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Subclass of univalent harmonic functions defined by dual convolution

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Abstract

In the present paper, we study a subclass of univalent harmonic functions defined by convolution and integral convolution. We obtain the basic properties such as coefficient characterization and distortion theorem, extreme points and convolution condition.

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1 Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic function in a simply connected complex domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D. It was shown by Clunie and Sheil-Small [1] that such a harmonic function can be represented by $f = h + \overline{g}$, where h and g are analytic in D. Also, a necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ (see also [2–4] and [5]).

Denote by S_H the class of functions f that are harmonic univalent and sense-preserving in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$, for which $f(0) = h(0) = f_z'(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \tag{1.1}$$

Clunie and Sheil-Small [1] investigated the class S_H as well as its geometric subclasses and obtained some coefficient bounds.

Also, let $S_{\overline{H}}$ denote the subclass of S_H consisting of functions $f = h + \overline{g}$ such that the functions h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \qquad g(z) = \sum_{n=1}^{\infty} |b_n| z^n, \quad |b_1| < 1. \tag{1.2}$$

Recently Kanas and Wisniowska [6] (see also Kanas and Srivastava [7]) studied the class of k-uniformly convex analytic functions, denoted by $k - UCV$, $k \geq 0$, so that $\phi \in k - UCV$ if and only if

$$\operatorname{Re}\left\{1 + \frac{(z-\zeta)\phi''(z)}{\phi'(z)}\right\} \geq 0 \quad \bigl(|\zeta| \leq k; z \in U\bigr). \tag{1.3}$$

For $\theta \in \mathbb{R}$, if we let $\zeta = -kze^{i\theta}$, then condition (1.3) can be written as

$$\operatorname{Re}\left\{1 + \left(1 + ke^{i\theta}\right)\frac{z\phi''(z)}{\phi'(z)}\right\} \geq 0. \tag{1.4}$$

Kim *et al.* [8] introduced and studied the class $HCV(k,\alpha)$ consisting of functions $f = h + \overline{g}$, such that h and g are given by (1.1), and satisfying the condition

$$\operatorname{Re}\left\{1 + \left(1 + ke^{i\theta}\right)\frac{z^2h''(z) + \overline{2zg'(z) + z^2g''(z)}}{zh'(z) - \overline{zg'(z)}}\right\} \geq \alpha \quad (0 \leq \alpha < 1; \theta \in \mathbb{R}; k \geq 0). \tag{1.5}$$

Also, the class of $k - UST$ uniformly starlike functions is defined by using (1.4) as the class of all functions $\psi(z) = z\phi'(z)$ such that $\phi \in k - UCV$, then $\psi(z) \in k - UST$ if and only if

$$\operatorname{Re}\left\{\left(1 + ke^{i\theta}\right)\frac{z\psi'(z)}{\psi(z)} - ke^{i\theta}\right\} \geq 0. \tag{1.6}$$

Generalizing the class $k - UST$ to include harmonic functions, we let $HST(k,\alpha)$ denote the class of functions $f = h + \overline{g}$, such that h and g are given by (1.1), which satisfies the condition

$$\operatorname{Re}\left\{\left(1 + ke^{i\theta}\right)\frac{zf'(z)}{z'f(z)} - ke^{i\theta}\right\} \geq \alpha \quad (0 \leq \alpha < 1; \theta \in \mathbb{R}; k \geq 0). \tag{1.7}$$

Replacing $h + \overline{g}$ for f in (1.7), we have

$$\operatorname{Re}\left\{\left(1 + ke^{i\theta}\right)\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} - ke^{i\theta}\right\} \geq \alpha \quad (0 \leq \alpha < 1; \theta \in \mathbb{R}; k \geq 0). \tag{1.8}$$

The convolution of two functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad F(z) = z + \sum_{n=2}^{\infty} A_n z^n$$

is defined as

$$(f * F)(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n, \tag{1.9}$$

while the integral convolution is defined by

$$(f \diamond F)(z) = z + \sum_{n=2}^{\infty} \frac{a_n A_n}{n} z^n. \tag{1.10}$$

