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Some new judgement theorems of Schur geometric and Schur harmonic convexities for a class of symmetric functions

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Abstract
The judgement theorems of Schur geometric and Schur harmonic convexities for a class of symmetric functions are given. As their application, some analytic inequalities are established.

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Keywords: Schur geometric convexity; Schur harmonic convexity; inequality; symmetric function

1 Introduction

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers, $\mathbf{x} = (x_1, x_2, \dots, x_n)$ denotes n-tuple (n-dimensional real vectors), the set of vectors can be written as

$$\mathbb{R}^n = \{\mathbf{x} = (x_1, \dots, x_n) : x_i \in \mathbb{R}, i = 1, \dots, n\},$$
$$\mathbb{R}^n_+ = \{\mathbf{x} = (x_1, \dots, x_n) : x_i > 0, i = 1, \dots, n\}.$$

In particular, the notations $\mathbb{R}$ and $\mathbb{R}_+$ denote $\mathbb{R}^1$ and $\mathbb{R}^1_+$, respectively.

Let $\pi = (\pi(1), \dots, \pi(n))$ be a permutation of $(1, \dots, n)$, all permutations are totally $n!$. The following conclusion is proved in [1, pp.127-129].

Theorem A *Let $A \subset \mathbb{R}^k$ be a symmetric convex set, and let φ be a Schur-convex function defined on A with the property that for each fixed $x_2, \dots, x_k$, $\varphi(z, x_2, \dots, x_k)$ is convex in z on $\{z : (z, x_2, \dots, x_k) \in A\}$. Then, for any $n > k$,*

$$\psi(x_1, \dots, x_n) = \sum_{\pi} \varphi(x_{\pi(1)}, \dots, x_{\pi(k)}) \tag{1}$$

is Schur-convex on

$$B = \{(x_1, \dots, x_n) : (x_{\pi(1)}, \dots, x_{\pi(k)}) \in A \text{ for all permutations } \pi\}.$$

Furthermore, the symmetric function

$$\overline{\psi}(\mathbf{x}) = \sum_{1 \le i_1 < \cdots < i_k \le n} \varphi(x_{i_1}, \dots, x_{i_k}) \tag{2}$$

is also Schur-convex on B.

Theorem A is very effective for judgement of the Schur-convexity of the symmetric functions of the form (2), see the references [1] and [2].

The Schur geometrically convex functions were proposed by Zhang [3] in 2004. Further, the Schur harmonically convex functions were proposed by Chu and Lü [4] in 2009. The theory of majorization was enriched and expanded by using these concepts [5–15]. Regarding Schur geometrically convex functions and Schur harmonically convex functions, the aim of this paper is to establish the following judgement theorems which are similar to Theorem A.

Theorem 1 *Let $A \subset \mathbb{R}^k$ be a symmetric geometrically convex set, and let φ be a Schur geometrically convex (concave) function defined on A with the property that for each fixed $x_2, \dots, x_k$, $\varphi(z, x_2, \dots, x_k)$ is GA convex (concave) in z on $\{z : (z, x_2, \dots, x_k) \in A\}$. Then, for any $n > k$,*

$$\psi(x_1, \dots, x_n) = \sum_{\pi} \varphi(x_{\pi(1)}, \dots, x_{\pi(k)})$$

is Schur geometrically convex (concave) on

$$B = \left\{(x_1, \dots, x_n) : (x_{\pi(1)}, \dots, x_{\pi(k)}) \in A \text{ for all permutations } \pi\right\}.$$

Furthermore, the symmetric function

$$\overline{\psi}(\mathbf{x}) = \sum_{1 \le i_1 < \cdots < i_k \le n} \varphi(x_{i_1}, \dots, x_{i_k})$$

is also Schur geometrically convex (concave) on B.

