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Some determinantal inequalities for accretive-dissipative matrices

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Abstract

This article aims to obtain some determinantal inequalities for accretive-dissipative matrices which are generalizations of the determinantal inequalities presented by Lin (Linear Algebra Appl. 438:2808-2812, 2013). At the same time, we give some numerical examples which show the effectiveness of our results.

Keywords: accretive-dissipative matrix; determinantal inequality

1 Introduction

Let $\mathbb{M}_n(\mathbf{C})$ be the space of complex matrices of size $n \times n$ matrices. $A \in \mathbb{M}_n(\mathbf{C})$ is said to be accretive-dissipative, if, in its Toeplitz decomposition

$$A = B + iC, \qquad B = B^*, \qquad C = C^*, \tag{1.1}$$

both matrices B and C are Hermitian positive definite. For simplicity, let A, B, C be partitioned as

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix} + i \begin{pmatrix} C_{11} & C_{12} \\ C_{12}^* & C_{22} \end{pmatrix} \tag{1.2}$$

such that the diagonal blocks A_{11} and A_{22} are of order k and l ($k > 0$, $l > 0$ and $k + l = n$), respectively, and let $m = \min\{k, l\}$.

If $A \in \mathbb{M}_n(\mathbf{C})$ is partitioned as

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} is a nonsingular submatrix, then the matrix $A/A_{11} := A_{22} - A_{21}A_{11}^{-1}A_{12}$ is called the *Schur complement* of the submatrix A_{11} in A.

If $A \in \mathbb{M}_n(\mathbf{C})$ is positive definite and partitioned as in (1.2), then the inequalities [1, Lemma 6] hold:

$$|\det A| = \left|\det(B + iC)\right| \le \left|\det(B + C)\right| \le 2^{\frac{n}{2}} \left|\det(B + iC)\right|. \tag{1.3}$$

If $A \in \mathbb{M}_n(\mathbf{C})$ is positive definite and partitioned as in (1.2), then the famous Fischer-type determinantal inequality is proved [2, p.478]:

$$\det A \leq \det A_{11} \det A_{22}. \tag{1.4}$$

If $A \in \mathbb{M}_n(\mathbf{C})$ is an accretive-dissipative matrix and partitioned as in (1.2), Ikramov [3] first proved the determinantal inequality for A:

$$|\det A| \leq 3^m |\det A_{11}||\det A_{22}|. \tag{1.5}$$

Lin [1, Theorem 8] got a stronger result than (1.5) as follows:
If $A \in \mathbb{M}_n(\mathbf{C})$ is an accretive-dissipative matrix, then

$$|\det A| \leq 2^{\frac{3}{2}m} |\det A_{11}||\det A_{22}|. \tag{1.6}$$

The purpose of this paper is to give some generalizations of (1.3) and (1.6). Our main results can be stated as follows.

Theorem 1 *Let $B, C \in \mathbb{M}_n(\mathbf{C})$ be positive definite and x, y be positive real numbers. Then*

$$\left|\det(B + iC)\right| \leq \det(B + C) \leq \left(x^2 + y^2\right)^{\frac{n}{2}} \left|\det\left(\frac{B}{x} + i\frac{C}{y}\right)\right|. \tag{1.7}$$

When $x = y$, the inequality $\det(B + C) \leq 2^{\frac{n}{2}} |\det(B + iC)|$ is a special case of Theorem 1. Thus (1.5) is a generalization of the inequality $|\det(B + iC)| \leq \det(B + C) \leq 2^{\frac{n}{2}} |\det(B + iC)|$ [1, Lemma 6].

Theorem 2 *Let $A \in \mathbb{M}_n(\mathbf{C})$ be accretive-dissipative and partitioned as in* (1.2), *and let x, y be positive real numbers. Then*

$$|\det A| \leq \left(x^2 + y^2\right)^{\frac{n}{2}} \left|\det\left(\frac{B_{11}}{x} + i\frac{C_{11}}{y}\right)\right| \left|\det\left(\frac{B_{22}}{x} + i\frac{C_{22}}{y}\right)\right|. \tag{1.8}$$

When $x = y$, we get the inequality $|\det A| \leq 2^{\frac{n}{2}} |\det A_{11}||\det A_{22}|$ [1, (3.1)], which is a special case of Theorem 2.