From (1.9) and (1.10), we have

$$(f \diamond F)(z) = \int_0^z \frac{(f * F)(t)}{t}\,dt.$$

Now we consider the subclass $HST(\phi,\psi,k,\alpha)$ consisting of functions $f = h+\overline{g}$, such that h and g are given by (1.1), and satisfying the condition

$$\operatorname{Re}\left\{(1+ke^{i\theta})\frac{h(z)*\varphi(z)-\overline{g(z)*\chi(z)}}{h(z)\diamond\varphi(z)+\overline{g(z)\diamond\chi(z)}}-ke^{i\theta}\right\}\ge\alpha\quad(0\le\alpha<1;k\ge0;\theta\text{ real}),\tag{1.11}$$

where

$$\varphi(z)=z+\sum_{n=2}^{\infty}\lambda_n z^n\quad(\lambda_n\ge0)\quad\text{and}\quad\chi(z)=z+\sum_{n=2}^{\infty}\mu_n z^n\quad(\mu_n\ge0).\tag{1.12}$$

We further consider the subclass $\overline{HST}(\phi,\chi,k,\alpha)$ of $HST(\phi,\chi,k,\alpha)$ for h and g given by (1.2).

We note that

(i) $\overline{HST}(\phi,\chi,0,\alpha)=\overline{HS}(\phi,\chi,\alpha)$ (see Dixit *et al.* [9]);
(ii) $\overline{HST}(\frac{z}{(1-z)^2},\frac{z}{(1-z)^2},1,\alpha)=G_{\overline{H}}(\alpha)$ (see Rosy *et al.* [10]);
(iii) $\overline{HST}(\frac{z+z^2}{(1-z)^3},\frac{z+z^2}{(1-z)^3},k,\alpha)=\overline{H}CV(k,\alpha)$ (see Kim *et al.* [8]);
(iv) $\overline{HST}(\frac{z}{(1-z)^2},\frac{z}{(1-z)^2},0,\alpha)=T_H^*(\alpha)$ (see Jahangiri [3], see also Joshi and Darus [11]);
(v) $\overline{HST}(\frac{z+z^2}{(1-z)^3},\frac{z+z^2}{(1-z)^3},0,\alpha)=C_H(\alpha)$ (see Jahangiri [3], see also Joshi and Darus [11]).

In this paper, we extend the results of the above classes to the classes $HST(\phi,\chi,k,\alpha)$ and $\overline{HST}(\phi,\chi,k,\alpha)$, we also obtain some basic properties for the class $\overline{HST}(\phi,\chi,k,\alpha)$.

2 Coefficient characterization and distortion theorem

Unless otherwise mentioned, we assume throughout this paper that $\varphi(z)$ and $\chi(z)$ are given by (1.12), $0\le\alpha<1$, $k\ge0$ and θ is real. We begin with a sufficient condition for functions in the class $HST(\phi,\chi,k,\alpha)$.

Theorem 1 *Let $f=h+\overline{g}$ be such that h and g are given by* (1.1). *Furthermore, let*

$$\sum_{n=2}^{\infty}\frac{\lambda_n}{n}\left(\frac{(1+k)n-(k+\alpha)}{(1-\alpha)}\right)|a_n|+\sum_{n=1}^{\infty}\frac{\mu_n}{n}\left(\frac{(1+k)n+(k+\alpha)}{(1-\alpha)}\right)|b_n|\le1,\tag{2.1}$$

where

$$n^2(1-\alpha)\le\lambda_n\big[(1+k)n-(k+\alpha)\big]\quad\textit{and}\quad n^2(1-\alpha)\le\mu_n\big[(1+k)n+(k+\alpha)\big]$$
$$\textit{for } n\ge2.$$

Then f is sense-preserving, harmonic univalent in U and $f\in HST(\phi,\chi,k,\alpha)$.

