Du *Journal of Inequalities and Applications* 2013, 2013:506
http://www.journalofinequalitiesandapplications.com/content/2013/1/506

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

A note on approximate fixed point property and Du-Karapinar-Shahzad's intersection theorems

Wei-Shih Du*

*Correspondence:
wsdu@nknucc.nknu.edu.tw
Department of Mathematics,
National Kaohsiung Normal
University, Kaohsiung, 824, Taiwan

Abstract
In this note, we give new short proofs of Du-Karapinar-Shahzad's intersection theorems for multivalued non-self-maps in complete metric spaces.
MSC: 47H10; 54H25

Keywords: τ-function; $\mathcal{MT}$-function ($\mathcal{R}$-function); coincidence point; fixed point; approximate fixed point property

1 Introduction and preliminaries

Let us begin with some basic definitions and notations that will be needed in this paper. Let (X,d) be a metric space. Denote by $\mathcal{N}(X)$ the family of all nonempty subsets of X and by $\mathcal{CB}(X)$ the family of all nonempty closed and bounded subsets of X. For each $x \in X$ and $A \subseteq X$, let $d(x,A) = \inf_{y\in A} d(x,y)$. A function $\mathcal{H} : \mathcal{CB}(X) \times \mathcal{CB}(X) \to [0,\infty)$ defined by

$$\mathcal{H}(A,B) = \max\Bigl\{\sup_{x\in B} d(x,A), \sup_{x\in A} d(x,B)\Bigr\}$$

is said to be the Hausdorff metric on $\mathcal{CB}(X)$ induced by the metric d on X. The symbols $\mathbb{N}$ and $\mathbb{R}$ are used to denote the sets of positive integers and real numbers, respectively.

Let K be a nonempty subset of X, $g : K \to X$ be a single-valued non-self-map and $T : K \to \mathcal{N}(X)$ be a multivalued non-self-map. A point v in X is a *coincidence point* (see, for instance, [1–6]) of g and T if $gv \in Tx$. If $g = \mathrm{id}$ is the identity map, then $v = gv \in Tv$ and call v a *fixed point* of T. The set of fixed points of T and the set of coincidence points of g and T are denoted by $\mathcal{F}_K(T)$ and $\mathcal{COP}_K(g,T)$, respectively. In particular, if $K \equiv X$, we use $\mathcal{F}(T)$ and $\mathcal{COP}(g,T)$ instead of $\mathcal{F}_K(T)$ and $\mathcal{COP}_K(g,T)$, respectively. The map T is said to have *approximate fixed point property* [1–5] on K provided $\inf_{x\in K} d(x,Tx) = 0$. It is obvious that $\mathcal{F}_K(T) \neq \emptyset$ implies that T has approximate fixed point property.

A function $\varphi : [0,\infty) \to [0,1)$ is said to be an $\mathcal{MT}$-*function* (or $\mathcal{R}$-*function*) [3–11] if $\limsup_{s\to t^+} \varphi(s) < 1$ for all $t \in [0,\infty)$. Clearly, if $\varphi : [0,\infty) \to [0,1)$ is a nondecreasing function or a nonincreasing function, then φ is an $\mathcal{MT}$-function. So, the set of $\mathcal{MT}$-functions is a rich class and has the questions many of which are worth studying.

The study of fixed points for single-valued non-self-maps or multivalued non-self-maps satisfying certain contractive conditions is an interesting and important direction of research in metric fixed point theory. A great deal of such research has been investigated by

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several authors, see, *e.g.*, [11–19] and the references therein. Very recently, Du, Karapinar and Shahzad [11] established the following intersection existence theorem of coincidence points and fixed points of multivalued non-self-maps of Kannan type and Chatterjea type.

Theorem 1.1 [11, Theorem 8] *Let (X,d) be a complete metric space, K be a nonempty closed subset of X, $T : K \to \mathcal{CB}(X)$ be a multivalued map and $g : K \to X$ be a continuous map. Suppose that*

(D1) *$Tx \cap K \neq \emptyset$ for all $x \in K$,*
(D2) *$Tx \cap K$ is g-invariant (i.e., $g(Tx \cap K) \subseteq Tx \cap K$) for each $x \in K$,*
(D3) *there exist a function $h : K \to [0,\infty)$ and $\gamma \in [0,\frac{1}{2})$ such that*

$$\mathcal{H}(Tx, Ty \cap K) \leq \gamma\bigl[d(x, Tx \cap K) + d(y, Tx \cap K) + d(y, Ty \cap K)\bigr] + h(y)d(gy, Tx \cap K) \quad \text{for all } x, y \in K. \tag{1.1}$$

Then $\mathcal{COP}_K(g,T) \cap \mathcal{F}_K(T) \neq \emptyset$.

