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On a strengthened Hardy-Hilbert type inequality

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Abstract
We derive a strengthenment of a Hardy-Hilbert type inequality by using the Euler-Maclaurin expansion for the zeta function and estimating the weight function effectively. As applications, some particular results are presented.
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1 Introduction

Let $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, $a_n,b_n\geq 0$, $0<\sum_{n=1}^{\infty}a_n^p<\infty$ and $0<\sum_{n=1}^{\infty}b_n^q<\infty$. Then one [1] has

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{\sin(\pi/p)}\left[\sum_{n=1}^{\infty}a_n^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}b_n^q\right]^{\frac{1}{q}}, \tag{1.1}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}}<pq\left[\sum_{n=1}^{\infty}a_n^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}b_n^q\right]^{\frac{1}{q}}, \tag{1.2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ and pq are best possible. Inequality (1.1) is well known as Hardy-Hilbert's inequality, and inequality (1.2) is named a Hardy-Hilbert type inequality. Both of them are important in analysis and applications [2]. In recent years, many results about generalizations of this type of inequality were established (see [3]). Under the same conditions as (1.1) and (1.2), some Hardy-Hilbert type inequalities, which are similar to (1.1) and (1.2), have been studied and generalized by some mathematicians.

By introducing a parameter, Yang gave a generalization of inequality (1.2) with the best constant factor as follows:

If $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, $2-\min\{p,q\}<\lambda\leq 2$, $a_n,b_n\geq 0$, such that $0<\sum_{n=1}^{\infty}n^{1-\lambda}a_n^p<\infty$ and $0<\sum_{n=1}^{\infty}n^{1-\lambda}b_n^q<\infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda,n^\lambda\}}<k_\lambda(p)\left\{\sum_{n=1}^{\infty}n^{1-\lambda}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}}, \tag{1.3}$$

where the constant factor $k_\lambda(p)=\frac{\lambda pq}{(p+\lambda-2)(q+\lambda-2)}$ is best possible.

Furthermore, by introducing a parameter and two pairs of conjugate exponents, Zhong gave a generalization of inequality (1.3) with the best constant factor as follows:

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $0 < \lambda \le \min\{r, s\}$, $a_n, b_n \ge 0$, such that $0 < \sum_{n=1}^{\infty} n^{p(1-\frac{\lambda}{r})-1} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{s})-1} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} < k_\lambda(r) \left\{\sum_{n=1}^{\infty} n^{p(1-\frac{\lambda}{r})-1} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{s})-1} b_n^q\right\}^{\frac{1}{q}}, \tag{1.4}$$

where the constant factor $k_\lambda(r) = \frac{rs}{\lambda}$ is best possible.

Recently, in [4], Jiang and Hua established an improvement of inequality (1.3) as follows: If $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - \min\{p, q\} < \lambda \le 2$, $a_n \ge 0$, $b_n \ge 0$, for $n \ge 1$, $n \in \mathrm{N}$ and $0 < \sum_{n=1}^{\infty} n^{1-\lambda} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} n^{1-\lambda} b_n^q < \infty$, then

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} &< \left\{\sum_{n=1}^{\infty} \left[k(\lambda) - \frac{q}{3(q+\lambda-2)n^{\frac{q+\lambda-2}{q}}}\right] n^{1-\lambda} a_n^p\right\}^{\frac{1}{p}} \\ &\quad \times \left\{\sum_{n=1}^{\infty} \left[k(\lambda) - \frac{p}{3(p+\lambda-2)n^{\frac{p+\lambda-2}{p}}}\right] n^{1-\lambda} b_n^q\right\},\end{aligned} \tag{1.5}$$

where $k(\lambda) = \frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)} > 0$.

In this paper, by introducing a parameter and estimating the weight coefficient, we obtain a strengthenment of inequality (1.4) and generalize inequality (1.5). As applications, some particular results are presented.