Proof First we note that f is locally univalent and sense-preserving in U. This is because

$$\begin{aligned}\left|h'(z)\right|&\ge1-\sum_{n=2}^{\infty}n|a_n|r^{n-1}>1-\sum_{n=2}^{\infty}n|a_n|\ge1-\sum_{n=2}^{\infty}\frac{\lambda_n}{n}\left(\frac{(1+k)n-(k+\alpha)}{(1-\alpha)}\right)|a_n|\\&\ge\sum_{n=1}^{\infty}\frac{\mu_n}{n}\left(\frac{(1+k)n+(k+\alpha)}{(1-\alpha)}\right)|b_n|\ge\sum_{n=1}^{\infty}n|b_n|\ge\sum_{n=1}^{\infty}n|b_n|r^{k-1}>\left|g'(z)\right|.\end{aligned}$$

To show that f is univalent in U, suppose $z_1, z_2 \in U$ so that $z_1 \neq z_2$, then

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| \geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| = 1-\left|\frac{\sum_{n=1}^{\infty} b_n(z_1^n - z_2^n)}{(z_1-z_2)+\sum_{n=2}^{\infty} a_n(z_1^n - z_2^n)}\right|$$

$$\geq 1-\frac{\sum_{n=1}^{\infty} n|b_n|}{1-\sum_{n=2}^{\infty} n|a_n|} > 1-\frac{\sum_{n=1}^{\infty}\frac{\mu_n}{n}(\frac{(1+k)n+(k+\alpha)}{(1-\alpha)})|b_n|}{1-\sum_{n=2}^{\infty}\frac{\lambda_n}{n}(\frac{(1+k)n-(k+\alpha)}{(1-\alpha)})|a_n|} \geq 0.$$

Now, we prove that $f \in HST(\phi, \psi, k, \alpha)$, by definition, we only need to show that if (2.1) holds, then condition (1.11) is satisfied. From (1.11), it suffices to show that

$$\operatorname{Re}\left\{\frac{(1+ke^{i\theta})(h(z)*\varphi(z)-\overline{g(z)*\chi(z)})-(ke^{i\theta}+\alpha)(h(z)\diamond\varphi(z)+\overline{g(z)\diamond\chi(z)})}{h(z)\diamond\varphi(z)+\overline{g(z)\diamond\chi(z)}}\right\}$$
$$\geq 0. \tag{2.2}$$

Substituting for h, g, φ and χ in (2.2) and dividing by $(1-\alpha)z$, we obtain $\operatorname{Re}\frac{A(z)}{B(z)} \geq 0$, where

$$A(z) = 1+\sum_{n=2}^{\infty}\frac{\lambda_n}{n}\frac{(1+ke^{i\theta})n-(ke^{i\theta}+\alpha)}{(1-\alpha)}a_n z^{n-1}$$
$$-\left(\frac{\overline{z}}{z}\right)\sum_{n=1}^{\infty}\frac{\mu_n}{n}\frac{(1+ke^{i\theta})n+(ke^{i\theta}+\alpha)}{(1-\alpha)}b_n\overline{z}^{n-1}$$

and

$$B(z) = 1+\sum_{n=2}^{\infty}\frac{\lambda_n}{n}a_n z^{n-1}+\left(\frac{\overline{z}}{z}\right)\sum_{n=1}^{\infty}\frac{\mu_n}{n}b_n\overline{z}^{n-1}.$$

Using the fact that $\operatorname{Re}(w) \geq 0$ if and only if $|1+w| \geq |1-w|$ in U, it suffices to show that $|A(z)+B(z)|-|A(z)-B(z)| \geq 0$. Substituting for $A(z)$ and $B(z)$ gives

$$\begin{aligned}
&\left|A(z)+B(z)\right|-\left|A(z)-B(z)\right| \\
&\quad= \left|2+\sum_{n=2}^{\infty}\frac{\lambda_n}{n}\frac{(1+ke^{i\theta})n-(ke^{i\theta}+2\alpha-1)}{(1-\alpha)}a_n z^{n-1}\right. \\
&\qquad\left.-\left(\frac{\overline{z}}{z}\right)\sum_{n=1}^{\infty}\frac{\mu_n}{n}\frac{(1+ke^{i\theta})n+(ke^{i\theta}+2\alpha-1)}{(1-\alpha)}b_n\overline{z}^{n-1}\right| \\
&\qquad-\left|\sum_{n=2}^{\infty}\frac{\lambda_n}{n}\frac{(1+ke^{i\theta})n-(1+ke^{i\theta})}{(1-\alpha)}a_n z^{n-1}\right. \\
&\qquad\left.-\left(\frac{\overline{z}}{z}\right)\sum_{n=1}^{\infty}\frac{\mu_n}{n}\frac{(1+ke^{i\theta})n+(1+ke^{i\theta})}{(1-\alpha)}b_n\overline{z}^{n-1}\right| \\
&\quad\geq 2-\sum_{n=2}^{\infty}\frac{\lambda_n}{n}\frac{(1+k)n-(k+2\alpha-1)}{(1-\alpha)}|a_n||z|^{n-1} \\
&\qquad-\sum_{n=1}^{\infty}\frac{\mu_n}{n}\frac{(1+k)n+(k+2\alpha-1)}{(1-\alpha)}|b_n||z|^{n-1}
\end{aligned}$$