Theorem 2 *Let $A \subset \mathbb{R}^k$ be a symmetric harmonically convex set, and let φ be a Schur harmonically convex (concave) function defined on A with the property that for each fixed $x_2, \dots, x_k$, $\varphi(z, x_2, \dots, x_k)$ is HA convex (concave) in z on $\{z : (z, x_2, \dots, x_k) \in A\}$. Then, for any $n > k$,*

$$\psi(x_1, \dots, x_n) = \sum_{\pi} \varphi(x_{\pi(1)}, \dots, x_{\pi(k)})$$

is Schur harmonically convex (concave) on

$$B = \left\{(x_1, \dots, x_n) : (x_{\pi(1)}, \dots, x_{\pi(k)}) \in A \text{ for all permutations } \pi\right\}.$$

Furthermore, the symmetric function

$$\overline{\psi}(\mathbf{x}) = \sum_{1 \le i_1 < \cdots < i_k \le n} \varphi(x_{i_1}, \dots, x_{i_k})$$

is also Schur harmonically convex (concave) on B.

2 Definitions and lemmas

In order to prove some further results, in this section we recall useful definitions and lemmas.

Definition 1 [1, 16] Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

(i) We say $\mathbf{y}$ majorizes $\mathbf{x}$ ($\mathbf{x}$ is said to be majorized by $\mathbf{y}$), denoted by $\mathbf{x} \prec \mathbf{y}$, if $\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$ for $k = 1, 2, \ldots, n-1$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, where $x_{[1]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq \cdots \geq y_{[n]}$ are rearrangements of $\mathbf{x}$ and $\mathbf{y}$ in a descending order.

(ii) Let $\Omega \subset \mathbb{R}^n$, a function $\varphi : \Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω if $\mathbf{x} \prec \mathbf{y}$ on Ω implies $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$. A function φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex function on Ω.

Definition 2 [1, 16] Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$, $0 \leq \alpha \leq 1$. A set $\Omega \subset \mathbb{R}^n$ is said to be a convex set if $\mathbf{x}, \mathbf{y} \in \Omega$ implies $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} = (\alpha x_1 + (1-\alpha)y_1, \ldots, \alpha x_n + (1-\alpha)y_n) \in \Omega$.

Definition 3 [1, 16]

(i) A set $\Omega \subset \mathbb{R}^n$ is called a symmetric set if $\mathbf{x} \in \Omega$ implies $\mathbf{x}P \in \Omega$ for every $n \times n$ permutation matrix P.

(ii) A function $\varphi : \Omega \to \mathbb{R}$ is called symmetric if for every permutation matrix P, $\varphi(\mathbf{x}P) = \varphi(\mathbf{x})$ for all $\mathbf{x} \in \Omega$.

Definition 4 Let $\Omega \subset \mathbb{R}_+^n$, $\mathbf{x} = (x_1, \ldots, x_n) \in \Omega$ and $\mathbf{y} = (y_1, \ldots, y_n) \in \Omega$.

(i) [3, p.64] A set Ω is called a geometrically convex set if $(x_1^\alpha y_1^\beta, \ldots, x_n^\alpha y_n^\beta) \in \Omega$ for all $\mathbf{x}, \mathbf{y} \in \Omega$ and $\alpha, \beta \in [0,1]$ such that $\alpha + \beta = 1$.

(ii) [3, p.107] A function $\varphi : \Omega \to \mathbb{R}_+$ is said to be a Schur geometrically convex function on Ω if $(\log x_1, \ldots, \log x_n) \prec (\log y_1, \ldots, \log y_n)$ on Ω implies $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$. A function φ is said to be a Schur geometrically concave function on Ω if and only if $-\varphi$ is a Schur geometrically convex function.

Definition 5 [17] Let $\Omega \subset \mathbb{R}_+^n$.