2 Some preliminary results

First, we need the following formula of the Riemann-ζ function (see [5]):

$$\begin{aligned}\zeta(\rho) &= \sum_{n=1}^{m} \frac{1}{n^\rho} - \frac{m^{1-\rho}}{1-\rho} - \frac{1}{2m^\rho} \\ &\quad - \sum_{n=1}^{l-1} \frac{B_{2n}}{2n} \binom{-\rho}{2n-1} \frac{1}{m^{\rho+2n-1}} - \frac{B_{2l}}{2l} \binom{-\rho}{2l-1} \frac{\varepsilon}{m^{\rho+2l-1}},\end{aligned} \tag{2.1}$$

where $\rho > 0$, $\rho \ne 1$, $m, l \ge 1$, $m, l \in \mathrm{N}$, $0 < \varepsilon = \varepsilon(\rho, l, m) < 1$. The numbers $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30, \ldots$ are Bernoulli numbers. In particular, $\zeta(\rho) = \sum_{n=1}^{\infty} \frac{1}{n^\rho}$ $(\rho > 1)$.

Since $\zeta(0) = -1/2$, the formula of the Riemann-ζ function (2.1) also holds for $\rho = 0$.

Lemma 2.1 *Let* $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $0 < \lambda \le \min\{r, s\}$, *define the weight coefficients* $\omega(m, \lambda, s)$ *and* $\omega(n, \lambda, r)$ *as*

$$\omega(m, \lambda, s) = \sum_{n=1}^{\infty} \frac{1}{\max\{m^\lambda, n^\lambda\}} \left(\frac{m}{n}\right)^{1-\frac{\lambda}{s}}, \tag{2.2}$$

$$\omega(n, \lambda, r) = \sum_{m=1}^{\infty} \frac{1}{\max\{m^\lambda, n^\lambda\}} \left(\frac{n}{m}\right)^{1-\frac{\lambda}{r}}. \tag{2.3}$$

Then we have

$$\omega(m, \lambda, s) < m^{1-\lambda} \left[k_\lambda - \frac{s}{3\lambda m^{\frac{\lambda}{s}}}\right] \tag{2.4}$$

and

$$\omega(n,\lambda,r) < n^{1-\lambda}\left[k_\lambda - \frac{r}{3\lambda m^{\frac{\lambda}{r}}}\right], \tag{2.5}$$

where $k_\lambda = \frac{rs}{\lambda}$.

Proof For $0 < \lambda \le \min\{r,s\}$, taking $\rho = 1 - \frac{\lambda}{s} \ge 0$, $l = 1$ in (2.1), we get

$$\zeta\left(1-\frac{\lambda}{s}\right) = \sum_{n=1}^{m} \frac{1}{n^{1-\frac{\lambda}{s}}} - \frac{sm^{\frac{\lambda}{s}}}{\lambda} - \frac{1}{2m^{1-\frac{\lambda}{s}}} + \frac{1-\frac{\lambda}{s}}{12m^{2-\frac{\lambda}{s}}}\varepsilon_1, \tag{2.6}$$

where $0 < \varepsilon_1 < 1$.

Set $\rho = 1 + \frac{\lambda}{r}$, $l = 1$, and we can derive

$$\zeta\left(1+\frac{\lambda}{r}\right) = \sum_{n=1}^{m-1} \frac{1}{n^{1+\frac{\lambda}{r}}} + \frac{rm^{-\frac{\lambda}{r}}}{\lambda} + \frac{1}{2m^{1+\frac{\lambda}{r}}} + \frac{1+\frac{\lambda}{r}}{12m^{2+\frac{\lambda}{r}}}\varepsilon_2, \tag{2.7}$$

where $0 < \varepsilon_2 < 1$.