$$\begin{aligned} & -\sum_{n=2}^{\infty}\frac{\lambda_n}{n}\frac{(1+k)n-(1+k)}{(1-\alpha)}|a_{n+1}||z|^{n-1} \\ & -\sum_{n=1}^{\infty}\frac{\mu_n}{n}\frac{(1+k)n+(1+k)}{(1-\alpha)}|b_n||z|^{n-1} \\ \geq{}& 2\left\{1-\sum_{n=2}^{\infty}\frac{\lambda_n}{n}\frac{(1+k)n-(k+\alpha)}{(1-\alpha)}|a_n|-\sum_{n=1}^{\infty}\frac{\mu_n}{n}\frac{(1+k)n+(k+\alpha)}{(1-\alpha)}|b_n|\right\} \\ \geq{}& 0 \quad \text{by (2.1)}. \end{aligned}$$

The harmonic functions

$$\begin{aligned} f(z) = z &+ \sum_{n=2}^{\infty}\frac{n}{\lambda_n}\frac{(1-\alpha)}{(1+k)n-(k+\alpha)}x_n z^n \\ &+ \sum_{n=1}^{\infty}\frac{n}{\mu_n}\frac{(1-\alpha)}{(1+k)n+(k+\alpha)}\overline{y}_n\overline{z}^n, \end{aligned} \tag{2.3}$$

where $\sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| = 1$, show that the coefficient bound given by (2.1) is sharp. The functions of the form (2.3) are in the class $HST(\phi,\chi,k,\alpha)$ because

$$\begin{aligned} & \sum_{n=2}^{\infty}\left[\frac{\lambda_n}{n}\frac{(1+k)n-(k+\alpha)}{(1-\alpha)}|a_n| + \sum_{n=1}^{\infty}\frac{\mu_n}{n}\frac{(1+k)n+(k+\alpha)}{(1-\alpha)}|b_n|\right] \\ &= \sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| = 1. \end{aligned}$$

This completes the proof of Theorem 1. □

In the following theorem, it is shown that condition (2.1) is also necessary for functions $f = h + \overline{g}$, where h and g are given by (1.2).

Theorem 2 *Let $f = h + \overline{g}$ be such that h and g are given by* (1.2). *Then $f \in \overline{HST}(\phi,\chi,k,\alpha)$ if and only if*

$$\sum_{n=2}^{\infty}\frac{\lambda_n}{n}\left(\frac{(1+k)n-(k+\alpha)}{(1-\alpha)}\right)|a_n| + \sum_{n=1}^{\infty}\frac{\mu_n}{n}\left(\frac{(1+k)n+(k+\alpha)}{(1-\alpha)}\right)|b_n| \leq 1. \tag{2.4}$$

Proof Since $\overline{HST}(\phi,\chi,k,\alpha) \subset HST(\phi,\chi,k,\alpha)$, we only need to prove the 'only if' part of the theorem. To this end, we notice that the necessary and sufficient condition for $f \in \overline{HST}(\phi,\chi,k,\alpha)$ is that

$$\operatorname{Re}\left\{(1+ke^{i\theta})\frac{h(z)*\varphi(z)-\overline{g(z)*\chi(z)}}{h(z)\diamond\varphi(z)+\overline{g(z)\diamond\chi(z)}} - ke^{i\theta}\right\} \geq \alpha.$$