In [11], they also gave some coincidence and fixed point theorems for multivalued non-self-maps of Mizoguchi-Takahashi type, Berinde-Berinde type and Du type.

Theorem 1.2 [11, Theorem 19] *Let (X,d) be a complete metric space, K be a nonempty closed subset of X, $T : K \to \mathcal{CB}(X)$ be a multivalued map and $g : K \to X$ be a continuous map. Suppose that conditions* (D1) *and* (D2) *as in Theorem* 1.1 *hold. If there exist an $\mathcal{MT}$-function $\varphi : [0,\infty) \to [0,1)$ and a function $h : K \to [0,\infty)$ such that*

$$\mathcal{H}(Tx, Ty \cap K) \leq \varphi\bigl(d(x,y)\bigr)d(x,y) + h(y)d(gy, Tx \cap K) \quad \text{for all } x, y \in K, \tag{1.2}$$

then $\mathcal{COP}_K(g,T) \cap \mathcal{F}_K(T) \neq \emptyset$.

In this work, we give new short proofs of Du-Karapinar-Shahzad's intersection theorems of $\mathcal{COP}_K(g,T)$ and $\mathcal{F}_K(T)$ for multivalued non-self-maps (*i.e.*, Theorems 1.1 and 1.2) by applying an existence theorem for approximate fixed point property.

2 Some auxiliary key results

Let (X,d) be a metric space. Recall that a function $p : X \times X \to [0,\infty)$ is said to be a *τ-function* [3–5, 7, 8, 20–22], first introduced and studied by Lin and Du, if the following conditions hold:

(τ1) $p(x,z) \leq p(x,y) + p(y,z)$ for all $x,y,z \in X$;
(τ2) if $x \in X$ and $\{y_n\}$ in X with $\lim_{n\to\infty} y_n = y$ such that $p(x,y_n) \leq M$ for some $M = M(x) > 0$, then $p(x,y) \leq M$;
(τ3) for any sequence $\{x_n\}$ in X with $\lim_{n\to\infty} \sup\{p(x_n,x_m) : m > n\} = 0$, if there exists a sequence $\{y_n\}$ in X such that $\lim_{n\to\infty} p(x_n,y_n) = 0$, then $\lim_{n\to\infty} d(x_n,y_n) = 0$;
(τ4) for $x,y,z \in X$, $p(x,y) = 0$ and $p(x,z) = 0$ imply $y = z$.

Note that with the additional condition

(τ5) $p(x,x) = 0$ for all $x \in X$,

a τ-function becomes a τ^0*-function* [3–5, 7, 8] introduced by Du.

Clearly, any metric d is a τ^0-function. Observe further that if p is a τ^0-function, then, from (τ4) and (τ5), $p(x,y) = 0$ if and only if $x = y$.

Example A [7] Let $X = \mathbb{R}$ with the metric $d(x,y) = |x-y|$ and $0 < a < b$. Define the function $p : X \times X \to [0,\infty)$ by

$$p(x,y) = \max\{a(y-x), b(x-y)\}.$$

Then p is nonsymmetric and hence p is not a metric. It is easy to see that p is a τ^0-function.

Lemma 2.1 [22, Lemma 2.1] *Let (X,d) be a metric space and $p : X \times X \to [0,\infty)$ be a function. Assume that p satisfies the condition (τ3). If a sequence $\{x_n\}$ in X with $\lim_{n\to\infty} \sup\{p(x_n,x_m) : m > n\} = 0$, then $\{x_n\}$ is a Cauchy sequence in X.*

Let (X,d) be a metric space and p be a τ-function. A multivalued map $T : X \to \mathcal{N}(X)$ is said to have *p-approximate fixed point property* on X provided

$$\inf_{x\in X} p(x, Tx) = 0.$$

The following characterizations of $\mathcal{MT}$-functions proved first by Du [6] are quite useful for proving our main results.