Thus we get

$$\begin{aligned}
\omega(m,\lambda,s) &= \sum_{n=1}^{\infty} \frac{1}{\max\{m^\lambda, n^\lambda\}}\left(\frac{m}{n}\right)^{1-\frac{\lambda}{s}} \\
&= \sum_{n=1}^{m} \frac{1}{\max\{m^\lambda, n^\lambda\}}\left(\frac{m}{n}\right)^{1-\frac{\lambda}{s}} - \frac{1}{m^\lambda} + \sum_{n=m}^{\infty} \frac{1}{\max\{m^\lambda, n^\lambda\}}\left(\frac{m}{n}\right)^{1-\frac{\lambda}{s}} \\
&= \sum_{n=1}^{m} \frac{1}{m^\lambda}\left(\frac{m}{n}\right)^{1-\frac{\lambda}{s}} - \frac{1}{m^\lambda} + \sum_{n=m}^{\infty} \frac{1}{n^\lambda}\left(\frac{m}{n}\right)^{1-\frac{\lambda}{s}} \\
&= \frac{1}{m^{\lambda+\frac{\lambda}{s}-1}} \sum_{n=1}^{m} \frac{1}{n^{1-\frac{\lambda}{s}}} - \frac{1}{m^\lambda} + m^{1-\frac{\lambda}{s}} \sum_{n=m}^{\infty} \frac{1}{n^{1+\frac{\lambda}{r}}}.
\end{aligned}$$

Combining (2.6) and (2.7), we have

$$\begin{aligned}
\omega(m,\lambda,s) &< \frac{1}{m^{\lambda+\frac{\lambda}{s}-1}}\left[\zeta\left(1-\frac{\lambda}{s}\right) + \frac{sm^{\frac{\lambda}{s}}}{\lambda} + \frac{1}{2m^{1-\frac{\lambda}{s}}}\right] - \frac{1}{m^\lambda} \\
&\quad + m^{1-\frac{\lambda}{s}}\left[\frac{rm^{-\frac{\lambda}{r}}}{\lambda} + \frac{1}{2m^{1+\frac{\lambda}{r}}} + \frac{1+\frac{\lambda}{r}}{12m^{2+\frac{\lambda}{r}}}\right] \\
&= \frac{1}{m^{\lambda+\frac{\lambda}{s}-1}}\zeta\left(1-\frac{\lambda}{s}\right) + \frac{sm^{1-\lambda}}{\lambda} + \frac{1}{2m^\lambda} - \frac{1}{m^\lambda} + \frac{rm^{1-\lambda}}{\lambda} + \frac{1}{2m^\lambda} + \frac{1+\frac{\lambda}{r}}{12m^{1+\lambda}} \\
&= \frac{1}{m^{\lambda+\frac{\lambda}{s}-1}}\zeta\left(1-\frac{\lambda}{s}\right) + \frac{rsm^{1-\lambda}}{\lambda} + \frac{1+\frac{\lambda}{r}}{12m^{1+\lambda}} \\
&= m^{1-\lambda}\left\{\frac{rs}{\lambda} - \frac{1}{m^{\frac{\lambda}{s}}}\left[-\zeta\left(1-\frac{\lambda}{s}\right) - \frac{1+\frac{\lambda}{r}}{12m^{2-\frac{\lambda}{s}}}\right]\right\}.
\end{aligned}$$

In (2.6), let $m = 1$, by $0 < \lambda \leq \min\{r, s\}$, we obtain

$$\zeta\left(1-\frac{\lambda}{s}\right) = 1-\frac{s}{\lambda}-\frac{1}{2}+\frac{(1-\frac{\lambda}{s})\varepsilon_1}{12} < \frac{1}{2}-\frac{s}{\lambda}+\frac{1-\frac{\lambda}{s}}{12} = \frac{6\lambda-12s-\lambda(1-\frac{\lambda}{s})}{12\lambda}$$
$$< \frac{6\lambda-12s-(\lambda-s)}{12\lambda} = \frac{5\lambda-11s}{12\lambda} = -\frac{11s-5\lambda}{12\lambda} < 0.$$

Therefore, for $m \geq 1$, $m \in \mathrm{N}$, $0 < \lambda \leq \min\{r, s\}$, we obtain

$$-\zeta\left(1-\frac{\lambda}{s}\right)-\frac{1+\frac{\lambda}{r}}{12m^{2-\frac{\lambda}{s}}} > \frac{11s-5\lambda}{12\lambda}-\frac{1+\frac{\lambda}{r}}{12} = \frac{11s-5\lambda-\lambda(1+\frac{\lambda}{r})}{12\lambda} \geq \frac{11s-5\lambda-2\lambda}{12\lambda}$$
$$= \frac{4s+7(s-\lambda)}{12\lambda} \geq \frac{4s}{12\lambda} = \frac{s}{3\lambda}.$$