This is equivalent to

$$\operatorname{Re}\left\{\frac{(1+ke^{i\theta})(h(z)*\varphi(z)-\overline{g(z)*\chi(z)}) - (ke^{i\theta}+\alpha)(h(z)\diamond\varphi(z)+\overline{g(z)\diamond\chi(z)})}{h(z)\diamond\varphi(z)+\overline{g(z)\diamond\chi(z)}}\right\} > 0,$$

which implies that

$$\begin{aligned}
&\operatorname{Re}\left\{\frac{(1-\alpha)z-\sum_{n=2}^{\infty}\frac{\lambda_n}{n}[(1+ke^{i\theta})n-(ke^{i\theta}+\alpha)]|a_n|z^n}{z-\sum_{n=2}^{\infty}\frac{\lambda_n}{n}|a_n|z^n+\sum_{n=1}^{\infty}\frac{\mu_n}{n}|b_n|\overline{z}^n}\right.\\
&\quad\left.-\frac{\sum_{n=1}^{\infty}\frac{\mu_n}{n}[(1+ke^{i\theta})n+(ke^{i\theta}+\alpha)]|b_n|\overline{z}^n}{z-\sum_{n=2}^{\infty}\frac{\lambda_n}{n}|a_n|z^n+\sum_{n=1}^{\infty}\frac{\mu_n}{n}|b_n|\overline{z}^n}\right\}\\
&\quad=\operatorname{Re}\left\{\frac{(1-\alpha)-\sum_{n=2}^{\infty}\frac{\lambda_n}{n}[(1+ke^{i\theta})n-(ke^{i\theta}+\alpha)]|a_n|z^{n-1}}{1-\sum_{n=2}^{\infty}\frac{\lambda_n}{n}|a_n|z^{n-1}+(\frac{\overline{z}}{z})\sum_{n=1}^{\infty}\frac{\mu_n}{n}|b_n|\overline{z}^{n-1}}\right.\\
&\quad\left.-\frac{(\frac{\overline{z}}{z})\sum_{n=1}^{\infty}\frac{\mu_n}{n}[(1+ke^{i\theta})n+(ke^{i\theta}+\alpha)]|b_n|\overline{z}^{n-1}}{1-\sum_{n=2}^{\infty}\frac{\lambda_n}{n}|a_n|z^{n}+(\frac{\overline{z}}{z})\sum_{n=1}^{\infty}\frac{\mu_n}{n}|b_n|\overline{z}^{n-1}}\right\}>0, \qquad (2.5)
\end{aligned}$$

since $\operatorname{Re}(e^{i\theta})\le|e^{i\theta}|=1$, the required condition (2.5) is equivalent to

$$\begin{aligned}
&\left\{\frac{1-\sum_{n=2}^{\infty}\frac{\lambda_n}{n}\frac{(1+k)n-(k+\alpha)}{(1-\alpha)}|a_n|r^{n-1}}{1-\sum_{n=2}^{\infty}\frac{\lambda_n}{n}|a_n|r^{n-1}+\sum_{n=1}^{\infty}\frac{\mu_n}{n}|b_n|r^{n-1}}-\frac{\sum_{n=1}^{\infty}\frac{\mu_n}{n}\frac{(1+k)n+(k+\alpha)}{(1-\alpha)}|b_n|r^{n-1}}{1-\sum_{n=2}^{\infty}\frac{\lambda_n}{n}|a_n|r^{n-1}+\sum_{n=1}^{\infty}\frac{\mu_n}{n}|b_n|r^{n-1}}\right\}\\
&\quad\ge 0. \qquad (2.6)
\end{aligned}$$

If condition (2.4) does not hold, then the numerator in (2.6) is negative for $z=r$ sufficiently close to 1. Hence there exists $z_0=r_0$ in $(0,1)$ for which the quotient in (2.6) is negative. This contradicts the required condition for $f\in\overline{HST}(\phi,\chi,k,\alpha)$, and so the proof of Theorem 2 is completed. □