Theorem 2.1 [6, Theorem 2.1] *Let $\varphi : [0,\infty) \to [0,1)$ be a function. Then the following statements are equivalent.*

(a) *φ is an $\mathcal{MT}$-function.*
(b) *For each $t \in [0,\infty)$, there exist $r_t^{(1)} \in [0,1)$ and $\varepsilon_t^{(1)} > 0$ such that $\varphi(s) \le r_t^{(1)}$ for all $s \in (t, t+\varepsilon_t^{(1)})$.*
(c) *For each $t \in [0,\infty)$, there exist $r_t^{(2)} \in [0,1)$ and $\varepsilon_t^{(2)} > 0$ such that $\varphi(s) \le r_t^{(2)}$ for all $s \in [t, t+\varepsilon_t^{(2)}]$.*
(d) *For each $t \in [0,\infty)$, there exist $r_t^{(3)} \in [0,1)$ and $\varepsilon_t^{(3)} > 0$ such that $\varphi(s) \le r_t^{(3)}$ for all $s \in (t, t+\varepsilon_t^{(3)}]$.*
(e) *For each $t \in [0,\infty)$, there exist $r_t^{(4)} \in [0,1)$ and $\varepsilon_t^{(4)} > 0$ such that $\varphi(s) \le r_t^{(4)}$ for all $s \in [t, t+\varepsilon_t^{(4)})$.*
(f) *For any nonincreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,\infty)$, we have $0 \le \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$.*
(g) *φ is a function of contractive factor; that is, for any strictly decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,\infty)$, we have $0 \le \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$.*

The following result was essentially proved by Du *et al.* in [4], but we give the proof for the sake of completeness and the readers convenience.

Lemma 2.2 [4, Lemma 3.1] *Let (X,d) be a metric space, p be a τ^0-function and $T : X \to \mathcal{N}(X)$ be a multivalued map. Then the following statements are equivalent.*

(Q1) *There exist a function $\xi : [0,\infty) \to [0,\infty)$ and an $\mathcal{MT}$-function $\varphi : [0,\infty) \to [0,1)$ such that for each $x \in X$, if $y \in Tx$ with $y \ne x$, then there exists $z \in Ty$ such that*

$$p(y,z) \le \varphi\bigl(\xi\bigl(p(x,y)\bigr)\bigr)p(x,y).$$

(Q2) *There exist a function* $\tau : [0,\infty) \to [0,\infty)$ *and an* $\mathcal{MT}$*-function* $\kappa : [0,\infty) \to [0,1)$ *such that for each* $x \in X$,

$$p(y, Ty) \le \kappa\big(\tau\big(p(x,y)\big)\big)p(x,y) \quad \textit{for all } y \in Tx.$$

Proof If (Q1) holds, then it is easy to verify that (Q2) also holds with $\kappa \equiv \varphi$ and $\tau \equiv \xi$. So it suffices to prove that '(Q2) $\Rightarrow$ (Q1)'. Suppose that (Q2) holds. Define $\varphi : [0,\infty) \to [0,1)$ by $\varphi(t) = \frac{1+\kappa(t)}{2}$. Then φ is also an $\mathcal{MT}$-function. Indeed, it is obvious that $0 \le \kappa(t) < \varphi(t) < 1$ for all $t \in [0,\infty)$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a strictly decreasing sequence in $[0,\infty)$. Since κ is an $\mathcal{MT}$-function, by (g) of Theorem 2.1, we get

$$0 \le \sup_{n\in\mathbb{N}} \kappa(x_n) < 1$$

and hence

$$0 < \sup_{n\in\mathbb{N}} \varphi(x_n) = \frac{1}{2}\Big[1 + \sup_{n\in\mathbb{N}} \kappa(x_n)\Big] < 1.$$

So, by Theorem 2.1 again, we prove that φ is an $\mathcal{MT}$-function.

For each $x \in X$, let $y \in Tx$ with $y \neq x$. Then $p(x,y) > 0$. By (Q2), we have

$$p(y, Ty) < \varphi\big(\tau\big(p(x,y)\big)\big)p(x,y).$$

Since $\varphi(t) > 0$ for all $t \in [0,\infty)$, there exists $z \in Ty$ such that

$$p(y,z) < \varphi\big(\tau\big(p(x,y)\big)\big)p(x,y),$$

which shows that (Q1) holds with $\xi \equiv \tau$. So, by above, we prove '(Q1) $\Longleftrightarrow$ (Q2)'. □

Now, we present an existence theorem for p-approximate fixed point property and approximate fixed point property, which is indeed a somewhat generalized form of [4, Theorem 3.3] and is one of the key technical devices in the new short proofs of Theorems 1.1 and 1.2.