Theorem 3 *Let $A \in \mathbb{M}_n(\mathbf{C})$ be accretive-dissipative and partitioned as in* (1.2), *and let x, y be positive real numbers. Then*

$$|\det A| \leq 2^m \left(x^2 + y^2\right)^{\frac{m}{2}} |\det A_{11}| \left|\det\left(\frac{B_{22}}{x} + i\frac{C_{22}}{y}\right)\right|. \tag{1.9}$$

When $x = y$, we get the inequality [1, (3.2)]

$$|\det A| \leq 2^{\frac{3}{2}m} |\det A_{22}||\det A_{11}|,$$

which is a special case of (1.7).

2 Proofs of main results

To achieve the proofs of Theorem 1, Theorem 2 and Theorem 3, we need the following lemmas.

Lemma 4 [4, Property 6] *Let $A \in \mathbb{M}_n(\mathbf{C})$ be accretive-dissipative and partitioned as in* (1.2). *Then $A/A_{11} := A_{22} - A_{21}A_{11}^{-1}A_{12}$, the Schur complement of A_{11} in A, is also accretive-dissipative.*

Lemma 5 [3, Lemma 1] *Let $A \in \mathbb{M}_n(\mathbf{C})$ be accretive-dissipative and partitioned as in* (1.2). *Then $A^{-1} = E - iF$ with $E = (B + CB^{-1}C)^{-1}$ and $F = (C + BC^{-1}B)^{-1}$.*

Lemma 6 [3, Lemma 4] *Let $B, C \in \mathbb{M}_n(\mathbf{C})$ be Hermitian and assume that $B > 0$. Then*

$$B + CB^{-1}C \geq 2C. \tag{2.1}$$

Remark 1 A stronger inequality than (2.4) was given in Lin [5, Lemma 2.2]: Let $A > 0$ and any Hermitian B. Then $A\sharp(BA^{-1}B) \geq B$.

Proof of Theorem 1 Let λ_j, $j = 1, \ldots, n$, be the eigenvalues of $B^{-\frac{1}{2}}CB^{-\frac{1}{2}}$, where $B^{\frac{1}{2}}$ means the unique positive definite square root of B. Then we have

$$|1 + i\lambda_j| \leq |1 + \lambda_j| \leq \sqrt{x^2 + y^2}\left|\frac{1}{x} + i\frac{\lambda_j}{y}\right|. \tag{2.2}$$

The first inequality follows from [6, Theorem 2.2], while the second one we prove is as follows:

$$\begin{aligned}
\left|\det(B + iC)\right| &\leq \left|\det(B + C)\right| \\
&= \left|\det B^{\frac{1}{2}}\left(I + B^{-\frac{1}{2}}CB^{-\frac{1}{2}}\right)B^{\frac{1}{2}}\right| \\
&= \left|\det B^{\frac{1}{2}}\right|\left|\det B^{\frac{1}{2}}\right|\left|\det\left(I + B^{-\frac{1}{2}}CB^{-\frac{1}{2}}\right)\right| \\
&= |\det B| \left|\prod_{j=1}^{n}|1 + \lambda_j|\right. \\
&\leq |\det B| \left|\prod_{j=1}^{n}\sqrt{x^2 + y^2}\left|\frac{1}{x} + i\frac{\lambda_j}{y}\right|\right. \quad \text{(by (2.2))} \\
&= \left(x^2 + y^2\right)^{\frac{n}{2}}|\det B|\left|\det\left(\frac{I}{x} + \frac{i}{y}B^{-\frac{1}{2}}CB^{-\frac{1}{2}}\right)\right| \\
&= \left(x^2 + y^2\right)^{\frac{n}{2}}\left|\det\left(\frac{B}{x} + \frac{i}{y}C\right)\right|.
\end{aligned}$$