Theorem 3 *Let $f\in\overline{HST}(\phi,\chi,k,\alpha)$. Then, for $|z|=r<1$, $|b_1|<\frac{1-\alpha}{2k+\alpha+1}$ and*

$$D_n\le\frac{\lambda_n}{n},\qquad E_n\le\frac{\mu_n}{n}\quad\text{for } n\ge 2\quad\text{and}\quad C=\min\{D_2,E_2\}, \qquad (2.7)$$

we have

$$\bigl|f(z)\bigr|\le\bigl(1+|b_1|\bigr)r+\left\{\frac{(1-\alpha)}{C(2+k-\alpha)}-\frac{2k+1+\alpha}{C(2+k-\alpha)}|b_1|\right\}r^2$$

and

$$\bigl|f(z)\bigr|\ge\bigl(1-|b_1|\bigr)r-\left\{\frac{(1-\alpha)}{C(2+k-\alpha)}-\frac{2k+1+\alpha}{C(2+k-\alpha)}|b_1|\right\}r^2.$$

The results are sharp.

Proof We prove the left-hand side inequality for $|f|$. The proof for the right-hand side inequality can be done by using similar arguments.

Let $f\in\overline{HST}(\phi,\chi,k,\alpha)$, then we have

$$\begin{aligned}
\bigl|f(z)\bigr|&=\left|z-\sum_{n=2}^{\infty}|a_n|z^n+\sum_{n=1}^{\infty}|b_n|\overline{z}^n\right|\\
&\ge r-|b_1|r-\sum_{n=2}^{\infty}\bigl(|a_n|+|b_n|\bigr)r^2
\end{aligned}$$

$$\begin{aligned}
&\geq r - |b_1|r \\
&\quad - \frac{(1-\alpha)}{C(2+k-\alpha)} \sum_{n=2}^{\infty} \frac{C((1+k)n-(k+\alpha))}{(1-\alpha)} \big(|a_n| + |b_n|\big) r^2 \\
&\geq r - |b_1|r \\
&\quad - \frac{(1-\alpha)}{C(2+k-\alpha)} \sum_{n=2}^{\infty} \left\{ \frac{C((1+k)n-(k+\alpha))}{(1-\alpha)} |a_n| \right. \\
&\quad \left. + \frac{C((1+k)n+(k+\alpha))}{(1-\alpha)} |b_n| \right\} r^2 \\
&\geq \big(1-|b_1|\big) r - \frac{(1-\alpha)}{C(2+k-\alpha)} \left\{ 1 - \frac{2k+1+\alpha}{(1-\alpha)} |b_1| \right\} r^2 \\
&\geq \big(1-|b_1|\big) r - \left\{ \frac{(1-\alpha)}{C(2+k-\alpha)} - \frac{2k+1+\alpha}{C(2+k-\alpha)} |b_1| \right\} r^2.
\end{aligned}$$

The bounds given in Theorem 3 are respectively attained for the following functions:

$$f(z) = z + |b_1|\bar{z} + \left(\frac{(1-\alpha)}{C(2+k-\alpha)} - \frac{2k+1+\alpha}{C(2+k-\alpha)} |b_1| \right) \bar{z}^2$$

and

$$f(z) = \big(1-|b_1|\big) z - \left(\frac{(1-\alpha)}{C(2+k-\alpha)} - \frac{2k+1+\alpha}{C(2+k-\alpha)} |b_1| \right) z^2. \qquad \square$$

The following covering result follows from the left side inequality in Theorem 3.

Corollary 1 *Let $f \in \overline{HST}(\phi, \chi, k, \alpha)$, then for $|b_1| < \frac{1-\alpha}{2k+\alpha+1}$ the set*

$$\left\{ w : |w| < 1 - \frac{(1-\alpha)}{C(2+k-\alpha)} - \left(1 - \frac{2k+1+\alpha}{C(2+k-\alpha)} \right) |b_1| \right\}$$

is included in $f(U)$, where C is given by (2.7).

3 Extreme points

Our next theorem is on the extreme points of convex hulls of the class $\overline{HST}(\phi, \chi, k, \alpha)$, denoted by $clco\overline{HST}(\phi, \chi, k, \alpha)$.

