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Global regularity for solutions of a class of quasilinear elliptic equations

Yueming Lu, Tingting Wang, Gejun Bao* and Guanfeng Li

*Correspondence: baogj@hit.edu.cn
Department of Mathematics, Harbin Institute of Technology, Harbin, 150001, P.R. China

Abstract

We derive interior and global Hölder estimates for solutions of a class of quasilinear elliptic equations. First, the interior Hölder continuity is obtained by the iteration of an oscillation estimate. Then, the Hölder continuity up to the boundary is established in domains with certain boundary constraints. Last, we prove the global Hölder continuity of solutions provided that their restrictions on boundary are Hölder continuous. The concluding section presents an application to illustrate our main results.

MSC: 35B65; 35J60

Keywords: oscillation estimate; global Hölder continuity; boundary condition

1 Introduction

In this paper we derive interior and global Hölder estimates for solutions of quasilinear elliptic equations of the form

$$-\operatorname{div} A(x,u,\nabla u) = f(x), \tag{1.1}$$

where Ω is a bounded open subset in $\mathbb{R}^n$, $n \geq 2$, $f(x) \in L^{\frac{q}{p-1}}(\Omega)$ for some $q > n$. Throughout the paper, the exponent p' is denoted as the Hölder conjugate of p, $1 < p \leq n$.

Suppose that the operator $A : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory mapping satisfying the following assumptions:

(a) for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$,

$$A(x,s,\xi)\cdot\xi \geq \alpha|\xi|^p, \qquad \bigl|A(x,s,\xi)\bigr| \leq b\bigl(|s|\bigr)\bigl(k(x) + |\xi|^{p-1}\bigr);$$

(b) for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$, $\xi_1, \xi_2 \in \mathbb{R}^n$, $\xi_1 \neq \xi_2$,

$$\bigl(A(x,s,\xi_1) - A(x,s,\xi_2)\bigr)\cdot(\xi_1 - \xi_2) > 0;$$

(c) for a.e. $x \in \Omega$, all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ and all $\lambda \in \mathbb{R}$,

$$A(x,s,\lambda\xi) = \lambda|\lambda|^{p-2}A(x,s,\xi),$$

where α is a positive real constant, $k(x)$ belongs to $L^{\frac{q}{p-1}}(\Omega)$ for some $q > n$ and $b : [0,+\infty) \to (0,+\infty)$ is a continuous function.

Definition 1.1 A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is called a weak solution of (1.1) if

$$\int_\Omega A(x,u,\nabla u)\nabla\phi\,dx = \int_\Omega f(x)\phi\,dx \tag{1.2}$$

for every $\phi \in W^{1,p}(\Omega)$ with compact support.

The Hölder continuity estimate for solutions of nonlinear elliptic equations has always been an important subject in the theory of differential equations and dynamical systems, see, *e.g.*, [1–4]. Numerous results on the Hölder regularity of elliptic equations with various conditions have been obtained, see [5–12] and references therein. In this paper, we derive interior and global Hölder estimates for solutions of a class of quasilinear elliptic equations. The key points are the choice of appropriate test functions, the method of iteration and the integral tests developed by Ladyzhenskaya and Ural'tseva, see [1]. Note that we restrict the exponent $1 < p \le n$ since for the case $p > n$, the Hölder estimate can be directly obtained by the Sobolev embedding theorem.

2 Preliminary results

The theorems of the following section require some preparatory results which we group together here.

Lemma 2.1 [9] *Let $f(\tau)$ be a non-negative bounded function defined for $0 \le R_0 \le \tau \le R_1$. Suppose that for $R_0 \le \tau < t \le R_1$ we have*

$$f(\tau) \le A(t-\tau)^{-\gamma} + B + \theta f(t),$$

where A, B, γ, θ are non-negative constants and $\theta < 1$. Then there exists a constant c depending only on γ and θ such that for every ρ, R, $R_0 \le \rho < R \le R_1$, we have

$$f(\rho) \le c\big[A(R-\rho)^{-\gamma} + B\big].$$

Now we present a very useful lemma which is fundamental in the proof of our theorems, it appears in [1] as follows.

Lemma 2.2 [1, Lemma 4.8, p.66] *Suppose that the function $u(x)$ is measurable and bounded in some ball B_{ρ_0} or in a portion of it $\Omega_{\rho_0} = B_{\rho_0} \cap \Omega$. Consider the balls B_ρ and $B_{b\rho}$, where b is a fixed constant greater than 1, which are concentric with B_{ρ_0}. Suppose that for arbitrary $\rho \le b^{-1}\rho_0$, at least one of the following inequalities regarding $u(x)$ is valid*:

$$\operatorname*{osc}_{\Omega_\rho} u \le c_1\rho^\varepsilon,$$

$$\operatorname*{osc}_{\Omega_\rho} u \le \vartheta \operatorname*{osc}_{\Omega_{b\rho}} u$$

for certain positive constants $c_1, \varepsilon \le 1$ and $\vartheta < 1$. Then, for $\rho \le \rho_0$,

$$\operatorname*{osc}_{\Omega_\rho} u \le c\rho_0^{-m}\rho^m,$$

where

$$m = \min\{-\log_b \vartheta, \varepsilon\}, \qquad c = b^m \max\{\omega_0, c_1\rho_0^{\varepsilon}\},$$

and

$$\omega_0 = \operatorname*{osc}_{\Omega_{\rho_0}} u.$$

3 Main results

3.1 Interior Hölder estimate

Let $x_0 \in \Omega$ and $t > 0$, we denote by B_t the ball of radius t centered at x_0. For $k > 0$, write

$$A_k = \{x \in \Omega : u(x) > k\}, \qquad A_{k,t} = A_k \cap B_t$$

and denote by $\mathcal{B}_p(\Omega, M, \gamma, \delta, 1/q)$ the class of functions $u(x)$ in $W^{1,p}(\Omega)$ with essential $\max_\Omega |u| \le M$ such that for $u(x)$ and $-u(x)$, the following inequalities are valid in an arbitrary sphere $B_\rho \subset \Omega$ for arbitrary $\sigma \in (0,1)$

$$\int_{A_{k,\rho-\sigma\rho}} |\nabla u|^p \, dx \le \gamma \left[\frac{1}{\sigma^p \rho^{p(1-\frac{n}{q})}} \max_{A_{k,\rho}} (u-k)^p + 1 \right] |A_{k,\rho}|^{1-\frac{p}{q}} \tag{3.1}$$

for $k \ge \max_{B_\rho} u(x) - \delta$, where the parameters of the class M, γ and δ are arbitrary positive numbers, $1 < p \le n$ and $q > n \ge 2$. Note that we do not exclude the case $q = \infty$.