The proof is completed. □

Proof of Theorem 2

$$\begin{aligned}
|\det A| &\leq \det(B + C) \quad \text{(by Theorem 1)} \\
&\leq \det(B_{11} + C_{11}) \cdot \det(B_{22} + C_{22}) \quad \text{(by (1.3))}
\end{aligned}$$

$$\leq \left(x^2+y^2\right)^{\frac{k}{2}} \left|\det \frac{B_{11}}{x} + \frac{i}{y}C_{11}\right| \left(x^2+y^2\right)^{\frac{l}{2}} \left|\det \frac{B_{22}}{x} + \frac{i}{y}C_{22}\right| \quad \text{(by (1.6))}$$
$$= \left(x^2+y^2\right)^{\frac{n}{2}} \left|\det \frac{B_{11}}{x} + \frac{i}{y}C_{11}\right| \left|\det \frac{B_{22}}{x} + \frac{i}{y}C_{22}\right|.$$

The proof is completed. □

Proof of Theorem 3 By Lemma 4, we obtain

$$\begin{aligned} A/A_{11} &= A_{22} - A_{21}A_{11}^{-1}A_{12} \\ &= B_{22} + iC_{22} - \left(B_{12}^* + iC_{12}^*\right)(B_{11} + iC_{11})^{-1}(B_{12} + iC_{12}) \\ &= B_{22} + iC_{22} - \left(B_{12}^* + iC_{12}^*\right)(E_k - iF_k)(B_{12} + iC_{12}). \end{aligned}$$

Furthermore, by Lemma 5, we have

$$E_k = \left(B_{11} + C_{11}B_{11}^{-1}C_{11}\right)^{-1}, \qquad F_k = \left(C_{11} + B_{11}C_{11}^{-1}B_{11}\right)^{-1},$$

where E_k and F_k are positive definite.

By a simple computation, we obtain

$$A/A_{11} = R + iS.$$

By Lemma 4, it is easy to know that R, S are positive definite and we have

$$\begin{aligned} R &= B_{22} - B_{12}^*E_kB_{12} + C_{12}^*E_kC_{12} - B_{12}^*F_kC_{12} - C_{12}^*F_kB_{12}, \\ S &= C_{22} + B_{12}^*F_kB_{12} - C_{12}^*F_kC_{12} - C_{12}^*E_kB_{12} - B_{12}^*E_kC_{12}. \end{aligned}$$

By the inequalities

$$\left(B_{12}^* \pm C_{12}^*\right)F_k(B_{12} \pm C_{12}) \geq 0, \qquad \left(B_{12}^* \pm C_{12}^*\right)E_k(B_{12} \pm C_{12}) \geq 0,$$

it can be proved that

$$\begin{aligned} &\pm\left(B_{12}^*F_kC_{12} + C_{12}^*F_kB_{12}\right) \leq B_{12}^*F_kB_{12} + C_{12}^*F_kC_{12}, \\ &\pm\left(C_{12}^*E_kB_{12} + B_{12}^*E_kC_{12}\right) \leq B_{12}^*E_kB_{12} + C_{12}^*E_kC_{12}. \end{aligned}$$

Thus

$$R + S \leq B_{22} + 2B_{12}^*F_kB_{12} + C_{22} + 2C_{12}^*E_kC_{12}. \tag{2.3}$$

By Lemma 6 and the operator reverse monotonicity of the inverse, we get

$$E_k \leq \frac{1}{2}C_{11}^{-1}, \qquad F_k \leq \frac{1}{2}B_{11}^{-1}. \tag{2.4}$$

As B, C are positive definite, we also have

$$B_{22} > B_{12}^*B_{11}^{-1}B_{12}, \qquad C_{22} > C_{12}^*C_{11}^{-1}C_{12}. \tag{2.5}$$

Without loss of generality, assume $m = l$. Then we have

$$\begin{aligned}
|\det A/A_{11}| &= |\det R + iS| \\
&\le \det(R+S) \quad (\text{by } (1.7)) \\
&\le \det\bigl(B_{22} + 2B_{12}^{*}F_k B_{12} + C_{22} + 2C_{12}^{*}E_k C_{12}\bigr) \quad (\text{by } (2.3)) \\
&\le \det\bigl(B_{22} + B_{12}^{*}B_{11}^{-1}B_{12} + C_{22} + C_{12}^{*}C_{11}^{-1}C_{12}\bigr) \quad (\text{by } (2.4)) \\
&< \det\bigl(2(B_{22}+C_{22})\bigr) \quad (\text{by } (2.5)) \\
&\le 2^m\bigl(x^2+y^2\bigr)^{\frac{m}{2}}\left|\det\left(\frac{B_{22}}{x} + i\frac{C_{22}}{y}\right)\right| \quad (\text{by } (1.6)).
\end{aligned}$$

By noting

$$|\det A| = |\det A_{11}|\bigl|\det(A/A_{11})\bigr|,$$

the proof is completed. □

3 Numerical examples

There are many upper bounds for the determinant of the accretive-dissipative matrices which are due to (1.6), (1.8) and (1.9). However, these bounds are incomparable.