Theorem 2.2 *Let* (X,d) *be a metric space,* p *be a* τ^0*-function and* $T : X \to \mathcal{N}(X)$ *be a multivalued map. Assume that one of* (L1) *and* (L2) *is satisfied, where*

(L1) *there exist a nondecreasing function* $\xi : [0,\infty) \to [0,\infty)$ *and an* $\mathcal{MT}$*-function* $\varphi : [0,\infty) \to [0,1)$ *such that for each* $x \in X$, *if* $y \in Tx$ *with* $y \neq x$, *then there exists* $z \in Ty$ *such that*

$$p(y,z) \le \varphi\big(\xi\big(p(x,y)\big)\big)p(x,y);$$

(L2) *there exist a nondecreasing function* $\tau : [0,\infty) \to [0,\infty)$ *and an* $\mathcal{MT}$*-function* $\kappa : [0,\infty) \to [0,1)$ *such that for each* $x \in X$,

$$p(y, Ty) \le \kappa\big(\tau\big(p(x,y)\big)\big)p(x,y) \quad \textit{for all } y \in Tx.$$

Then the following statements hold.

(a) *There exists a Cauchy sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* X *such that*
   (i) $x_{n+1} \in Tx_n$ *for all* $n \in \mathbb{N}$,
   (ii) $\inf_{n\in X} p(x_n, x_{n+1}) = \lim_{n\to\infty} p(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = 0$.
(b) $\inf_{x\in X} p(x, Tx) = \inf_{x\in X} d(x, Tx) = 0$; *that is*, T *has* p*-approximate fixed point property and approximate fixed point property on* X.

Proof By Lemma 2.2, it suffices to prove that the conclusions hold under assumption (L1). Let $u \in X$ be given. If $u \in Tu$, then

$$\inf_{x\in X} p(x, Tx) \le p(u, Tu) \le p(u, u) = 0,$$

and

$$\inf_{x\in X} d(x, Tx) \le d(u, u) = 0,$$

which implies that $\inf_{x\in X} p(x, Tx) = \inf_{x\in X} d(x, Tx) = 0$. Let $w_n = u$ for all $n \in \mathbb{N}$. Thus we have

$$w_{n+1} = u \in Tu = Tw_n \quad \text{for all } n \in \mathbb{N},$$

$$\lim_{n\to\infty} p(w_n, w_{n+1}) = \inf_{n\in\mathbb{N}} p(w_n, w_{n+1}) = p(u, u) = 0,$$

and

$$\lim_{n\to\infty} d(w_n, w_{n+1}) = \inf_{n\in\mathbb{N}} d(w_n, w_{n+1}) = d(u, u) = 0.$$

Clearly,

$$p(w_{n+1}, w_{n+2}) = 0 = \varphi\big(\xi\big(p(w_n, w_{n+1})\big)\big)p(w_n, w_{n+1}) \quad \text{for all } n \in \mathbb{N}.$$

So, conclusions (a) and (b) hold in this case $u \in Tu$, no matter what condition one begins with. Suppose that $u \notin Tu$. Put $x_1 = u$ and $x_2 \in Tx_1$. Then $x_2 \neq x_1$ and hence $p(x_1, x_2) > 0$. Assume that condition (L1) is satisfied. Then there exists $x_3 \in Tx_2$ such that

$$p(x_2, x_3) \le \varphi\big(\xi\big(p(x_1, x_2)\big)\big)p(x_1, x_2).$$

If $x_2 = x_3 \in Tx_2$, then, following a similar argument as above, the conclusions are also proved. If $x_3 \neq x_2$, then there exists $x_4 \in Tx_3$ such that

$$p(x_3, x_4) \le \varphi\big(\xi\big(p(x_2, x_3)\big)\big)p(x_2, x_3).$$

By induction, we can obtain a sequence $\{x_n\}$ in X satisfying $x_{n+1} \in Tx_n$ and

$$p(x_{n+1}, x_{n+2}) \le \varphi\big(\xi\big(p(x_n, x_{n+1})\big)\big)p(x_n, x_{n+1}) \quad \text{for all } n \in \mathbb{N}. \tag{2.1}$$