Applying the above inequality, we obtain (2.4). Similarly, we can prove (2.5). The lemma is proved. □

3 Main results

Theorem 3.1 *Assume that* $p, q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $r > 1$, $\frac{1}{r}+\frac{1}{s} = 1$, $0 < \lambda \leq \min\{r, s\}$, $a_n \geq 0$, $b_n \geq 0$, *such that* $0 < \sum_{n=1}^{\infty} n^{p(1-\frac{\lambda}{r})-1} a_n^p < \infty$ *and* $0 < \sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{s})-1} b_n^q < \infty$, *then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} < \left\{\sum_{n=1}^{\infty}\left[k_\lambda-\frac{s}{3\lambda n^{\frac{\lambda}{s}}}\right] n^{p(1-\frac{\lambda}{r})-1} a_n^p\right\}^{\frac{1}{p}} \times \left\{\sum_{n=1}^{\infty}\left[k_\lambda-\frac{r}{3\lambda n^{\frac{\lambda}{r}}}\right] n^{q(1-\frac{\lambda}{s})-1} b_n^q\right\}^{\frac{1}{q}}, \tag{3.1}$$

$$\sum_{n=1}^{\infty}\frac{n^{\frac{p\lambda}{s}-1}}{[k_\lambda-\frac{r}{3\lambda n^{\frac{\lambda}{r}}}]^{p-1}}\left[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda, n^\lambda\}}\right]^p < \sum_{n=1}^{\infty}\left[k_\lambda-\frac{s}{3\lambda n^{\frac{\lambda}{s}}}\right] n^{p(1-\frac{\lambda}{r})-1} a_n^p, \tag{3.2}$$

where $k_\lambda = \frac{rs}{\lambda} > 0$. *Inequality* (3.1) *is equivalent to* (3.2). *In particular, we have the following equivalent inequalities*:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} < k_\lambda\left\{\sum_{n=1}^{\infty}\left[1-\frac{s}{k_\lambda 3\lambda n^{\frac{\lambda}{s}}}\right] n^{p(1-\frac{\lambda}{r})-1} a_n^p\right\}^{\frac{1}{p}} \times \left\{\sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{s})-1} b_n^q\right\}^{\frac{1}{q}}, \tag{3.3}$$

$$\sum_{n=1}^{\infty} n^{\frac{p\lambda}{s}-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda, n^\lambda\}}\right]^p < k_\lambda^p \sum_{n=1}^{\infty}\left[1-\frac{s}{3k_\lambda\lambda n^{\frac{\lambda}{s}}}\right] n^{p(1-\frac{\lambda}{r})-1} a_n^p. \tag{3.4}$$

Proof From Hölder inequality (see [6]), we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}}$$
$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}}\frac{n^{(\lambda/s-1)/p}}{m^{(\lambda/r-1)/q}}\frac{m^{(\lambda/r-1)/q}}{n^{(\lambda/s-1)/p}}$$

$$\leq \left\{ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m^p m^{p(1-\lambda/r)+\lambda-2}}{\max\{m^\lambda, n^\lambda\}} \left(\frac{m}{n}\right)^{1-\frac{\lambda}{s}} \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{b_n^q n^{q(1-\lambda/s)+\lambda-2}}{\max\{m^\lambda, n^\lambda\}} \left(\frac{n}{m}\right)^{1-\frac{\lambda}{r}} \right\}^{\frac{1}{q}}$$

$$= \left\{ \sum_{m=1}^{\infty} \omega(m,\lambda,s) m^{p(1-\lambda/r)+\lambda-2} a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \omega(n,\lambda,r) n^{q(1-\lambda/s)+\lambda-2} b_n^q \right\}^{\frac{1}{q}}.$$

Hence, by (2.4), (2.5), inequality (3.1) is true.

