t}\right)^2;$$
$$\frac{P - \sqrt{P^2 - 4\pi A}}{2\pi} \leq r \leq t \leq R \leq \frac{P + \sqrt{P^2 - 4\pi A}}{2\pi}.$$

Each equality holds if and only if K is a disc.

It should be noted that the first inequality in (27) is due to Bonnesen, and he only derived some inequalities for 2-dimensional case and never had any progress for higher dimensions or 2-dimensional surface of constant curvature. One would be interested in the situations in higher dimensional space R^n and in the surface of constant curvature. Related development in those areas can be found in [26, 35, 37, 63–66] and [58]. More details for the isoperimetric inequality and Bonnesen style inequalities can be found in [67–80].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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