Since $\varphi(t) < 1$ for all $t \in [0,\infty)$, inequality (2.1) implies that the sequence $\{p(x_n, x_{n+1})\}_{n\in\mathbb{N}}$ is strictly decreasing in $[0,\infty)$. Hence

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} p(x_n, x_{n+1}) \geq 0 \quad \text{exists.} \tag{2.2}$$

Since ξ is nondecreasing, $\{\xi(p(x_n, x_{n+1}))\}_{n\in\mathbb{N}}$ is a nonincreasing sequence in $[0,\infty)$. Since φ is an $\mathcal{MT}$-function, by (g) of Theorem 2.1, we have

$$0 \leq \sup_{n\in\mathbb{N}} \varphi\bigl(\xi\bigl(p(x_n, x_{n+1})\bigr)\bigr) < 1.$$

Let $\lambda := \sup_{n\in\mathbb{N}} \varphi(\xi(p(x_n, x_{n+1})))$. So $\lambda \in [0,1)$ and we get from (2.1) that

$$p(x_{n+1}, x_{n+2}) \leq \lambda p(x_n, x_{n+1}) \leq \cdots \leq \lambda^n p(x_1, x_2) \quad \text{for each } n \in \mathbb{N}. \tag{2.3}$$

Since $\lambda \in [0,1)$, $\lim_{n\to\infty} \lambda^n = 0$ and hence the last inequality implies

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{2.4}$$

By (2.2) and (2.4), we obtain

$$\inf_{n\in\mathbb{N}} p(x_n, x_{n+1}) = \lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{2.5}$$

Now, we claim that $\{x_n\}$ is a Cauchy sequence in X. Let $\alpha_n = \frac{\lambda^{n-1}}{1-\lambda} p(x_1, x_2)$, $n \in \mathbb{N}$. For $m, n \in \mathbb{N}$ with $m > n$, by (2.3), we have

$$p(x_n, x_m) \leq \sum_{j=n}^{m-1} p(x_j, x_{j+1}) < \alpha_n.$$

Since $\lambda \in [0,1)$, $\lim_{n\to\infty} \alpha_n = 0$ and hence

$$\lim_{n\to\infty} \sup\bigl\{p(x_n, x_m) : m > n\bigr\} = 0.$$

Applying Lemma 2.1, we show that $\{x_n\}$ is a Cauchy sequence in X. Hence $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Since $\inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) \leq d(x_m, x_{m+1})$ for all $m \in \mathbb{N}$ and $\lim_{m\to\infty} d(x_m, x_{m+1}) = 0$, one also obtains

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = 0. \tag{2.6}$$

So conclusion (a) is proved. To see (b), since $x_{n+1} \in Tx_n$ for each $n \in \mathbb{N}$, we have

$$\inf_{x\in X} p(x, Tx) \leq p(x_n, Tx_n) \leq p(x_n, fx_{n+1}) \tag{2.7}$$

and

$$\inf_{x\in X} d(x, Tx) \leq d(x_n, Tx_n) \leq d(x_n, fx_{n+1}) \tag{2.8}$$

for all $n \in \mathbb{N}$. Combining (2.6), (2.7) and (2.8), we get

$$\inf_{x \in X} p(x, Tx) = \inf_{x \in X} d(x, Tx) = 0.$$

The proof is completed. □

The following existence theorem is obviously an immediate result from Theorem 2.2.

Theorem 2.3 *Let (X,d) be a metric space, p be a τ^0-function and $T : X \to \mathcal{N}(X)$ be a multivalued map. Assume that one of* (H1) *and* (H2) *is satisfied, where*

(H1) *there exists an $\mathcal{MT}$-function $\alpha : [0,\infty) \to [0,1)$ such that for each $x \in X$, if $y \in Tx$ with $y \neq x$, then there exists $z \in Ty$ such that*

$$p(y,z) \leq \alpha\big(p(x,y)\big)p(x,y);$$

(H2) *there exists an $\mathcal{MT}$-function $\beta : [0,\infty) \to [0,1)$ such that for each $x \in X$,*

$$p(y, Ty) \leq \beta\big(p(x,y)\big)p(x,y) \quad \text{for all } y \in Tx.$$

Then the following statements hold.