Lemma 3.1 [1, Lemma 6.2, p.85] *There exists a positive number s such that for an arbitrary ball B_ρ belonging to Ω together with the ball $B_{4\rho}$ concentric with it and for an arbitrary function $u(x)$ in $\mathcal{B}_p(\Omega, M, \gamma, \delta, 1/q)$, at least one of the following two inequalities holds*:

$$\operatorname*{osc}_{B_\rho} u \le 2^s \rho^{1-\frac{n}{q}},$$

$$\operatorname*{osc}_{B_\rho} u \le \left(1 - \frac{1}{2^{s-1}}\right) \operatorname*{osc}_{B_{4\rho}} u.$$

To prove the interior Hölder continuity, firstly, by choosing an appropriate test function and making full use of fundamental inequalities, together with Lemma 2.1, we can obtain the following result.

Theorem 3.2 *Let $1 < p \le n$ and $f(x) \in L^{\frac{q}{p-1}}(\Omega)$ for some $q > n$. Suppose that u is a bounded solution of* (1.1), *then $u \in \mathcal{B}_p(\Omega, M, \gamma, \delta, 1/q)$.*

Proof Let $B_1 \Subset \Omega$ and $0 \le R_0 \le \tau < t \le R_1$ be arbitrarily fixed. Let η be a cutoff function such that

$$\eta \in C_0^\infty(B_t), \qquad 0 \le \eta \le 1, \qquad \eta|_{B_\tau} \equiv 1, \qquad |\nabla \eta| \le \frac{2}{t-\tau}.$$

Set $M > 0$ satisfies $|u| \le M$. For every $k > 0$, let $h > M + k$ and take

$$\phi = T_h\big(\eta(u-k)^+\big) = \max\big\{-h, \min\{h, \eta(u-k)^+\}\big\}$$

as a test function in (1.2) and obtain

$$\begin{aligned}
&\int_{A_{k,t}} f(x)T_h\big(\eta(u-k)\big)\,dx \\
&\quad= \int_{\Omega} f(x)T_h\big(\eta(u-k)^+\big)\,dx \\
&\quad= \int_{\Omega} A(x,u,\nabla u)\nabla T_h\big(\eta(u-k)^+\big)\,dx \\
&\quad= \int_{A_{k,t}} A(x,u,\nabla u)\nabla T_h\big(\eta(u-k)\big)\,dx \\
&\quad= \int_{A_{k,t}} A(x,u,\nabla u)\nabla\big(\eta(u-k)\big)\,dx \\
&\quad= \int_{A_{k,\tau}} A(x,u,\nabla u)\nabla u\,dx + \int_{A_{k,t}\setminus A_{k,\tau}} A(x,u,\nabla u)\big((u-k)\nabla\eta + \eta\nabla u\big)\,dx.
\end{aligned} \tag{3.2}$$

Then, by applying the structure conditions of mapping A, (3.2) yields

$$\begin{aligned}
&\int_{A_{k,t}} f(x)T_h\big(\eta(u-k)\big)\,dx \\
&\quad\geq \alpha\int_{A_{k,\tau}} |\nabla u|^p\,dx - b\int_{A_{k,t}\setminus A_{k,\tau}} \big(k(x) + |\nabla u|^{p-1}\big)\big|(u-k)\nabla\eta + \eta\nabla u\big|\,dx,
\end{aligned}$$

thus

$$\begin{aligned}
\alpha\int_{A_{k,\tau}} |\nabla u|^p\,dx \leq{}& \int_{A_{k,t}} \big|f(x)\big|\big|\eta(u-k)\big|\,dx \\
&+ b\int_{A_{k,t}\setminus A_{k,\tau}} \big(k(x) + |\nabla u|^{p-1}\big)\big|(u-k)\nabla\eta + \eta\nabla u\big|\,dx,
\end{aligned} \tag{3.3}$$

where $b = \max_{[0,M]} b(|s|)$. Hence it follows from (3.3) and Young's inequality that

$$\begin{aligned}
\alpha\int_{A_{k,\tau}} |\nabla u|^p\,dx \leq{}& \varepsilon\int_{A_{k,t}} (u-k)^p\,dx + c(p,\varepsilon)\int_{A_{k,t}} |f|^{p'}\,dx \\
&+ b\varepsilon\int_{A_{k,t}\setminus A_{k,\tau}} \big(k(x) + |\nabla u|^{p-1}\big)^{p'}\,dx \\
&+ bc(n,\varepsilon)\int_{A_{k,t}\setminus A_{k,\tau}} \big|(u-k)\nabla\eta + \eta\nabla u\big|^p\,dx \\
\leq{}& c(b,\varepsilon,p)\int_{A_{k,t}\setminus A_{k,\tau}} |\nabla u|^p\,dx + c(b,p)\varepsilon\int_{A_{k,t}\setminus A_{k,\tau}} |\nabla u|^p\,dx \\
&+ c(n,\varepsilon,b,p)(t-\tau)^{-p}\int_{A_{k,t}\setminus A_{k,\tau}} (u-k)^p\,dx + \varepsilon\int_{A_{k,t}} (u-k)^p\,dx \\
&+ c(p,\varepsilon)\int_{A_{k,t}} |f|^{p'}\,dx + c(b,\varepsilon,p)\int_{A_{k,t}\setminus A_{k,\tau}} \big|k(x)\big|^{p'}\,dx.
\end{aligned} \tag{3.4}$$