Setting b_n as

$$b_n = \frac{n^{p\lambda/s-1}}{[k_\lambda - \frac{r}{3\lambda n^{\frac{\lambda}{r}}}]^{p-1}} \left[\sum_{m=1}^{\infty} \frac{a_m}{\max\{m^\lambda, n^\lambda\}} \right]^{p-1},$$

by using (3.1), we have

$$\begin{aligned} &\sum_{n=1}^{\infty} \left[k_\lambda - \frac{r}{3\lambda n^{\frac{\lambda}{r}}} \right] n^{q(1-\frac{\lambda}{s})-1} b_n^q \\ &\quad = \sum_{n=1}^{\infty} \frac{n^{p\lambda/s-1}}{[k_\lambda - \frac{r}{3\lambda n^{\frac{\lambda}{r}}}]^{p-1}} \left[\sum_{m=1}^{\infty} \frac{a_m}{\max\{m^\lambda, n^\lambda\}} \right]^p \\ &\quad = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} \leq \left\{ \sum_{n=1}^{\infty} \left[k_\lambda - \frac{s}{3\lambda n^{\frac{\lambda}{s}}} \right] n^{p(1-\frac{\lambda}{r})-1} a_n^p \right\}^{\frac{1}{p}} \\ &\qquad \times \left\{ \sum_{n=1}^{\infty} \left[k_\lambda - \frac{r}{3\lambda n^{\frac{\lambda}{r}}} \right] n^{q(1-\frac{\lambda}{s})-1} b_n^q \right\}^{\frac{1}{q}}. \end{aligned} \tag{3.5}$$

Hence, we obtain

$$\begin{aligned} 0 &< \sum_{n=1}^{\infty} \frac{n^{\frac{p\lambda}{s}-1}}{[k_\lambda - \frac{r}{3\lambda n^{\frac{\lambda}{r}}}]^{p-1}} \left[\sum_{m=1}^{\infty} \frac{a_m}{\max\{m^\lambda, n^\lambda\}} \right]^p \\ &< \sum_{n=1}^{\infty} \left[k_\lambda - \frac{s}{3\lambda n^{\frac{\lambda}{s}}} \right] n^{p(1-\frac{\lambda}{r})-1} a_n^p < \infty. \end{aligned} \tag{3.6}$$

By (3.1), both (3.5) and (3.6) take the form of strict inequality, and we have (3.2).

On the other hand, suppose that (3.2) is valid, from Hölder inequality, we find

$$\begin{aligned} &\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} \\ &\quad = \sum_{n=1}^{\infty} \frac{n^{[q(\lambda/s-1)+1]/q}}{[k_\lambda - \frac{r}{3\lambda n^{\frac{\lambda}{r}}}]^{\frac{1}{q}}} \left[\sum_{m=1}^{\infty} \frac{a_m}{\max\{m^\lambda, n^\lambda\}} \right] \left[k_\lambda - \frac{r}{3\lambda n^{\frac{\lambda}{r}}} \right]^{\frac{1}{q}} n^{[q(1-\lambda/s)-1]/q} b_n \\ &\quad \leq \left\{ \sum_{n=1}^{\infty} \frac{n^{p\lambda/s-1}}{[k_\lambda - \frac{r}{3\lambda n^{\frac{\lambda}{r}}}]^{p-1}} \left[\sum_{m=1}^{\infty} \frac{a_m}{\max\{m^\lambda, n^\lambda\}} \right]^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \left[k_\lambda - \frac{r}{3\lambda n^{\frac{\lambda}{r}}} \right] n^{q(1-\lambda/s)-1} b_n^q \right\}^{\frac{1}{q}}. \end{aligned}$$

Then, by using (3.2), we have (3.1). Hence, (3.2) and (3.1) are equivalent. The proof of Theorem 3.1 is completed. □

Since $0 < \lambda \leq \min\{r, s\}$, by Theorem 3.1, we have the following.