(a) *There exists a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that*
   (i) $x_{n+1} \in Tx_n$ *for all* $n \in \mathbb{N}$,
   (ii) $\inf_{n\in X} p(x_n, x_{n+1}) = \lim_{n\to\infty} p(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = 0$.

(b) $\inf_{x\in X} p(x, Tx) = \inf_{x\in X} d(x, Tx) = 0$; *that is, T has p-approximate fixed point property and approximate fixed point property on X.*

Lemma 2.3 *Let $\tau : [0,\infty) \to [0,\infty)$ be a nondecreasing function and $\kappa : [0,\infty) \to [0,1)$ be an $\mathcal{MT}$-function. Then $\kappa \circ \tau$ is an $\mathcal{MT}$-function.*

Proof Let $\{x_n\}_{n\in\mathbb{N}}$ be a strictly decreasing sequence in $[0,\infty)$. Since τ is a nondecreasing function, $\{\tau(x_n)\}_{n\in\mathbb{N}}$ is a nonincreasing sequence in $[0,\infty)$. Since κ is an $\mathcal{MT}$-function, by (f) of Theorem 2.1, we get

$$0 \leq \sup_{n\in\mathbb{N}} \kappa\big(\tau(x_n)\big) < 1,$$

or, equivalently,

$$0 \leq \sup_{n\in\mathbb{N}} (\kappa \circ \tau)(x_n) < 1.$$

So, by Theorem 2.1 again, we prove that $\kappa \circ \tau$ is an $\mathcal{MT}$-function. □

Applying Lemma 2.3, we conclude that Theorem 2.2 is also a special case of Theorem 2.3. Therefore we obtain the following important fact.

Theorem 2.4 *Theorem* 2.2 *and Theorem* 2.3 *are equivalent.*

3 Short proofs of Theorems 1.1 and 1.2

Let us see how we can utilize Theorem 2.3 to prove Theorem 1.1.

Short proof of Theorem 1.1 Since K is a nonempty closed subset of X and X is complete, (K,d) is also a complete metric space. Let $x \in K$. Put $k = \frac{\gamma}{1-\gamma}$ and $\lambda = \frac{1+k}{2}$. So, $0 \leq k < \lambda < 1$. Let $y \in Tx \cap K$ be arbitrary. So, $d(y, Tx \cap K) = 0$. By (D2), we have $d(gy, Tx \cap K) = 0$. Hence inequality (1.1) implies

$$\mathcal{H}(Tx, Ty \cap K) \leq \gamma \bigl[d(x, Tx \cap K) + \mathcal{H}(Tx, Ty \cap K)\bigr] \quad \text{for all } y \in Tx \cap K. \tag{3.1}$$

Inequality (3.1) shows that

$$d(y, Ty \cap K) \leq \mathcal{H}(Tx, Ty \cap K) \leq kd(x, Tx \cap K) < \lambda d(x,y) \quad \text{for all } y \in Tx \cap K. \tag{3.2}$$

Define $G : K \to \mathcal{CB}(K)$ by

$$Gx = Tx \cap K \quad \text{for all } x \in K,$$

and let $\mu : [0,\infty) \to [0,1)$ be defined by

$$\eta(t) = \lambda \quad \text{for all } t \in [0,\infty).$$

Then μ is an $\mathcal{MT}$-function. By (3.2), we obtain

$$d(y, Gy) \leq \mu\bigl(d(x,y)\bigr)d(x,y) \quad \text{for all } y \in Gx.$$

Applying Theorem 2.3 with $p \equiv d$, there exists a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in K such that

$$x_{n+1} \in Gx_n = Tx_n \cap K \quad \text{for all } n \in \mathbb{N} \tag{3.3}$$

and

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = 0. \tag{3.4}$$

By the completeness of K, there exists $v \in K$ such that $x_n \to v$ as $n \to \infty$. By (3.3) and (D2), we have

$$gx_{n+1} \in Tx_n \cap K \quad \text{for each } n \in \mathbb{N}. \tag{3.5}$$

Since g is continuous and $\lim_{n\to\infty} x_n = v$, we have

$$\lim_{n\to\infty} gx_n = gv. \tag{3.6}$$

Since the function $x \mapsto d(x, Tv)$ is continuous, by (1.1), (3.3), (3.4), (3.5) and (3.6), we get