Since $p' < \frac{q}{p-1}$ for some $q > n \geq p > 1$, by applying Hölder's inequality, we have estimates for the last two integrals on the right-hand side of (3.4)

$$\int_{A_{k,t}} |f|^{p'}\,dx \leq |A_{k,t}|^{1-\frac{p}{q}} \|f\|^{\frac{p}{p-1}}_{\frac{q}{p-1};A_{k,t}} \leq \|f\|^{\frac{p}{p-1}}_{\frac{q}{p-1};\Omega} |A_{k,t}|^{1-\frac{p}{q}} \tag{3.5}$$

and

$$\begin{aligned}\int_{A_{k,t}\setminus A_{k,\tau}} |k|^{p'}\,dx &\leq \int_{A_{k,t}} |k|^{p'}\,dx \\ &\leq |A_{k,t}|^{1-\frac{p}{q}} \|k\|^{\frac{p}{p-1}}_{\frac{q}{p-1};A_{k,t}} \leq \|k\|^{\frac{p}{p-1}}_{\frac{q}{p-1};\Omega} |A_{k,t}|^{1-\frac{p}{q}}.\end{aligned} \tag{3.6}$$

Since $|\operatorname{supp}(u-k)^+| = |A_k| < \frac{1}{k^p}\|u\|^p_{p;\Omega}$, then there is $k_0 > 0$ such that for $k > k_0$, we have $|A_k| \leq \frac{1}{2}|B_t|$, then we get $(u-k)^+ \in W^{1,p}(B_t)$ and $|\operatorname{supp}(u-k)^+| \leq \frac{1}{2}|B_t|$.

Take $\hat{n} = n$ when $p < n$, and $\hat{n}$ to be any real number satisfying $n < \hat{n} < q$ when $p = n$. Let $\tilde{p} = \frac{p\hat{n}}{\hat{n}-p}$, then

$$\tilde{p} = \frac{p\hat{n}}{\hat{n}-p} = \begin{cases} \frac{np}{n-p}, & p < n, \\ \frac{n\cdot\frac{\hat{n}n}{2\hat{n}-n}}{n-\frac{\hat{n}n}{2\hat{n}-n}} \ (1 < \frac{\hat{n}n}{2\hat{n}-n} < n = p), & p = n. \end{cases}$$

According to the Sobolev imbedding inequality, we obtain

$$\begin{aligned}\left(\int_{B_t} \left|(u-k)^+\right|^{\tilde{p}}\,dx\right)^{\frac{1}{\tilde{p}}} &\leq c(n,p)\left(\int_{B_t} \left|\nabla\left((u-k)^+\right)\right|^p\,dx\right)^{\frac{1}{p}} \\ &= c(n,p)\left(\int_{A_{k,t}} |\nabla u|^p\,dx\right)^{\frac{1}{p}}.\end{aligned}$$

It then follows from Hölder's inequality that

$$\begin{aligned}\int_{A_{k,t}} (u-k)^p\,dx &= \int_{B_t} \left|(u-k)^+\right|^{\tilde{p}\cdot\frac{p}{\tilde{p}}} \cdot 1\,dx \\ &\leq \left(\int_{B_t} (u-k)^{\tilde{p}}\,dx\right)^{\frac{p}{\tilde{p}}} \cdot |B_t|^{1-\frac{p}{\tilde{p}}} \\ &\leq c(n,p)|B_t|^{1-\frac{p}{\tilde{p}}} \int_{A_{k,t}} |\nabla u|^p\,dx.\end{aligned} \tag{3.7}$$

By substituting (3.5)-(3.7) into (3.4), we see that for $k > k_0$,

$$\begin{aligned}\alpha \int_{A_{k,\tau}} |\nabla u|^p\,dx &\leq c_1 \int_{A_{k,t}\setminus A_{k,\tau}} |\nabla u|^p\,dx + c(b,p)\varepsilon \int_{A_{k,t}\setminus A_{k,\tau}} |\nabla u|^p\,dx \\ &\quad + c(n,\varepsilon,b,p)(t-\tau)^{-p} \int_{A_{k,t}\setminus A_{k,\tau}} (u-k)^p\,dx + \varepsilon c(n,p)|B_t|^{1-\frac{p}{\tilde{p}}} \\ &\quad \times \int_{A_{k,t}} |\nabla u|^p\,dx \\ &\quad + c(p,\varepsilon)\|f\|^{\frac{p}{p-1}}_{\frac{q}{p-1};\Omega} |A_{k,t}|^{1-\frac{p}{q}} + c(b,\varepsilon,p)\|k\|^{\frac{p}{p-1}}_{\frac{q}{p-1};\Omega} |A_{k,t}|^{1-\frac{p}{q}},\end{aligned} \tag{3.8}$$

where $c_1 = c(b, \varepsilon, p)$. Adding to (3.8) both sides

$$c_1 \int_{A_{k,\tau}} |\nabla u|^p \, dx,$$

we obtain

$$\begin{aligned}\int_{A_{k,\tau}} |\nabla u|^p \, dx \le{}& \frac{c_1}{\alpha + c_1} \int_{A_{k,t}} |\nabla u|^p \, dx + \frac{c(b,p)\varepsilon}{\alpha + c_1} \int_{A_{k,t} \setminus A_{k,\tau}} |\nabla u|^p \, dx \\ &+ \frac{c(n,\varepsilon,b,p)}{\alpha + c_1} (t-\tau)^{-p} \max_{B_t} (u-k)^p |A_{k,t}| \\ &+ \frac{\varepsilon c(n,p)}{\alpha + c_1} |B_t|^{1-\frac{p}{\bar{p}}} \int_{A_{k,t}} |\nabla u|^p \, dx \\ &+ \frac{c(b,\varepsilon,p)}{\alpha + c_1} \bigl(\|f\|^{\frac{p}{p-1}}_{\frac{q}{p-1};\Omega} + \|k\|^{\frac{p}{p-1}}_{\frac{q}{p-1};\Omega} \bigr) |A_{k,t}|^{1-\frac{p}{q}}. \end{aligned} \tag{3.9}$$

For $k > k_0$, we can choose R_1 and ε sufficiently small such that for $t \le R_1$, we get

$$\frac{c(b,p)\varepsilon}{\alpha + c_1} + \frac{c(n,p)\varepsilon}{\alpha + c_1} |B_t|^{1-\frac{p}{\bar{p}}} \le \frac{1}{2} \cdot \frac{\alpha}{\alpha + c_1} \tag{3.10}$$

and

$$|A_{k,t}| = |A_{k,t}|^{1-\frac{p}{q}} |A_{k,t}|^{\frac{p}{q}} \le |A_{k,t}|^{1-\frac{p}{q}} |B_t|^{\frac{p}{q}} = |A_{k,t}|^{1-\frac{p}{q}} \cdot t^{n \cdot \frac{p}{q}}. \tag{3.11}$$