Corollary 3.2 *Assume that* $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $0 < \lambda \leq \min\{r, s\}$, $a_n \geq 0$, $b_n \geq 0$, *such that* $0 < \sum_{n=1}^{\infty} n^{p(1-\frac{\lambda}{r})-1} a_n^p < \infty$ *and* $0 < \sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{s})-1} b_n^q < \infty$, *then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} < \left\{\sum_{n=1}^{\infty}\left[k_\lambda - \frac{1}{3n^{\frac{\lambda}{s}}}\right] n^{p(1-\frac{\lambda}{r})-1} a_n^p\right\}^{\frac{1}{p}} \times \left\{\sum_{n=1}^{\infty}\left[k_\lambda - \frac{1}{3n^{\frac{\lambda}{r}}}\right] n^{q(1-\frac{\lambda}{s})-1} b_n^q\right\}^{\frac{1}{q}}, \tag{3.7}$$

$$\sum_{n=1}^{\infty} \frac{n^{\frac{p\lambda}{s}-1}}{[k_\lambda - \frac{1}{3n^{\frac{\lambda}{r}}}]^{p-1}} \left[\sum_{m=1}^{\infty} \frac{a_m}{\max\{m^\lambda, n^\lambda\}}\right]^p < \sum_{n=1}^{\infty}\left[k_\lambda - \frac{1}{3n^{\frac{\lambda}{s}}}\right] n^{p(1-\frac{\lambda}{r})-1} a_n^p, \tag{3.8}$$

where $k_\lambda = \frac{rs}{\lambda} > 0$. *Inequality* (3.7) *is equivalent to* (3.8).

For $r = s = 2$, by using (3.1) and (3.2), we have the following.

Corollary 3.3 *Assume that* $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \lambda \leq 2$, $a_n \geq 0$, $b_n \geq 0$, *such that* $0 < \sum_{n=1}^{\infty} n^{p(1-\frac{\lambda}{2})-1} a_n^p < \infty$ *and* $0 < \sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{2})-1} b_n^q < \infty$, *then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} < \left\{\sum_{n=1}^{\infty}\left[k_\lambda - \frac{2}{3\lambda n^{\frac{\lambda}{2}}}\right] n^{p(1-\frac{\lambda}{2})-1} a_n^p\right\}^{\frac{1}{p}} \times \left\{\sum_{n=1}^{\infty}\left[k_\lambda - \frac{2}{3\lambda n^{\frac{\lambda}{2}}}\right] n^{q(1-\frac{\lambda}{2})-1} b_n^q\right\}^{\frac{1}{q}}, \tag{3.9}$$

$$\sum_{n=1}^{\infty} \frac{n^{\frac{p\lambda}{2}-1}}{[k_\lambda - \frac{2}{3\lambda n^{\frac{\lambda}{2}}}]^{p-1}} \left[\sum_{m=1}^{\infty} \frac{a_m}{\max\{m^\lambda, n^\lambda\}}\right]^p < \sum_{n=1}^{\infty}\left[k_\lambda - \frac{2}{3\lambda n^{\frac{\lambda}{2}}}\right] n^{p(1-\frac{\lambda}{2})-1} a_n^p, \tag{3.10}$$

where $k_\lambda = \frac{4}{\lambda} > 0$. *Inequality* (3.9) *is equivalent to* (3.10). *In particular, we have the equivalent inequalities as follows.*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} < k_\lambda \left\{\sum_{n=1}^{\infty}\left[1 - \frac{2}{k_\lambda 3\lambda n^{\frac{\lambda}{2}}}\right] n^{p(1-\frac{\lambda}{2})-1} a_n^p\right\}^{\frac{1}{p}} \times \left\{\sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{2})-1} b_n^q\right\}^{\frac{1}{q}}, \tag{3.11}$$

$$\sum_{n=1}^{\infty} n^{\frac{p\lambda}{2}-1} \left[\sum_{m=1}^{\infty} \frac{a_m}{\max\{m^\lambda, n^\lambda\}}\right]^p < k_\lambda^p \sum_{n=1}^{\infty}\left[1 - \frac{2}{3k_\lambda \lambda n^{\frac{\lambda}{2}}}\right] n^{p(1-\frac{\lambda}{2})-1} a_n^p. \tag{3.12}$$

For $r = q$, $s = p$, by using (3.1) and (3.2), we have the following.