(i) A set Ω is said to be a harmonically convex set if $\frac{\mathbf{xy}}{\lambda\mathbf{x}+(1-\lambda)\mathbf{y}} \in \Omega$ for every $\mathbf{x}, \mathbf{y} \in \Omega$ and $\lambda \in [0,1]$, where $\mathbf{xy} = \sum_{i=1}^{n} x_i y_i$ and $\frac{1}{\mathbf{x}} = (\frac{1}{x_1}, \ldots, \frac{1}{x_n})$.

(ii) A function $\varphi : \Omega \to \mathbb{R}_+$ is said to be a Schur harmonically convex function on Ω if $\frac{1}{\mathbf{x}} \prec \frac{1}{\mathbf{y}}$ implies $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$. A function φ is said to be a Schur harmonically concave function on Ω if and only if $-\varphi$ is a Schur harmonically convex function.

Definition 6 [18] Let $I \subset \mathbb{R}_+$, $\varphi : I \to \mathbb{R}_+$ be continuous.

(i) A function φ is said to be a GA convex (concave) function on I if

$$\varphi(\sqrt{xy}) \leq (\geq) \frac{\varphi(x) + \varphi(y)}{2}$$

for all $x, y \in I$.

(ii) A function φ is said to be a HA convex (concave) function on I if

$$\varphi\left(\frac{2xy}{x+y}\right) \leq (\geq) \frac{\varphi(x) + \varphi(y)}{2}$$

for all $x, y \in I$.

Lemma 1 [16, p.57] *Let $\Omega \subset \mathbb{R}^n$ be a symmetric convex set with a nonempty interior Ω^0. $\varphi : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable on Ω^0. Then φ is a Schur-convex (Schur-concave) function if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial\varphi}{\partial x_1} - \frac{\partial\varphi}{\partial x_2}\right) \geq 0 \ (\leq 0) \tag{3}$$

holds for any $\mathbf{x} = (x_1, \dots, x_n) \in \Omega^0$.

Lemma 2 [3, p.108] *Let $\Omega \subset \mathbb{R}^n_+$ be a symmetric geometrically convex set with a nonempty interior Ω^0. Let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable on Ω^0. Then φ is a Schur geometrically convex (Schur geometrically concave) function if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(x_1\frac{\partial\varphi}{\partial x_1} - x_2\frac{\partial\varphi}{\partial x_2}\right) \geq 0 \ (\leq 0) \tag{4}$$

holds for any $\mathbf{x} = (x_1, \dots, x_n) \in \Omega^0$.

Lemma 3 [17, 19] *Let $\Omega \subset \mathbb{R}^n_+$ be a symmetric harmonically convex set with a nonempty interior Ω^0. Let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable on Ω^0. Then φ is a Schur harmonically convex (Schur harmonically concave) function if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(x_1^2\frac{\partial\varphi}{\partial x_1} - x_2^2\frac{\partial\varphi}{\partial x_2}\right) \geq 0 \ (\leq 0) \tag{5}$$

holds for any $\mathbf{x} = (x_1, \dots, x_n) \in \Omega^0$.

Lemma 4 [18] *Let $I \subset \mathbb{R}_+$ be an open subinterval, and let $\varphi : I \to \mathbb{R}_+$ be differentiable.*
(i) *φ is GA-convex (concave) if and only if $x\varphi'(x)$ is increasing (decreasing).*
(ii) *φ is HA-convex (concave) if and only if $x^2\varphi'(x)$ is increasing (decreasing).*