By substituting (3.10)-(3.11) into (3.9), we obtain that

$$\begin{aligned}\int_{A_{k,\tau}} |\nabla u|^p \, dx \le{}& \theta \int_{A_{k,t}} |\nabla u|^p \, dx \\ &+ \gamma \Bigl[(t-\tau)^{-p} \cdot t^{n \cdot \frac{p}{q}} \max_{B_t} (u-k)^p + 1 \Bigr] |A_{k,t}|^{1-\frac{p}{q}}, \end{aligned} \tag{3.12}$$

where $\theta = (\frac{1}{2}\alpha + c_1)/(\alpha + c_1) < 1$. Thus, let ρ, R be arbitrarily fixed with $R_0 \le \rho < R \le R_1$, we obtain

$$\begin{aligned}\int_{A_{k,\tau}} |\nabla u|^p \, dx \le{}& \theta \int_{A_{k,t}} |\nabla u|^p \, dx \\ &+ \gamma \Bigl[(t-\tau)^{-p} \cdot R^{n \cdot \frac{p}{q}} \max_{B_R} (u-k)^p + 1 \Bigr] |A_{k,R}|^{1-\frac{p}{q}}. \end{aligned} \tag{3.13}$$

Therefore we have deduced that for every t and τ such that $R_0 \le \rho \le \tau < t \le R \le R_1$, inequality (3.13) holds. Therefore we have from Lemma 2.1 that

$$\int_{A_{k,\rho}} |\nabla u|^p \, dx \le C \Bigl[(R-\rho)^{-p} \cdot R^{n \cdot \frac{p}{q}} \max_{B_R} (u-k)^p + 1 \Bigr] |A_{k,R}|^{1-\frac{p}{q}}. \tag{3.14}$$

Since u is a solution to equation (1.1), we have that $-u$ is a solution to the equation

$$-\operatorname{div} \tilde{A}(x, v, \nabla v) = f(x),$$

where $\tilde{A}(x,v,\nabla v) = A(x,-v,-\nabla v)$. And the operator $\tilde{A}$ satisfies the same structure conditions (a)-(c), hence the same inequality (3.14) holds with u replaced by $-u$. Therefore we get that the function $u \in \mathcal{B}_p(\Omega, M, \gamma, \delta, 1/q)$. □

Remark Especially, if $q = \infty$, *i.e.*, $f(x), k(x) \in L^\infty(\Omega)$, then condition (a) of A simplifies into $|A(x,u,\nabla u)| \le b(M)|\nabla u|^{p-1} + b(M)$. Proceeding the process of the proof in Theorem 3.2, we finally have

$$\int_{A_{k,\tau}} |\nabla u|^p \, dx \le \theta \int_{A_{k,t}} |\nabla u|^p \, dx + (t-\tau)^{-p} \int_{A_{k,R}} (u-k)^p \, dx + |A_{k,R}|.$$

Therefore we have from Lemma 2.1 that

$$\int_{A_{k,\rho}} |\nabla u|^p \, dx \le C\left((R-\rho)^{-p} \int_{A_{k,R}} (u-k)^p \, dx + |A_{k,R}|\right).$$

Similarly, we get that the same inequality holds with u replaced by $-u$, hence the function $u \in \mathcal{B}_p(\Omega, M, \gamma, \delta, 0)$.

Then, by applying Lemma 3.1 and Lemma 2.2, we obtain, for arbitrary $\rho \le \rho_0$, that

$$\operatorname*{osc}_{B_\rho} u \le 4^m \left(\frac{\rho}{\rho_0}\right)^m \left(\operatorname*{osc}_{B_{\rho_0}} u + 2^s \rho_0^{1-\frac{n}{q}}\right),$$

where $m = \min\{-\log_4(1 - \frac{1}{2^{s-1}}), 1 - \frac{n}{q}\}$. Choosing $\rho = \rho_0/5$, we have the following oscillation estimate which is important and fundamental in our main results.

Proposition 3.3 *Suppose that* $u(x) \in W^{1,p}_{\mathrm{loc}}(\Omega)$ *is a bounded weak solution of* (1.1), *then*

$$\operatorname*{osc}_{B_{\frac{R}{5}}} u \le \gamma \operatorname*{osc}_{B_R} u + CR^{1-\frac{n}{q}} \tag{3.15}$$

holds for any ball $B_R \subset \Omega$, *where* $\gamma = \gamma(n,q,s) \in (0,1)$, *and C is a positive constant depending on n, q and s.*

The oscillation estimate Proposition 3.3 can be used to obtain the following interior Hölder estimate by the method of iteration.

Theorem 3.4 *Suppose that* $u(x) \in W^{1,p}_{\mathrm{loc}}(\Omega)$ *is a bounded weak solution of* (1.1), *then*

$$\operatorname*{osc}_{B_r} u \le 5^\kappa \left(\frac{r}{R}\right)^\kappa \operatorname*{osc}_{B_R} u + Cr^\kappa \tag{3.16}$$

for any ball $B_R \subset \Omega$ *and* $0 < r < R < \infty$, *where* $\kappa \in (0,1)$ *depends on n, q and s, and* $C = C(n,q,s,\Omega)$.

Proof Let $0 < \rho \le R$, $B_R \subset \Omega$, then Proposition 3.3 yields

$$\operatorname*{osc}_{B_{\frac{\rho}{5}}} u \le \gamma \operatorname*{osc}_{B_\rho} u + C\rho^{1-\frac{n}{q}}. \tag{3.17}$$

Let $\rho_0 = R$, $\rho_\iota = 5^{-\iota}\rho_0$ for $\iota = 0, 1, 2, \ldots$ and take $b = \frac{\gamma+1}{2} \in (0,1)$. Let $\omega_\iota = \operatorname{osc}_{B_{\rho_\iota}} u$, where spheres B_{ρ_ι} are concentric with B_{ρ_0}, and

$$\kappa = \min\left\{\log_5 \frac{b}{\gamma}, 1 - \frac{n}{q}\right\}.$$

Then we have $0 < \kappa < 1$, $5^\kappa \gamma \le b$. Moreover, denote $c_1 = 5^\kappa \cdot CR^{1-\frac{n}{q}}$.