$$\begin{aligned} d(v, Tv \cap K) &= \lim_{n\to\infty} d(x_{n+1}, Tv \cap K) \\ &\leq \lim_{n\to\infty} \mathcal{H}(Tx_n, Tv \cap K) \end{aligned}$$

$$\begin{aligned}
&\leq \lim_{n\to\infty}\bigl\{\gamma\bigl[d(x_n, Tx_n\cap K) + d(v, Tx_n\cap K) + d(v, Tv\cap K)\bigr]\\
&\quad + h(v)d(gv, Tx_n\cap K)\bigr\}\\
&\leq \lim_{n\to\infty}\bigl\{\gamma\bigl[d(x_n, x_{n+1}) + d(v, x_{n+1}) + d(v, Tv\cap K)\bigr] + h(v)d(gv, gx_{n+1})\bigr\}\\
&= \gamma d(v, Tv\cap K),
\end{aligned}$$

which implies $d(v, Tv\cap K) = 0$. By the closedness of Tv, we have $v \in Tv\cap K$. From (D2), $gv \in Tv\cap K \subseteq Tv$. Hence we verify $v \in \mathcal{COP}_K(g,T)\cap\mathcal{F}_K(T)$. The proof is complete. □

In order to finish off our work, let us prove Theorem 1.2 by applying Theorem 2.3.

Short proof of Theorem 1.2 Since K is a nonempty closed subset of X and X is complete, (K,d) is also a complete metric space. Note first that for each $x \in K$, by (D2), we have $d(gy, Tx\cap K) = 0$ for all $y \in Tx\cap K$. So, for each $x \in K$, by (1.2), we obtain

$$d(y, Ty\cap K) \leq \varphi\bigl(d(x,y)\bigr)d(x,y) \quad \text{for all } y \in Tx\cap K. \tag{3.7}$$

Define $G: K \to \mathcal{CB}(K)$ by

$$Gx = Tx\cap K \quad \text{for all } x \in K.$$

From (3.7), we obtain

$$d(y, Gy) \leq \varphi\bigl(d(x,y)\bigr)d(x,y) \quad \text{for all } y \in Gx.$$

By using Theorem 2.3, there exists a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in K such that

$$x_{n+1} \in Gx_n = Tx_n\cap K \quad \text{for all } n \in \mathbb{N} \tag{3.8}$$

and

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = 0. \tag{3.9}$$

By the completeness of K, there exists $v \in K$ such that $x_n \to v$ as $n \to \infty$. Thanks to (3.8) and (D2), we have

$$gx_{n+1} \in Tx_n\cap K \quad \text{for each } n \in \mathbb{N}. \tag{3.10}$$

Since g is continuous and $\lim_{n\to\infty} x_n = v$, we have

$$\lim_{n\to\infty} gx_n = gv. \tag{3.11}$$

Since the function $x \mapsto d(x, Tv)$ is continuous, by (1.2), (3.8), (3.10) and (3.11), we get

$$\begin{aligned}
d(v, Tv\cap K) &= \lim_{n\to\infty} d(x_{n+1}, Tv\cap K)\\
&\leq \lim_{n\to\infty} \mathcal{H}(Tx_n, Tv\cap K)
\end{aligned}$$

$$\leq \lim_{n\to\infty}\bigl\{\varphi\bigl(d(x_n,v)\bigr)d(x_n,v)+h(v)d(gv,Tx_n\cap K)\bigr\}$$
$$\leq \lim_{n\to\infty}\bigl\{\varphi\bigl(d(x_n,v)\bigr)d(x_n,v)+h(v)d(gv,gx_{n+1})\bigr\}=0,$$

which implies $d(v,Tv\cap K)=0$. By the closedness of Tv, we have $v\in Tv\cap K$. By (D2), $gv\in Tv\cap K\subseteq Tv$ and hence $v\in\mathcal{COP}_K(g,T)\cap\mathcal{F}_K(T)$. The proof is complete. □

Competing interests
The author declares that he has no competing interests.

Acknowledgements
In this research, the author was supported by grant No. NSC 102-2115-M-017-001 of the National Science Council of the Republic of China.

Received: 30 July 2013 Accepted: 24 September 2013 Published: 08 Nov 2013

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10.1186/1029-242X-2013-506
Cite this article as: Du: A note on approximate fixed point property and Du-Karapinar-Shahzad's intersection theorems. *Journal of Inequalities and Applications* 2013, 2013:506