Theorem 4 *Let $f = h + \bar{g}$ be such that h and g are given by* (1.2). *Then $f \in clco\overline{HST}(\phi, \chi, k, \alpha)$ if and only if f can be expressed as*

$$f(z) = \sum_{n=1}^{\infty} \big[X_n h_n(z) + Y_n g_n(z) \big], \tag{3.1}$$

where

$$h_1(z) = z,$$

$$h_n(z) = z - \frac{n(1-\alpha)}{\lambda_n((1+k)n-(k+\alpha))} z^n \quad (n \geq 2),$$

$$g_n(z) = z + \frac{n(1-\alpha)}{\mu_n((1+k)n+(k+\alpha))}\overline{z}^n \quad (n \geq 1),$$

$$X_n \geq 0, \qquad Y_n \geq 0, \quad \sum_{n=1}^{\infty}[X_n + Y_n] = 1.$$

In particular, the extreme points of the class $\overline{HST}(\phi, \chi, k, \alpha)$ *are* $\{h_n\}$ *and* $\{g_n\}$, *respectively.*

Proof For functions $f(z)$ of the form (3.1), we have

$$f(z) = \sum_{n=1}^{\infty}[X_n + Y_n]z - \sum_{n=2}^{\infty}\frac{n(1-\alpha)}{\lambda_n((1+k)n-(k+\alpha))}X_n z^n + \sum_{n=1}^{\infty}\frac{n(1-\alpha)}{\mu_n((1+k)n+(k+\alpha))}Y_n\overline{z}^n.$$

Then

$$\begin{aligned}
&\sum_{n=2}^{\infty}\frac{\lambda_n((1+k)n-(k+\alpha))}{n(1-\alpha)}\left(\frac{n(1-\alpha)}{\lambda_n((1+k)n-(k+\alpha))}\right)X_n \\
&\quad + \sum_{n=1}^{\infty}\frac{\mu_n((1+k)n+(k+\alpha))}{n(1-\alpha)}\left(\frac{n(1-\alpha)}{\mu_n((1+k)n+(k+\alpha))}\right)Y_n \\
&= \sum_{n=2}^{\infty}X_n + \sum_{n=1}^{\infty}Y_n = 1 - X_1 \leq 1,
\end{aligned}$$

and so $f(z) \in clco\overline{HST}(\phi, \chi, k, \alpha)$. Conversely, suppose that $f(z) \in clco\overline{HST}(\phi, \chi, k, \alpha)$. Set

$$X_n = \frac{\lambda_n((1+k)n-(k+\alpha))}{n(1-\alpha)}|a_n| \quad (n \geq 2)$$

and

$$Y_n = \frac{\mu_n((1+k)n+(k+\alpha))}{n(1-\alpha)}|b_n| \quad (n \geq 1),$$

then note that by Theorem 2, $0 \leq X_n \leq 1$ $(n \geq 2)$ and $0 \leq Y_n \leq 1$ $(n \geq 1)$.

Consequently, we obtain

$$f(z) = \sum_{n=1}^{\infty}\bigl[X_n h_n(z) + Y_n g_n(z)\bigr].$$

Using Theorem 2, it is easily seen that the class $\overline{HST}(\phi, \chi, k, \alpha)$ is convex and closed and so $clco\overline{HST}(\phi, \chi, k, \alpha) = \overline{HST}(\phi, \chi, k, \alpha)$. □

4 Convolution result

For harmonic functions of the form

$$f(z) = z - \sum_{n=2}^{\infty}|a_n|z^n + \sum_{n=1}^{\infty}|b_n|\overline{z}^n \tag{4.1}$$

and

$$G(z) = z - \sum_{n=2}^{\infty}A_n z^n + \sum_{n=1}^{\infty}B_n\overline{z}^n \quad (A_n, B_n \geq 0), \tag{4.2}$$

we define the convolution of two harmonic functions f and G as

$$(f * G)(z) = f(z) * G(z) = z - \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} b_n B_n \overline{z}^n.$$

Using this definition, we show that the class $\overline{HST}(\phi, \chi, k, \alpha)$ is closed under convolution.

Theorem 5 *For* $0 \leq \alpha < 1$, *let* $f \in \overline{HST}(\phi, \chi, k, \alpha)$ *and* $G \in \overline{HST}(\phi, \chi, k, \alpha)$. *Then* $f(z) * G(z) \in \overline{HST}(\phi, \chi, k, \alpha)$.