Observe that (3.17) implies that

$$\operatorname*{osc}_{B_{\frac{\rho_{\iota-1}}{5}}} u \le \gamma \operatorname*{osc}_{B_{\rho_{\iota-1}}} u + C\rho_{\iota-1}^{1-\frac{n}{q}}. \tag{3.18}$$

From the notations above, we can get from (3.18) that

$$\omega_\iota \le \gamma \omega_{\iota-1} + C\rho_\iota^{1-\frac{n}{q}}. \tag{3.19}$$

Write $y_\iota = 5^{\iota\kappa}\omega_\iota$, then we have from (3.19) that

$$y_\iota \le b y_{\iota-1} + c_1 5^{-\kappa}. \tag{3.20}$$

We obtain

$$\begin{aligned}
y_\iota &\le b y_{\iota-1} + c_1 5^{-\kappa} \\
&\le b\left(b y_{\iota-2} + c_1 5^{-\kappa}\right) + c_1 5^{-\kappa} \\
&= b^2 y_{\iota-2} + b c_1 5^{-\kappa} + c_1 5^{-\kappa} \\
&\le \cdots \\
&\le b^\iota y_0 + \left(b^{\iota-1} + \cdots + b + 1\right) c_1 5^{-\kappa} \\
&\le y_0 + \frac{1}{1-b} c_1 5^{-\kappa}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\omega_\iota &= 5^{-\iota\kappa} y_\iota \le 5^{-\iota\kappa} y_0 + \frac{1}{1-b} c_1 5^{-\kappa} 5^{-\iota\kappa} \\
&= y_0 \left(\frac{\rho_\iota}{\rho_0}\right)^\kappa + \frac{1}{1-b} c_1 5^{-\kappa} \left(\frac{\rho_\iota}{\rho_0}\right)^\kappa.
\end{aligned} \tag{3.21}$$

For arbitrary $0 < r < R = \rho_0 < \infty$, there exists $\iota_0 \ge 1$ such that $\rho_{\iota_0} \le r \le \rho_{\iota_0 - 1}$. We obtain

$$\begin{aligned}
\operatorname*{osc}_{B_r} u &\le \operatorname*{osc}_{B_{\rho_{\iota_0-1}}} u \\
&\le y_0 5^\kappa \rho_0^{-\kappa} \rho_{\iota_0}^\kappa + \frac{1}{1-b} c_1 \rho_0^{-\kappa} \rho_{\iota_0}^\kappa \\
&\le 5^\kappa y_0 \rho_0^{-\kappa} r^\kappa + \frac{1}{1-b} c_1 \rho_0^{-\kappa} r^\kappa \\
&= 5^\kappa \left(\frac{r}{R}\right)^\kappa \operatorname*{osc}_{B_R} u + \frac{1}{1-b} 5^\kappa \cdot CR^{1-\frac{n}{q}} R^{-\kappa} r^\kappa
\end{aligned}$$

$$\le 5^{\kappa}\left(\frac{r}{R}\right)^{\kappa}\operatorname*{osc}_{B_R} u + CR^{(1-\frac{n}{q})-\kappa}r^{\kappa}$$
$$\le 5^{\kappa}\left(\frac{r}{R}\right)^{\kappa}\operatorname*{osc}_{B_R} u + Cr^{\kappa},$$

where $C = C(n, q, s, \Omega)$. □

As an important application of Theorem 3.4, we investigate the global Hölder continuity of weak solutions of (1.1), which is the main result of the paper.

Theorem 3.5 *Suppose that* $u(x) \in C(\overline{\Omega}) \cap W^{1,p}_{\mathrm{loc}}(\Omega)$ *is a weak solution of* (1.1). *If there are constants* $L \ge 0$ *and* $0 < \delta \le 1$ *such that*

$$\bigl|u(x) - u(x_0)\bigr| \le L|x - x_0|^{\delta} \tag{3.22}$$

for all $x \in \Omega$ *and* $x_0 \in \partial\Omega$, *then there exist constants* $L_1 \ge 0$ *and* $0 < \delta_1 \le 1$ *such that*

$$\bigl|u(x) - u(y)\bigr| \le L_1|x - y|^{\delta_1} \tag{3.23}$$

for all $x, y \in \overline{\Omega}$.

Proof For arbitrary $x, y \in \overline{\Omega}$, we discuss the following three cases:

(A) $x \in \partial\Omega$, $y \in \Omega$ or $x \in \Omega$, $y \in \partial\Omega$:

We have by (3.22) that (3.23) holds with $\delta_1 = \delta$, and $L_1 = L$.

(B) $x, y \in \partial\Omega$:

For $x \in \partial\Omega$, there exists $\{x_k\} \subset \Omega$ such that $x_k \to x$ as $k \to \infty$. Since $u \in C(\overline{\Omega})$, we have $u(x_k) \to u(x)$ and $|x_k - y|^{\delta} \to |x - y|^{\delta}$ as $k \to \infty$. Then we get

$$\begin{aligned}\bigl|u(x) - u(y)\bigr| &\le \bigl|u(x) - u(x_k)\bigr| + \bigl|u(x_k) - u(y)\bigr| \\ &\le \bigl|u(x) - u(x_k)\bigr| + L|x_k - y|^{\delta}.\end{aligned}$$

Let $k \to \infty$, then (3.23) is obtained with $\delta_1 = \delta$ and $L_1 = L$.

(C) $x, y \in \Omega$:

For case (C), we consider two cases (I) and (II):

(I) $|x - y| \le \frac{1}{2}\operatorname{dist}(x, \partial\Omega)$.