3 Proofs of main results

Proof of Theorem 1 To verify condition (4) of Lemma 2, denote by $\sum_{\pi(i,j)}$ the summation over all permutations π such that $\pi(i) = 1$, $\pi(j) = 2$. Because φ is symmetric,

$$\begin{aligned}
&\psi(x_1, \dots, x_n) \\
&\quad = \sum_{\substack{i,j\leq k\\ i\neq j}} \sum_{\pi(i,j)} \varphi(x_1, x_2, x_{\pi(1)}, \dots, x_{\pi(i-1)}, x_{\pi(i+1)}, \dots, x_{\pi(j-1)}, x_{\pi(j+1)}, \dots, x_{\pi(k)}) \\
&\qquad + \sum_{i\leq k<j} \sum_{\pi(i,j)} \varphi(x_1, x_{\pi(1)}, \dots, x_{\pi(i-1)}, x_{\pi(i+1)}, \dots, x_{\pi(k)}) \\
&\qquad + \sum_{j\leq k<i} \sum_{\pi(i,j)} \varphi(x_2, x_{\pi(1)}, \dots, x_{\pi(j-1)}, x_{\pi(j+1)}, \dots, x_{\pi(k)}) \\
&\qquad + \sum_{\substack{k<i,j\\ i\neq j}} \sum_{\pi(i,j)} \varphi(x_{\pi(1)}, \dots, x_{\pi(k)}).
\end{aligned}$$

Then

$$\begin{aligned}\Delta_1 &:= \left(x_1\frac{\partial\psi}{\partial x_1} - x_2\frac{\partial\psi}{\partial x_2}\right)(x_1 - x_2) \\ &= \sum_{\substack{i,j\le k\\ i\ne j}}\sum_{\pi(i,j)}\bigl[x_1\varphi_{(1)}(x_1, x_2, x_{\pi(1)}, \ldots, x_{\pi(i-1)}, x_{\pi(i+1)}, \ldots, x_{\pi(j-1)}, x_{\pi(j+1)}, \ldots, x_{\pi(k)}) \\ &\quad - x_2\varphi_{(2)}(x_1, x_2, x_{\pi(1)}, \ldots, x_{\pi(i-1)}, x_{\pi(i+1)}, \ldots, x_{\pi(j-1)}, x_{\pi(j+1)}, \ldots, x_{\pi(k)})\bigr](x_1 - x_2) \\ &\quad + \sum_{i\le k<j}\sum_{\pi(i,j)}\bigl[x_1\varphi_{(1)}(x_1, x_{\pi(1)}, \ldots, x_{\pi(i-1)}, x_{\pi(i+1)}, \ldots, x_{\pi(k)}) \\ &\quad - x_2\varphi_{(1)}(x_2, x_{\pi(1)}, \ldots, x_{\pi(i-1)}, x_{\pi(i+1)}, \ldots, x_{\pi(k)})\bigr](x_1 - x_2).\end{aligned}$$

Here,

$$(x_1\varphi_{(1)} - x_2\varphi_{(2)})(x_1 - x_2) \ge 0\ (\le 0)$$

because φ is Schur geometrically convex (concave), and

$$\bigl[x_1\varphi_{(1)}(x_1, z) - x_2\varphi_{(1)}(x_2, z)\bigr](x_1 - x_2) \ge 0\ (\le 0)$$

because $\varphi(z, x_2, \ldots, x_k)$ is GA convex (concave) in its first argument on $\{z : (z, x_2, \ldots, x_k) \in A\}$. Accordingly, $\Delta_1 \ge 0\ (\le 0)$. This shows that ψ is Schur geometrically convex (concave) on

$$B = \bigl\{(x_1, \ldots, x_n) : (x_{\pi(1)}, \ldots, x_{\pi(k)}) \in A \textit{ for all permutations } \pi\bigr\}.$$

Notice that

$$\overline{\psi}(\mathbf{x}) = \psi(\mathbf{x})/k!(n-k)!.$$

Of course, $\overline{\psi}$ is Schur geometrically convex (concave) whenever ψ is Schur geometrically convex (concave).

The proof of Theorem 1 is completed. □

Proof of Theorem 2 We only need to verify condition (5) of Lemma 3, the proof is similar to that of Theorem 1 and is omitted. □

Remark 1 In most applications, A has the form I^k for some interval $I \subset R$ and in this case $B = I^n$. Notice that the convexity of φ in its first argument also implies that φ is convex in each argument, the other arguments being fixed, because φ is symmetric.