Proof Let the functions $f(z)$ defined by (4.1) be in the class $\overline{HST}(\phi, \chi, k, \alpha)$, and let the functions $G(z)$ defined by (4.2) be in the class $\overline{HST}(\phi, \chi, k, \alpha)$. Obviously, the coefficients of f and G must satisfy a condition similar to inequality (2.4). So, for the coefficients of $f(z) * G(z)$, we can write

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{\lambda_n}{n} \frac{(1+k)n - (k+\alpha)}{(1-\alpha)} |a_n| A_n + \sum_{n=1}^{\infty} \frac{\mu_n}{n} \frac{(1+k)n + (k+\alpha)}{(1-\alpha)} |b_n| B_n \\
&\quad \leq \sum_{n=2}^{\infty} \left[\frac{\lambda_n}{n} \frac{(1+k)n - (k+\alpha)}{(1-\alpha)} |a_n| + \sum_{n=1}^{\infty} \frac{\mu_n}{n} \frac{(1+k)n + (k+\alpha)}{(1-\alpha)} |b_n| \right],
\end{aligned}$$

the right-hand side of this inequality is bounded by 1 because $f \in \overline{HST}(\phi, \chi, k, \alpha)$. Then $f(z) * G(z) \in \overline{HST}(\phi, \chi, k, \alpha)$. □

Finally, we show that $\overline{HST}(\phi, \chi, k, \alpha)$ is closed under convex combinations of its members.

Theorem 6 *The class* $\overline{HST}(\phi, \chi, k, \alpha)$ *is closed under convex linear combination.*

Proof For $i = 1, 2, 3, \ldots$, let $f_i \in \overline{HST}(\phi, \chi, k, \alpha)$, where the functions f_i are given by

$$f_i(z) = z - \sum_{n=2}^{\infty} |a_{n,i}| z^n + \sum_{n=1}^{\infty} |b_{n,i}| \overline{z}^n.$$

For $\sum_{i=1}^{\infty} t_i = 1$; $0 \leq t_i \leq 1$, the convex linear combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{n,i}| \right) z^n + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{n,i}| \right) \overline{z}^n,$$

then by (2.4) we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{\lambda_n}{n} \frac{(1+k)n - (k+\alpha)}{(1-\alpha)} \sum_{i=1}^{\infty} t_i |a_{n,i}| + \sum_{n=1}^{\infty} \frac{\mu_n}{n} \frac{(1+k)n + (k+\alpha)}{(1-\alpha)} \sum_{i=1}^{\infty} t_i |b_{n,i}| \\
&\quad = \sum_{i=1}^{\infty} t_i \left\{ \sum_{n=2}^{\infty} \left[\frac{\lambda_n}{n} \frac{(1+k)n - (k+\alpha)}{(1-\alpha)} |a_{n,i}| + \sum_{n=1}^{\infty} \frac{\mu_n}{n} \frac{(1+k)n + (k+\alpha)}{(1-\alpha)} |b_{n,i}| \right] \right\} \\
&\quad \leq \sum_{i=1}^{\infty} t_i = 1.
\end{aligned}$$

This condition is required by (2.4) and so $\sum_{i=1}^{\infty} t_i f_i(z) \in \overline{HST}(\phi, \chi, k, \alpha)$. This completes the proof of Theorem 6. □

Remarks

(i) Putting $k = 0$ in our results, we obtain the results obtained by Dixit *et al.* [9];

(ii) Putting $\varphi(z) = \chi(z) = \frac{z}{(1-z)^2}$ and $k = 1$ in our results, we obtain the results obtained by Rosy *et al.* [10];

(iii) Putting $\varphi(z) = \chi(z) = \frac{z+z^2}{(1-z)^3}$ in our results, we obtain the results obtained by Kim *et al.* [8];

(iv) Putting $\varphi(z) = \chi(z) = \frac{z}{(1-z)^2}$ and $k = 0$ in our results, we obtain the results obtained by Jahangiri [3];

(v) Putting $\varphi(z) = \chi(z) = \frac{z+z^2}{(1-z)^3}$ and $k = 0$ in our results, we obtain the results obtained by Jahangiri [2].

Competing interests

The author declares that they have no competing interests.

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