Corollary 3.4 *Assume that* $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, $0<\lambda\leq\min\{p,q\}$, $a_n\geq 0$, $b_n\geq 0$, *such that* $0<\sum_{n=1}^{\infty}n^{(p-1)(1-\lambda)}a_n^p<\infty$ *and* $0<\sum_{n=1}^{\infty}n^{(q-1)(1-\lambda)}b_n^q<\infty$, *then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda,n^\lambda\}}<\left\{\sum_{n=1}^{\infty}\left[k_\lambda-\frac{p}{3\lambda n^{\frac{\lambda}{p}}}\right]n^{(p-1)(1-\lambda)}a_n^p\right\}^{\frac{1}{p}}\times\left\{\sum_{n=1}^{\infty}\left[k_\lambda-\frac{q}{3\lambda n^{\frac{\lambda}{q}}}\right]n^{(q-1)(1-\lambda)}b_n^q\right\}^{\frac{1}{q}},\tag{3.13}$$

$$\sum_{n=1}^{\infty}\frac{n^{\lambda-1}}{[k_\lambda-\frac{q}{3\lambda n^{\frac{\lambda}{q}}}]^{p-1}}\left[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda,n^\lambda\}}\right]^p<\sum_{n=1}^{\infty}\left[k_\lambda-\frac{p}{3\lambda n^{\frac{\lambda}{p}}}\right]n^{(p-1)(1-\lambda)}a_n^p,\tag{3.14}$$

where $k_\lambda=\frac{pq}{\lambda}>0$. *Inequality* (3.13) *is equivalent to* (3.14). *In particular, we have the equivalent inequalities as follows.*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda,n^\lambda\}}<k_\lambda\left\{\sum_{n=1}^{\infty}\left[1-\frac{p}{k_\lambda 3\lambda n^{\frac{\lambda}{p}}}\right]n^{(p-1)(1-\lambda)}a_n^p\right\}^{\frac{1}{p}}\times\left\{\sum_{n=1}^{\infty}n^{(q-1)(1-\lambda)}b_n^q\right\}^{\frac{1}{q}},\tag{3.15}$$

$$\sum_{n=1}^{\infty}n^{\lambda-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda,n^\lambda\}}\right]^p<k_\lambda^p\sum_{n=1}^{\infty}\left[1-\frac{p}{3k_\lambda\lambda n^{\frac{\lambda}{p}}}\right]n^{(p-1)(1-\lambda)}a_n^p.\tag{3.16}$$

For $r=p$, $s=q$, by using (3.1) and (3.2), we have the following.

Corollary 3.5 *Assume that* $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, $0<\lambda\leq\min\{p,q\}$, $a_n\geq 0$, $b_n\geq 0$, *such that* $0<\sum_{n=1}^{\infty}n^{p-\lambda-1}a_n^p<\infty$ *and* $0<\sum_{n=1}^{\infty}n^{q-\lambda-1}b_n^q<\infty$, *then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda,n^\lambda\}}<\left\{\sum_{n=1}^{\infty}\left[k_\lambda-\frac{q}{3\lambda n^{\frac{\lambda}{q}}}\right]n^{p-\lambda-1}a_n^p\right\}^{\frac{1}{p}}\times\left\{\sum_{n=1}^{\infty}\left[k_\lambda-\frac{p}{3\lambda n^{\frac{\lambda}{p}}}\right]n^{q-\lambda-1}b_n^q\right\}^{\frac{1}{q}},\tag{3.17}$$

$$\sum_{n=1}^{\infty}\frac{n^{(p-1)\lambda-1}}{[k_\lambda-\frac{p}{3\lambda n^{\frac{\lambda}{p}}}]^{p-1}}\left[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda,n^\lambda\}}\right]^p<\sum_{n=1}^{\infty}\left[k_\lambda-\frac{q}{3\lambda n^{\frac{\lambda}{q}}}\right]n^{p-\lambda-1}a_n^p,\tag{3.18}$$