4 Applications

Let

$$E_k\left(\frac{\mathbf{x}}{1-\mathbf{x}}\right) = \sum_{1\le i_1<\cdots<i_k\le n}\prod_{j=1}^{k}\frac{x_{i_j}}{1-x_{i_j}}. \tag{6}$$

In 2011, Guan and Guan [20] proved the following theorem through Lemma 2.

Theorem 3 *The symmetric function* $E_k(\frac{\mathbf{x}}{1-\mathbf{x}})$, $k = 1, \ldots, n$, *is Schur geometrically convex on* $(0,1)^n$.

Now, we give a new proof of Theorem 3 by using Theorem 1. Furthermore, we prove the following theorem through Theorem 2.

Theorem 4 *The symmetric function* $E_k(\frac{\mathbf{x}}{1-\mathbf{x}})$, $k = 1, \ldots, n$, *is Schur harmonically convex on* $(0,1)^n$.

Proof of Theorem 3 Let $\varphi(\mathbf{z}) = \prod_{i=1}^k [z_i/(1-z_i)]$. Then

$$\log\varphi(\mathbf{z}) = \sum_{i=1}^k \bigl[\log z_i - \log(1-z_i)\bigr]$$

and

$$\frac{\partial\varphi(\mathbf{z})}{\partial z_1} = \varphi(\mathbf{z})\left(\frac{1}{z_1} + \frac{1}{1-z_1}\right), \qquad \frac{\partial\varphi(\mathbf{z})}{\partial z_2} = \varphi(\mathbf{z})\left(\frac{1}{z_2} + \frac{1}{1-z_2}\right), \tag{7}$$

$$\begin{aligned}\Delta &:= (z_1 - z_2)\left(z_1\frac{\partial\varphi(\mathbf{z})}{\partial z_1} - z_2\frac{\partial\varphi(\mathbf{z})}{\partial z_2}\right)\\ &= (z_1 - z_2)\varphi(\mathbf{z})\left(\frac{z_1}{1-z_1} - \frac{z_2}{1-z_2}\right)\\ &= (z_1 - z_2)^2\varphi(\mathbf{z})\frac{1}{(1-z_2)(1-z_1)}.\end{aligned}$$

This shows that $\Delta \geq 0$ when $0 < z_i < 1$, $i = 1, \ldots, k$. According to Lemma 2, φ is Schur geometrically convex on $A = \{\mathbf{z} : \mathbf{z} \in (0,1)^k\}$. Let $g(t) = \frac{t}{1-t}$, then $h(t) := tg'(t) = \frac{t}{(1-t)^2}$. From $t \in (0,1)$, it follows that $h'(t) = \frac{1+t}{(1-t)^3} \geq 0$. According to Lemma 4(i), φ is GA convex in its single variable on $(0,1)$. So $E_k(\frac{\mathbf{x}}{1-\mathbf{x}})$ is Schur geometrically convex on $(0,1)^n$ from Theorem 1. The proof of Theorem 3 is completed. □

Proof of Theorem 4 Let $\varphi(\mathbf{z}) = \prod_{i=1}^k (z_i/1 - z_i)$, then

$$\log\varphi(\mathbf{z}) = \sum_{i=1}^k \bigl[\log z_i - \log(1-z_i)\bigr].$$

From (7), we get

$$\begin{aligned}\Delta_1 &:= (z_1 - z_2)\left(z_1^2\frac{\partial\varphi(\mathbf{z})}{\partial z_1} - z_2^2\frac{\partial\varphi(\mathbf{z})}{\partial z_2}\right)\\ &= (z_1 - z_2)\varphi(\mathbf{z})\left(z_1 - z_2 + \frac{z_1^2}{1-z_1} - \frac{z_2^2}{1-z_2}\right)\\ &= (z_1 - z_2)^2\varphi(\mathbf{z})\left[1 + \frac{z_1 + z_2 - z_1z_2}{(1-z_2)(1-z_1)}\right].\end{aligned}$$

