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Convergence analysis of Agarwal *et al.* iterative scheme for Lipschitzian hemicontractive mappings

Shin Min Kang[1], Arif Rafiq[2], Faisal Ali[3] and Young Chel Kwun[4*]

*Correspondence:
yckwun@dau.ac.kr
[4]Department of Mathematics, Dong-A University, Pusan, 614-714, Korea
Full list of author information is available at the end of the article

Abstract

In this paper, we establish strong convergence for the Agarwal *et al.* iterative scheme associated with Lipschitzian hemicontractive mappings in Hilbert spaces.

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1 Introduction and preliminaries

Let K be a nonempty subset of a Hilbert space H and $T : K \to K$ be a mapping.

The mapping T is called *Lipschitzian* if there exists $L > 0$ such that

$$\|Tx - Ty\| \le L\|x - y\|, \quad \forall x, y \in K.$$

If $L = 1$, then T is called *nonexpansive* and if $0 \le L < 1$, then T is called *contractive*.

The mapping $T : K \to K$ is said to be *pseudocontractive* (see, for example, [1, 2]) if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in K,$$

and it is said to be *strongly pseudocontractive* if there exists $k \in (0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in K.$$

Let $F(T) := \{x \in H : Tx = x\}$, and the mapping $T : K \to K$ is called *hemicontractive* if $F(T) \neq \emptyset$ and

$$\|Tx - x^*\|^2 \le \|x - x^*\|^2 + \|x - Tx\|^2, \quad \forall x \in K, x^* \in F(T).$$

It is easy to see that the class of pseudocontractive mappings with fixed points is a subclass of the class of hemicontractions. For the importance of fixed points of pseudocontractions, the reader may consult [1].

In 1974, Ishikawa [3] proved the following result.

Theorem 1.1 *Let K be a compact convex subset of a Hilbert space H, and let $T : K \to K$ be a Lipschitzian pseudocontractive mapping. For arbitrary $x_1 \in K$, let $\{x_n\}$ be a sequence defined iteratively by*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n, \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n, \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying

(i) $0 \leq \alpha_n \leq \beta_n \leq 1$;
(ii) $\lim_{n\to\infty} \beta_n = 0$;
(iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Another iteration scheme has been studied extensively in connection with fixed points of pseudocontractive mappings.

In 2007, Agarwal *et al.* [4] introduced the new iterative scheme as in the following.

The sequence $\{x_n\}$ defined by, for arbitrary $x_1 \in K$,

$$\begin{cases} x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n, \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n, \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$, is known as the Agarwal *et al.* iterative scheme.

In this paper, we establish the strong convergence for the Agarwal *et al.* iterative scheme associated with Lipschitzian hemicontractive mappings in Hilbert spaces.

2 Main results

We need the following lemma.

Lemma 2.1 [5] *For all $x, y \in H$ and $\lambda \in [0,1]$, we have*

$$\|(1-\lambda)x + \lambda y\|^2 = (1-\lambda)\|x\|^2 + \lambda\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2.$$

Now we prove our main results.

Theorem 2.2 *Let K be a compact convex subset of a real Hilbert space H, and let $T : K \to K$ be a Lipschitzian hemicontractive mapping satisfying*

$$\|x - Ty\| \leq \|Tx - Ty\|, \quad \forall x, y \in K. \tag{C}$$

Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$ satisfying

(ii) $\lim_{n\to\infty} \beta_n = 0$;
(iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$;
(iv) $\lim_{n\to\infty} \alpha_n = 1$.

For arbitrary $x_1 \in K$, let $\{x_n\}$ be a sequence iteratively defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n, \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n, \quad n \geq 1. \end{cases} \tag{2.1}$$

Then the sequence $\{x_n\}$ converges strongly to the fixed point x^* of T.

Proof From Schauder's fixed point theorem, $F(T)$ is nonempty since K is a convex compact set and T is continuous, let $x^* \in F(T)$.