Choose $x_0 \in \partial\Omega$ such that $|x_0 - x| = \operatorname{dist}(x, \partial\Omega) = r$. Then, for arbitrary $z \in B_{\frac{r}{2}}(x)$,

$$\begin{aligned}\bigl|u(z) - u(x)\bigr| &\le \bigl|u(z) - u(x_0)\bigr| + \bigl|u(x_0) - u(x)\bigr| \\ &\le L|z - x_0|^{\delta} + L|x_0 - x|^{\delta} \\ &\le L\bigl(|z - x| + |x - x_0|\bigr)^{\delta} + L|x_0 - x|^{\delta} \\ &\le L\left(\frac{3}{2}r\right)^{\delta} + Lr^{\delta} \\ &= Lr^{\delta}\left(1 + \left(\frac{3}{2}\right)^{\delta}\right) \\ &\le \frac{5}{2}Lr^{\delta};\end{aligned}$$

therefore we have, for all $z_1, z_2 \in B_{\frac{r}{2}}(x)$,

$$\begin{aligned}|u(z_1) - u(z_2)| &\le |u(z_1) - u(x)| + |u(x) - u(z_2)| \\ &\le 5Lr^\delta.\end{aligned}$$

Therefore we obtain $\operatorname{osc}_{B_{\frac{r}{2}}(x)} u \le 5Lr^\delta$. Since $x, y \in \overline{B}(x, |x-y|) \subset B_{\frac{r}{2}}(x) \subset \Omega$, it follows from (3.16) that

$$\begin{aligned}|u(x) - u(y)| &\le \operatorname*{osc}_{\overline{B}(x,|x-y|)} u = \operatorname*{osc}_{B(x,|x-y|)} u \\ &\le 5^\kappa \left(\frac{|x-y|}{r/2}\right)^\kappa \operatorname*{osc}_{B_{\frac{r}{2}}(x)} u + C|x-y|^\kappa \\ &\le 50L|x-y|^\kappa r^{\delta-\kappa} + C|x-y|^\kappa. \qquad (3.24)\end{aligned}$$

To estimate (3.24), we consider two cases (i) and (ii).

(i) If $\delta < \kappa$, then $\delta = \min\{\delta, \kappa\} \triangleq \delta_1$. Since $\kappa - \delta > 0$ and $|x-y| < r/2 < r < \operatorname{diam}\Omega$, thus $|x-y|^{\kappa-\delta} < r^{\kappa-\delta}$ and $|x-y|^{\kappa-\delta} < (\operatorname{diam}\Omega)^{\kappa-\delta}$. Then we obtain

$$|u(x) - u(y)| \le 50L|x-y|^{\delta_1} + C(\operatorname{diam}\Omega)^{\kappa-\delta_1}|x-y|^{\delta_1}. \qquad (3.25)$$

(ii) If $\delta \ge \kappa$, then $\kappa = \min\{\delta, \kappa\} \triangleq \delta_1$. Since $\delta - \kappa \ge 0$ and $r = \operatorname{dist}(x, \partial\Omega) \le \operatorname{diam}\Omega$, thus $r^{\delta-\kappa} \le (\operatorname{diam}\Omega)^{\delta-\kappa}$. Then we obtain

$$|u(x) - u(y)| \le 50L(\operatorname{diam}\Omega)^{\delta-\delta_1}|x-y|^{\delta_1} + C|x-y|^{\delta_1}. \qquad (3.26)$$

Therefore we have the estimate for case (I) by substituting (3.25) and (3.26) into (3.24) so that

$$\begin{aligned}&|u(x) - u(y)| \\ &\quad \le \left(50L \max\left(1, (\operatorname{diam}\Omega)^{\delta-\delta_1}\right) + C \max\left(1, (\operatorname{diam}\Omega)^{\kappa-\delta_1}\right)\right) \cdot |x-y|^{\delta_1}. \qquad (3.27)\end{aligned}$$

Next, we estimate case (II) of case (C).

(II) $|x-y| \ge \frac{1}{2}\operatorname{dist}(x, \partial\Omega)$.

Choose $x_0 \in \partial\Omega$ such that $|x_0 - x| = \operatorname{dist}(x, \partial\Omega) = r$. Then we have

$$\begin{aligned}|u(x) - u(y)| &\le |u(x) - u(x_0)| + |u(x_0) - u(y)| \\ &\le L|x-x_0|^\delta + L|x_0-y|^\delta \\ &\le L|x-x_0|^\delta + L\left(|x_0-x| + |x-y|\right)^\delta \\ &\le L2^\delta|x-y|^\delta + L3^\delta|x-y|^\delta \le 5L|x-y|^\delta. \qquad (3.28)\end{aligned}$$

Similarly, to estimate (3.28), we consider two cases (iii) and (iv).

(iii) If $\delta < \kappa$, then $\delta = \min\{\delta, \kappa\} = \delta_1$. Then we obtain

$$|u(x) - u(y)| \le 5L|x-y|^{\delta_1}. \qquad (3.29)$$

(iv) If $\delta \geq \kappa$, then $\delta_1 = \kappa$, thus we have $|x-y|^{\delta} = |x-y|^{\delta_1} \cdot |x-y|^{\delta-\delta_1} \leq (\operatorname{diam}\Omega)^{\delta-\delta_1} \cdot |x-y|^{\delta_1}$. Then we obtain

$$\left|u(x)-u(y)\right| \leq 5L(\operatorname{diam}\Omega)^{\delta-\delta_1} \cdot |x-y|^{\delta_1}. \tag{3.30}$$

Therefore we have the estimate for case (II) by substituting (3.29) and (3.30) into (3.28) so that

$$\left|u(x)-u(y)\right| \leq 5L\max\left(1, (\operatorname{diam}\Omega)^{\delta-\delta_1}\right) \cdot |x-y|^{\delta_1}. \tag{3.31}$$

Finally, combined with (3.27) and (3.31), we have the estimate for case (C)

$$\begin{aligned} &\left|u(x)-u(y)\right| \\ &\quad \leq \left(50L\max\left(1, (\operatorname{diam}\Omega)^{\delta-\delta_1}\right) + C\max\left(1, (\operatorname{diam}\Omega)^{\kappa-\delta_1}\right)\right) \cdot |x-y|^{\delta_1}. \end{aligned} \tag{3.32}$$

Therefore the theorem follows with

$$\delta_1 = \min\{\delta, \kappa\} \in (0,1]$$

and

$$L_1 = 50L\max\left(1, (\operatorname{diam}\Omega)^{\delta-\delta_1}\right) + C\max\left(1, (\operatorname{diam}\Omega)^{\kappa-\delta_1}\right). \qquad \square$$

3.2 Global Hölder estimate

In order to extend the above results to a global Hölder estimate, we need to place an additional constraint on Ω.

Definition 3.6 [1] We shall say that the boundary $\partial\Omega$ of Ω satisfies condition (A) if there exist two positive numbers a_0 and θ_0 such that for an arbitrary ball B_ρ with center on $\partial\Omega$ of radius $\rho \leq a_0$ and for an arbitrary component $\widehat{\Omega}_\rho$ of $B_\rho \cap \Omega$, the inequality

$$|\widehat{\Omega}_\rho| \leq (1-\theta_0)|B_\rho|$$

holds.