This shows that $\Delta_1 \geq 0$ when $0 < z_i < 1$, $i = 1, \ldots, k$. According to Lemma 3, φ is Schur harmonically convex on $A = \{\mathbf{z} : \mathbf{z} \in (0,1)^k\}$. Let $g(t) = \frac{t}{1-t}$, then $p(t) := t^2g'(t) = \frac{t^2}{(1-t)^2}$. From

$t \in (0,1)$, it follows that $p'(t) = \frac{2t}{(1-t)^3} \geq 0$. According to Lemma 4(ii), φ is HA convex in its single variable on $(0,1)$. So $E_k(\frac{\mathbf{x}}{1-\mathbf{x}})$ is Schur harmonically convex on $(0,1)^n$ from Theorem 2. The proof of Theorem 4 is completed. □

By using Theorem A, the following conclusion is proved in [1, p.129].

The symmetric function

$$\overline{\psi}(\mathbf{x}) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{x_{i_1} + \cdots + x_{i_k}}{x_{i_1} \cdots x_{i_k}} \tag{8}$$

is Schur-convex on $\mathbb{R}^n_+$.

Now we use Theorem 1 and Theorem 2, respectively, to study Schur geometric convexity and Schur harmonic convexity of $\overline{\psi}(\mathbf{x})$.

Theorem 5 *The symmetric function $\overline{\psi}(\mathbf{x})$ is Schur geometrically convex and Schur harmonically concave on $\mathbb{R}^n_+$.*

Proof Let $\varphi(\mathbf{y}) = \sum_{i=1}^k y_i / \prod_{i=1}^k y_i$, then $\log \varphi(\mathbf{y}) = \log(\sum_{i=1}^k y_i) - \sum_{i=1}^k \log y_i$. Thus,

$$\frac{\partial \varphi(\mathbf{y})}{\partial y_1} = \varphi(\mathbf{y})\left(\frac{1}{\sum_{i=1}^k y_i} - \frac{1}{y_1}\right), \qquad \frac{\partial \varphi(\mathbf{y})}{\partial y_2} = \varphi(\mathbf{y})\left(\frac{1}{\sum_{i=1}^k y_i} - \frac{1}{y_2}\right),$$

$$\begin{aligned} \Delta &:= (y_1 - y_2)\left(y_1 \frac{\partial \varphi(\mathbf{y})}{\partial y_1} - y_2 \frac{\partial \varphi(\mathbf{y})}{\partial y_2}\right) \\ &= (y_1 - y_2)\varphi(\mathbf{y})\left(\frac{y_1 - y_2}{\sum_{i=1}^k y_i}\right) \\ &= \frac{(y_1 - y_2)^2}{\prod_{i=1}^k y_i} \geq 0. \end{aligned}$$

According to Lemma 2, $\varphi(\mathbf{y})$ is Schur geometrically convex on $\mathbb{R}^k_+$. Let $g(z) = \varphi(z, x_2, \dots, x_k) = \frac{z+a}{bz} = \frac{1}{b} + \frac{a}{bz}$, where $a = \sum_{i=2}^k x_i$, $b = \prod_{i=2}^k x_i$, then $h(z) := zg'(z) = -\frac{a}{bz}$. From $z \in \mathbb{R}_+$, it follows that $h'(z) = \frac{a}{bz^2} \geq 0$. According to Lemma 4(i), φ is GA convex in its single variable on $\mathbb{R}_+$. So $\overline{\psi}(\mathbf{x})$ is Schur geometrically convex on $\mathbb{R}_+$ from Theorem 1.