By using condition (C), we have

$$\begin{aligned}\|x - Tx\| &\le \|x - Ty\| + \|Tx - Ty\| \\ &\le 2\|Tx - Ty\|. \end{aligned} \tag{2.2}$$

Using the fact that T is hemicontractive, we obtain

$$\|Tx_n - x^*\|^2 \le \|x_n - x^*\|^2 + \|x_n - Tx_n\|^2 \tag{2.3}$$

and

$$\|Ty_n - x^*\|^2 \le \|y_n - x^*\|^2 + \|y_n - Ty_n\|^2. \tag{2.4}$$

With the help of (2.1), (2.3) and Lemma 2.1, we obtain

$$\begin{aligned}\|y_n - x^*\|^2 &= \|(1-\beta_n)x_n + \beta_n Tx_n - x^*\|^2 \\ &= \|(1-\beta_n)(x_n - x^*) + \beta_n(Tx_n - x^*)\|^2 \\ &= (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|Tx_n - x^*\|^2 \\ &\quad - \beta_n(1-\beta_n)\|x_n - Tx_n\|^2 \\ &\le (1-\beta_n)\|x_n - x^*\|^2 + \beta_n(\|x_n - x^*\|^2 + \|x_n - Tx_n\|^2) \\ &\quad - \beta_n(1-\beta_n)\|x_n - Tx_n\|^2 \\ &= \|x_n - x^*\|^2 + \beta_n^2\|x_n - Tx_n\|^2 \end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\|y_n - Ty_n\|^2 &= \|(1-\beta_n)x_n + \beta_n Tx_n - Ty_n\|^2 \\ &= \|(1-\beta_n)(x_n - Ty_n) + \beta_n(Tx_n - Ty_n)\|^2 \\ &= (1-\beta_n)\|x_n - Ty_n\|^2 + \beta_n\|Tx_n - Ty_n\|^2 \\ &\quad - \beta_n(1-\beta_n)\|x_n - Tx_n\|^2. \end{aligned} \tag{2.6}$$

Substituting (2.5) and (2.6) in (2.4), we obtain

$$\begin{aligned}\|Ty_n - x^*\|^2 &\le \|x_n - x^*\|^2 + (1-\beta_n)\|x_n - Ty_n\|^2 \\ &\quad + \beta_n\|Tx_n - Ty_n\|^2 - \beta_n(1-2\beta_n)\|x_n - Tx_n\|^2. \end{aligned} \tag{2.7}$$

Also, with the help of conditions (2.2) and (2.7), we have

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &\quad = \|(1-\alpha_n)Tx_n + \alpha_n Ty_n - x^*\|^2 \end{aligned}$$

$$\begin{aligned}
&= \|(1-\alpha_n)(Tx_n - x^*) + \alpha_n(Ty_n - x^*)\|^2 \\
&= (1-\alpha_n)\|Tx_n - x^*\|^2 + \alpha_n\|Ty_n - x^*\|^2 \\
&\quad - \alpha_n(1-\alpha_n)\|Tx_n - Ty_n\|^2 \\
&\le (1-\alpha_n)(\|x_n - x^*\|^2 + \|x_n - Tx_n\|^2) + \alpha_n(\|x_n - x^*\|^2 \\
&\quad + (1-\beta_n)\|x_n - Ty_n\|^2 + \beta_n\|Tx_n - Ty_n\|^2 \\
&\quad - \beta_n(1-2\beta_n)\|x_n - Tx_n\|^2) \\
&= \|x_n - x^*\|^2 + (1-\alpha_n)\|x_n - Tx_n\|^2 + \alpha_n\beta_n\|Tx_n - Ty_n\|^2 \\
&\quad - \alpha_n\beta_n(1-2\beta_n)\|x_n - Tx_n\|^2 + \alpha_n(1-\beta_n)\|x_n - Ty_n\|^2 \\
&\le \|x_n - x^*\|^2 + (4(1-\alpha_n) + \alpha_n\beta_n + \alpha_n(1-\beta_n))\|Tx_n - Ty_n\|^2 \\
&\quad - \alpha_n\beta_n(1-2\beta_n)\|x_n - Tx_n\|^2 \\
&\le \|x_n - x^*\|^2 + \theta\alpha_n\|Tx_n - Ty_n\|^2 - \alpha_n\beta_n(1-2\beta_n)\|x_n - Tx_n\|^2,
\end{aligned} \tag{2.8}$$

because by (iv), there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$,

$$1 - \alpha_n \le \frac{\theta - 1}{\theta + 3}, \tag{2.9}$$

where $\theta > 1$, which implies that

$$4(1-\alpha_n) + \alpha_n\beta_n + \alpha_n(1-\beta_n) \le \theta\alpha_n. \tag{2.10}$$