Now let

$$\Omega_t = \Omega \cap B_t, \qquad \Omega_{k,t} = A_k \cap \Omega_t,$$

and let $\mathcal{B}_p(\overline{\Omega}, M, \gamma, \delta, 1/q)$ be the class of functions $u(x)$ in $\mathcal{B}_p(\Omega, M, \gamma, \delta, 1/q)$ that, together with their negatives, satisfy inequality (3.1) for the balls B_ρ with $B_\rho \cap \Omega \neq \emptyset$, the integration region $\Omega_{k,\rho}$, and for $k \geq \max_{\Omega_\rho} u(x) - \delta$ and $k \geq \max_{B_\rho \cap \partial\Omega} u(x)$.

Lemma 3.7 [1, Lemma 7.1, p.92] *If* $\partial\Omega$ *satisfies condition* (A) *and if the function* $u(x)$ *in* $\mathcal{B}_p(\overline{\Omega}, M, \gamma, \delta, 1/q)$ *satisfies on* $\partial\Omega$ *a Hölder condition, more precisely, if*

$$\operatorname*{osc}_{\partial\Omega \cap B_\rho} u \leq L\rho^{\epsilon}, \quad \epsilon > 0, \tag{3.33}$$

for balls B_ρ (where $\rho \le a_0$) with centers on $\partial\Omega$, then there exists a positive number s such that for an arbitrary ball B_ρ, for a ball $B_{4\rho}$ (where $4\rho \le \frac{1}{4}\min\{a_0,1\}$) with center on $\partial\Omega$ that is concentric with it, at least one of the following inequalities holds:

$$\operatorname*{osc}_{\Omega_\rho} u \le 2^s \rho^{\epsilon_1}, \quad \epsilon_1 = \min\left\{1 - \frac{n}{q}, \epsilon\right\},$$

$$\operatorname*{osc}_{\Omega_\rho} u \le \left(1 - \frac{1}{2^{s-1}}\right) \operatorname*{osc}_{\Omega_{4\rho}} u.$$

The number s is determined by the parameters of the class $\mathcal{B}_p$, by the numbers ϵ and L in (3.33), *and by the numbers a_0 and θ_0 in condition* (A).

Analogously, we proceed the proof basically the same [10] as Theorem 3.2, and we can prove that $u \in \mathcal{B}_p(\overline{\Omega}, M, \gamma, \delta, 1/q)$. By applying Lemma 3.7 and Lemma 2.2, for

$$\rho \le \rho_0 = \frac{1}{4}\min\{a_0, 1\},$$

we obtain that

$$\operatorname*{osc}_{\Omega_\rho} u \le 4^m \left(\frac{\rho}{\rho_0}\right)^m \left(\operatorname*{osc}_{\Omega_{\rho_0}} u + 2^s \rho_0^{\epsilon_1}\right),$$

where $\epsilon_1 = \min\{1 - \frac{n}{q}, \epsilon\}$, $m = \min\{-\log_4(1 - \frac{1}{2^{s-1}}), \epsilon_1\}$ and ϵ is in (3.33). Choosing $\rho = \rho_0/5$, we have the following oscillation estimate.

Proposition 3.8 *Suppose that $u(x) \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a bounded weak solution of* (1.1), *then*

$$\operatorname*{osc}_{\Omega_{\frac{R}{5}}} u \le \gamma \operatorname*{osc}_{\Omega_R} u + CR^{\epsilon_1},$$

where $\gamma \in (0,1)$ and C are positive constants depending on n, q, s and ϵ in (3.33), *holds*.

Proceeding completely analogously to the proof of Theorem 3.4, we obtain the following.

Theorem 3.9 *Suppose that $u(x) \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a bounded weak solution of* (1.1), *then*

$$\operatorname*{osc}_{\Omega_r} u \le 5^\kappa \left(\frac{r}{R}\right)^\kappa \operatorname*{osc}_{\Omega_R} u + Cr^\kappa \tag{3.34}$$

for any ball B_R and $0 < r < R < \rho_0$, where $\kappa \in (0,1)$ and $C > 0$ depends on n, q, s and ϵ.

We now have the following global Hölder estimate based on the boundary Hölder continuity.

Theorem 3.10 *Suppose that Ω is bounded and satisfies condition* (A). *Let $u(x) \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ is a weak solution of* (1.1). *If there are constants $M \ge 0$ and $0 < \gamma \le 1$ such that*

$$\left|u(x) - u(y)\right| \le M|x - y|^\gamma \tag{3.35}$$

for all $x, y \in \partial\Omega$, *then there exist constants* $M_1 \geq 0$ *and* $\gamma_1 > 0$ *such that*

$$\left|u(x) - u(y)\right| \leq M_1|x - y|^{\gamma_1} \tag{3.36}$$

for all $x, y \in \overline{\Omega}$.

Proof It is clear that we just need to prove (3.22) according to Theorem 3.5. For all $x \in \Omega$ and $x_0 \in \partial\Omega$, we consider the following two cases:

(i) If $r = |x - x_0| < 1$, then there exists r_0 such that $r = |x - x_0| < r_0 \leq 1$. The boundary Hölder estimate (3.34) with $R = r^{\frac{1}{2}}$ yields

$$\begin{aligned} \left|u(x) - u(x_0)\right| &\leq \operatorname*{osc}_{\Omega\cap B_r(x_0)} u \\ &\leq Cr^{\frac{\kappa}{2}} \operatorname*{osc}_{\Omega\cap B_R(x_0)} u + Cr^{\kappa}. \end{aligned} \tag{3.37}$$

Since $u(x) \in C(\overline{\Omega})$ and Ω is bounded, we have

$$\sup_{\Omega} |u| \leq C_1 < \infty. \tag{3.38}$$

Therefore we get by substituting (3.38) into (3.37) that

$$\begin{aligned} \left|u(x) - u(x_0)\right| &\leq \operatorname*{osc}_{\Omega\cap B_r(x_0)} u \\ &\leq C2C_1|x - x_0|^{\frac{\kappa}{2}} + C|x - x_0|^{\kappa} \\ &\leq C(2C_1 + 1)|x - x_0|^{\frac{\kappa}{2}}. \end{aligned} \tag{3.39}$$