It is easy to check that

$$\begin{aligned} \Delta_1 &:= (y_1 - y_2)\left(y_1^2 \frac{\partial \varphi(\mathbf{y})}{\partial y_1} - y_2^2 \frac{\partial \varphi(\mathbf{y})}{\partial y_2}\right) \\ &= \frac{(y_1 - y_2)^2 (y_1 + y_2 - \sum_{i=1}^k y_i)}{\prod_{i=1}^k y_i} \leq 0. \end{aligned}$$

According to Lemma 3, $\varphi(\mathbf{y})$ is Schur harmonically concave on $\mathbb{R}^k_+$. Let $h(z) := z^2 g'(z) = -\frac{a}{b}$. $h'(z) = 0$ when $z \in \mathbb{R}_+$. According to Lemma 4(ii), φ is HA concave in its single variable on $\mathbb{R}_+$. So $\overline{\psi}(\mathbf{x})$ is Schur harmonically concave on $\mathbb{R}^n_+$ from Theorem 2. □

Remark 2 Let

$$H = \frac{n}{\sum_{i=1}^n \frac{1}{x_i}}, \qquad G = \left(\prod_{i=1}^n x_i\right)^{\frac{1}{n}},$$

where $x_i > 0$, $i = 1, \dots, n$. Then

$$(\log G, \dots, \log G) \prec (\log x_1, \dots, \log x_n), \tag{9}$$

$$\left(\frac{1}{H}, \dots, \frac{1}{H}\right) \prec \left(\frac{1}{x_1}, \dots, \frac{1}{x_n}\right). \tag{10}$$

From Theorem 5, it follows that

$$\frac{kC_n^k}{H^{k-1}} \geq \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{x_{i_1} + \cdots + x_{i_k}}{x_{i_1} \cdots x_{i_k}} \geq \frac{kC_n^k}{G^{k-1}}. \tag{11}$$

By using Theorem A, the following conclusion is proved in [1, p.129].

The symmetric function

$$\psi(\mathbf{x}) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{x_{i_1} \cdots x_{i_k}}{x_{i_1} + \cdots + x_{i_k}}$$

is Schur-concave on $\mathbb{R}_+^n$.

By applying Theorem 2, we further obtain the following result.

Theorem 6 *The symmetric function $\psi(\mathbf{x})$ is Schur harmonically convex on $\mathbb{R}_+^n$.*

Proof Let $\lambda(\mathbf{y}) = \prod_{i=1}^k y_i / \sum_{i=1}^k y_i$. According to the proof of Theorem 5, $\varphi(\mathbf{y})$ is Schur harmonically concave on $\mathbb{R}_+^k$. Let $\lambda(\mathbf{y}) = \frac{1}{\varphi(\mathbf{y})}$. From the definition of Schur harmonically convex, it follows that $\lambda(\mathbf{y})$ is Schur harmonically convex on $\mathbb{R}_+^k$. Let $g(z) = \lambda(z, x_2, \dots, x_k) = \frac{bz}{z+a}$, where $a = \sum_{i=2}^k x_i$, $b = \prod_{i=2}^k x_i$. Then $h(z) := z^2 g'(z) = \frac{z^2 ab}{(z+a)^2}$. With the fact that $h'(z) = \frac{2za^2 b}{(z+a)^3} \geq 0$ for $z \in \mathbb{R}_+$, it follows that φ is HA convex in its single variable on $\mathbb{R}_+$. So, from Theorem 2, $\psi(\mathbf{x})$ is Schur harmonically convex on $\mathbb{R}_+^n$. □

Remark 3 From Theorem 6 and (10), it follows that

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{x_{i_1} \cdots x_{i_k}}{x_{i_1} + \cdots + x_{i_k}} \geq \frac{H^{k-1} C_n^k}{k}, \tag{12}$$

where $x_i > 0$, $i = 1, \dots, n$.

Remark 4 It needs further discussion that $\psi(\mathbf{x})$ is Schur geometrically convex on $\mathbb{R}_+^n$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors co-authored this paper together. All authors read and approved the final manuscript.

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