Hence (2.8) yields

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad \le \|x_n - x^*\|^2 + \theta\alpha_n L^2\|x_n - y_n\|^2 - \alpha_n\beta_n(1-2\beta_n)\|x_n - Tx_n\|^2 \\
&\quad = \|x_n - x^*\|^2 + \theta\alpha_n\beta_n^2 L^2\|x_n - Tx_n\|^2 - \alpha_n\beta_n(1-2\beta_n)\|x_n - Tx_n\|^2 \\
&\quad = \|x_n - x^*\|^2 - \alpha_n\beta_n(1 - (2+\theta L^2)\beta_n)\|x_n - Tx_n\|^2.
\end{aligned} \tag{2.11}$$

Now, by (ii), since $\lim_{n\to\infty}\beta_n = 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$,

$$\beta_n \le \frac{1}{2(2+\theta L^2)}. \tag{2.12}$$

With the help of (iii) and (2.12), (2.11) yields

$$\|x_{n+1} - x^*\|^2 \le \|x_n - x^*\|^2 - \frac{1}{2}\alpha_n\beta_n\|x_n - Tx_n\|^2,$$

which implies that

$$\frac{1}{2}\alpha_n\beta_n\|x_n - Tx_n\|^2 \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2,$$

so that

$$\frac{1}{2}\sum_{j=N}^{n}\alpha_j\beta_j\|x_j - Tx_j\|^2 \le \|x_N - x^*\|^2 - \|x_{n+1} - x^*\|^2.$$

The rest of the argument follows exactly as in the proof of theorem of [3]. This completes the proof. □

Theorem 2.3 *Let K be a compact convex subset of a real Hilbert space H; let $T : K \to K$ be a Lipschitzian hemicontractive mapping satisfying condition* (C). *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequence in* $[0,1]$ *satisfying conditions* (ii)-(iv).

Let $P_K : H \to K$ be the projection operator of H onto K. Let $\{x_n\}$ be a sequence defined iteratively by

$$\begin{cases} x_{n+1} = P_K((1-\alpha_n)Tx_n + \alpha_n Ty_n), \\ y_n = P_K((1-\beta_n)x_n + \beta_n Tx_n), \quad n \ge 1. \end{cases}$$

Then the sequence$\{x_n\}$ converges strongly to a fixed point of T.

Proof The operator P_K is nonexpansive (see, *e.g.*, [2]). K is a Chebyshev subset of H so that P_K is a single-valued mapping. Hence, we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|P_K((1-\alpha_n)Tx_n + \alpha_n Ty_n) - P_K x^*\|^2 \\ &\le \|(1-\alpha_n)Tx_n + \alpha_n Ty_n - x^*\|^2 \\ &= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(Ty_n - x^*)\|^2 \\ &\le \|x_n - x^*\|^2 - \alpha_n\beta_n(1 - (2+\theta L^2)\beta_n)\|x_n - Tx_n\|^2. \end{aligned}$$

The set $K = K \cup T(K)$ is compact and so the sequence $\{\|x_n - Tx_n\|\}$ is bounded. The rest of the argument follows exactly as in the proof of Theorem 2.2. This completes the proof. □

Example 2.4 The choice for the control parameters is $\alpha_n = \frac{n}{n+1}$ and $\beta_n = \frac{1}{n}$.

Remark 2.5 (1) We remove the condition $\alpha_n \le \beta_n$ as introduced in [3].

(2) The condition (C) is not new and it is due to [6].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors read and approved the final manuscript.

Author details

[1]Department of Mathematics and RINS, Gyeongsang National University, Jinju, 660-701, Korea. [2]Department of Mathematics, Lahore Leads University, Lahore, Pakistan. [3]Centre for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University, Multan, 54000, Pakistan. [4]Department of Mathematics, Dong-A University, Pusan, 614-714, Korea.

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