In this section, we give some numerical examples to show that (1.8) and (1.9) are better than (1.6) in some cases.

Example 3.1 Let

$$\begin{aligned}
A = B + i * C &= \begin{pmatrix} 1.01 & -1 \\ -1 & 1.01 \end{pmatrix} + i * \begin{pmatrix} 1.01 & 1 \\ 1 & 1.01 \end{pmatrix} \\
&= \begin{pmatrix} 1.01 + 1.01i & -1 + i \\ -1 + i & 1.01 + 1.01i \end{pmatrix}.
\end{aligned}$$

We calculate that $|\det A| = 4.0402$.

By the upper bound of $|\det A|$ in (1.6), we have

$$2^{\frac{3}{2}m}|\det A_{11}||\det A_{22}| = 5.7706,$$

where $A_{11} = 1.01 + 1.01i$, $A_{22} = 1.01 + 1.01i$.

Let $x = 4$, $y = 5$. From the upper bound of $|\det A|$ in (1.8), we have

$$\bigl(x^2+y^2\bigr)^{\frac{n}{2}}\left|\det\left(\frac{B_{11}}{x} + i\frac{C_{11}}{y}\right)\right|\left|\det\left(\frac{B_{22}}{x} + i\frac{C_{22}}{y}\right)\right| = 4.2870,$$

where $B_{11} = 1.01$, $B_{22} = 1.01$, $C_{11} = 1.01$, $C_{22} = 1.01$.

Meanwhile, by the upper bound of $|\det A|$ in (1.9), we get

$$2^m\bigl(x^2+y^2\bigr)^{\frac{m}{2}}|\det A_{11}|\left|\det\left(\frac{B_{22}}{x} + i\frac{C_{22}}{y}\right)\right| = 5.9148.$$

Example 3.2 Let

$$A = B + i * C = \begin{pmatrix} 5 & -1 & 3 \\ -1 & 2 & -2 \\ 3 & -2 & 3 \end{pmatrix} + i * \begin{pmatrix} 5 & 3 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix}$$
$$= \begin{pmatrix} 5+5i & -1+3i & 3+i \\ -1+3i & 2+2i & -2+i \\ 3+i & -2+i & 3+4i \end{pmatrix}.$$

We calculate that $|\det A| = 119.0378$.

By the upper bound of $|\det A|$ in (1.6), we have

$$2^{\frac{3}{2}m} |\det A_{11}| |\det A_{22}| = 384.7077,$$

where $A_{11} = \left(\begin{smallmatrix} 5+5i & -1+3i \\ -1+3i & 2+2i \end{smallmatrix}\right)$, $A_{22} = 3 + 4i$.

Let $x = 4$, $y = 5$. Then, by the upper bound of $|\det A|$ in (1.8), we have

$$\left(x^2 + y^2\right)^{\frac{n}{2}} \left|\det\left(\frac{B_{11}}{x} + i\frac{C_{11}}{y}\right)\right| \left|\det\left(\frac{B_{22}}{x} + i\frac{C_{22}}{y}\right)\right| = 403.3473,$$

where $B_{11} = \left(\begin{smallmatrix} 5 & -1 \\ -1 & 2 \end{smallmatrix}\right)$, $C_{11} = \left(\begin{smallmatrix} 5 & 3 \\ 3 & 2 \end{smallmatrix}\right)$, $B_{22} = 3$, $C_{22} = 4$.

Meanwhile, by the upper bound of $|\det A|$ in (1.9), we get

$$2^m \left(x^2 + y^2\right)^{\frac{m}{2}} |\det A_{11}| \left|\det\left(\frac{B_{22}}{x} + i\frac{C_{22}}{y}\right)\right| = 382.0149.$$

From the two examples above, we can obtain that (1.8) and (1.9) are better than (1.6) in some cases.

Competing interests

The author declares that they have no competing interests.

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