where $k_\lambda=\frac{pq}{\lambda}>0$. *Inequality* (3.17) *is equivalent to* (3.18). *In particular, we have the equivalent inequalities as follows.*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda,n^\lambda\}}<k_\lambda\left\{\sum_{n=1}^{\infty}\left[1-\frac{q}{k_\lambda 3\lambda n^{\frac{\lambda}{q}}}\right]n^{p-\lambda-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{q-\lambda-1}b_n^q\right\}^{\frac{1}{q}},\tag{3.19}$$

$$\sum_{n=1}^{\infty}n^{(p-1)\lambda-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda,n^\lambda\}}\right]^p<k_\lambda^p\sum_{n=1}^{\infty}\left[1-\frac{q}{3k_\lambda\lambda n^{\frac{\lambda}{q}}}\right]n^{p-\lambda-1}a_n^p.\tag{3.20}$$

Set $\lambda = 1$, combining (3.1) and (3.2), we have the following.

Corollary 3.6 *Assume that* $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $a_n \geq 0$, $b_n \geq 0$, *such that* $0 < \sum_{n=1}^{\infty} n^{\frac{p}{s}-1} a_n^p < \infty$ *and* $0 < \sum_{n=1}^{\infty} n^{\frac{q}{r}-1} b_n^q < \infty$, *then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m,n\}} < \left\{ \sum_{n=1}^{\infty} \left[rs - \frac{s}{3n^{\frac{1}{s}}} \right] n^{\frac{p}{s}-1} a_n^p \right\}^{\frac{1}{p}} \times \left\{ \sum_{n=1}^{\infty} \left[rs - \frac{r}{3n^{\frac{1}{r}}} \right] n^{\frac{q}{r}-1} b_n^q \right\}^{\frac{1}{q}}, \tag{3.21}$$

$$\sum_{n=1}^{\infty} \frac{n^{\frac{p}{s}-1}}{[rs - \frac{r}{3n^{\frac{1}{r}}}]^{p-1}} \left[\sum_{m=1}^{\infty} \frac{a_m}{\max\{m,n\}} \right]^p < \sum_{n=1}^{\infty} \left[rs - \frac{s}{3n^{\frac{1}{s}}} \right] n^{\frac{p}{s}-1} a_n^p. \tag{3.22}$$

In particular, we have the equivalent inequalities as follows.

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m,n\}} < rs \left\{ \sum_{n=1}^{\infty} \left[1 - \frac{1}{3rn^{\frac{1}{s}}} \right] n^{\frac{p}{s}-1} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{\frac{q}{r}-1} b_n^q \right\}^{\frac{1}{q}}, \tag{3.23}$$

$$\sum_{n=1}^{\infty} n^{\frac{p}{s}-1} \left[\sum_{m=1}^{\infty} \frac{a_m}{\max\{m,n\}} \right]^p < (rs)^p \sum_{n=1}^{\infty} \left[1 - \frac{1}{3rn^{\frac{1}{s}}} \right] n^{\frac{p}{s}-1} a_n^p. \tag{3.24}$$

Taking $p = q = r = s = 2$, in (3.23) and (3.24), we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m,n\}} < 4 \left\{ \sum_{n=1}^{\infty} \left[1 - \frac{1}{6\sqrt{n}} \right] a_n^2 \right\}^{\frac{1}{2}} \left\{ \sum_{n=1}^{\infty} \left[1 - \frac{1}{6\sqrt{n}} \right] b_n^2 \right\}^{\frac{1}{2}}, \tag{3.25}$$

$$\sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \frac{a_m}{\max\{m,n\}} \right]^2 < 16 \sum_{n=1}^{\infty} \left[1 - \frac{1}{6\sqrt{n}} \right] a_n^2. \tag{3.26}$$

Remark 3.1 *For* $r = \frac{\lambda p}{\lambda + p - 2}$ *and* $s = \frac{\lambda q}{\lambda + q - 2}$ *in Theorem* 3.1, *we get the results of* [4].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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