(ii) If $r = |x - x_0| \geq 1$, then

$$\left|u(x) - u(x_0)\right| \leq 2\sup_{\Omega} |u| \leq 2C_1|x - x_0|^{\frac{\kappa}{2}}. \tag{3.40}$$

Combining (3.39) and (3.40), we have

$$\left|u(x) - u(x_0)\right| \leq M_1|x - x_0|^{\frac{\kappa}{2}}$$

with

$$M_1 = \max(C2C_1 + C, 2C_1).$$

Therefore the theorem follows from Theorem 3.5. □

4 Application

We conclude this paper with an application of Theorem 3.10 in a simple case of (1.1). We consider the following equation:

$$-D_i\left(a^{ij}(x)D_j u\right) = f(x), \tag{4.1}$$

where $f(x) \in L^{\frac{q}{2}}(\Omega)$ for some $q > n$, and the coefficients a^{ij} $(i,j = 1,\ldots,n)$ are assumed to be measurable functions on Ω, and there exist positive constants λ and Λ such that

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad \forall x \in \Omega, \xi \in \mathbb{R}^n \tag{4.2}$$

and

$$\Sigma\left|a^{ij}(x)\right|^2 \leq \Lambda^2. \tag{4.3}$$

Let

$$A_i(x, \nabla u) = a^{ij}(x)D_j u,$$

then we can easily prove that the operator A satisfies the structural assumption (a)-(c).

To apply Theorem 3.10 for (4.1), we put a more general constraint on Ω to obtain the Hölder continuity up to boundary.

Definition 4.1 [13] We shall say that Ω satisfies an exterior cone condition at a point $x_0 \in \partial\Omega$ if there exists a finite right circular cone $V = V_{x_0}$ with vertex x_0 such that $\overline{\Omega} \cap V_{x_0} = x_0$.

Definition 4.2 [13] Let us say that Ω satisfies a uniform exterior cone condition on $\partial\Omega$ if Ω satisfies an exterior cone condition at every $x_0 \in \partial\Omega$ and the cones V_{x_0} are all congruent to some fixed cone V.

We now have the following Hölder estimate at the boundary.

Theorem 4.3 [13] *Suppose that $u(x)$ is a $W^{1,2}(\Omega)$ solution of* (4.1) *in Ω and Ω satisfies an exterior cone condition at a point $x_0 \in \partial\Omega$. We have, for any* $0 < r \leq \rho$,

$$\operatorname*{osc}_{\Omega\cap B_r(x_0)} u \leq C\left[\left(\frac{r}{\rho}\right)^{\kappa} \sup_{\Omega\cap B_\rho(x_0)} |u| + r^{\kappa} \cdot \lambda^{-1}\|f\|_{L^{\frac{q}{2}}(\Omega)} + \operatorname*{osc}_{\partial\Omega\cap B_{\sqrt{r\rho}}(x_0)} u\right], \tag{4.4}$$

where $C = C(n, \lambda, \Lambda, q, \rho, V_{x_0})$, $\kappa = \kappa(n, \lambda, \Lambda, q, V_{x_0})$ are positive constants.

Theorem 4.4 *Suppose that Ω is bounded and satisfies the uniform exterior cone condition. Let $u(x) \in C(\overline{\Omega}) \cap W^{1,2}(\Omega)$ be a weak solution of* (4.1). *If there are constants $M \geq 0$ and $0 < \gamma \leq 1$ such that*

$$\left|u(x) - u(y)\right| \leq M|x - y|^{\gamma} \tag{4.5}$$

for all $x, y \in \partial\Omega$, then there exist constants $M_1 \geq 0$ and $\gamma_1 > 0$ such that

$$\left|u(x) - u(y)\right| \leq M_1|x - y|^{\gamma_1} \tag{4.6}$$

for all $x, y \in \overline{\Omega}$.

Proof It is clear that we just need to prove (3.22) according to Theorem 3.5. For all $x \in \Omega$ and $x_0 \in \partial\Omega$, we consider the following two cases:

(i) If $r = |x - x_0| < 1$, then there exists r_0 such that $r = |x - x_0| < r_0 \leq 1$. The boundary Hölder estimate (4.4) with $\rho = r^{\frac{1}{2}}$ yields

$$\begin{aligned} \left|u(x) - u(x_0)\right| &\leq \operatorname*{osc}_{\Omega\cap B_r(x_0)} u \\ &\leq C\Big[r^{\frac{\kappa}{2}} \sup_{\Omega\cap B_\rho(x_0)} |u| + r^\kappa \cdot \lambda^{-1}\|f\|_{L^{\frac{q}{2}}(\Omega)} + \operatorname*{osc}_{\partial\Omega\cap B_{r^{\frac{3}{4}}}(x_0)} u\Big]. \end{aligned} \tag{4.7}$$

Since $u(x) \in C(\overline{\Omega})$ and Ω is bounded, we have

$$\sup_\Omega |u| \leq C_1 < \infty, \tag{4.8}$$

and for all $z_1, z_2 \in \partial\Omega \cap B_{r^{\frac{3}{4}}(x_0)}$, we have from (4.5) that

$$\begin{aligned} \left|u(z_1) - u(z_2)\right| &\leq M|z_1 - z_2|^\gamma \\ &\leq M\big(|z_1 - x_0| + |z_2 - x_0|\big)^\gamma \\ &\leq 2Mr^{\frac{3}{4}\gamma}. \end{aligned} \tag{4.9}$$

Thus (4.9) yields that

$$\operatorname*{osc}_{\partial\Omega\cap B_{r^{\frac{3}{4}}}(x_0)} u \leq 2Mr^{\frac{3}{4}\gamma}. \tag{4.10}$$

Therefore we get by substituting (4.8) and (4.10) into (4.7) that

$$\begin{aligned} \left|u(x) - u(x_0)\right| &\leq \operatorname*{osc}_{\Omega\cap B_r(x_0)} u \\ &\leq CC_1|x - x_0|^{\frac{\kappa}{2}} + C\|f\|_{L^{\frac{q}{2}}(\Omega)}|x - x_0|^\kappa + CM|x - x_0|^{\frac{3}{4}\gamma} \\ &\leq C\max\big(C_1, \|f\|_{L^{\frac{q}{2}}(\Omega)}, M\big)|x - x_0|^{\gamma_1}, \end{aligned} \tag{4.11}$$

where $\gamma_1 = \min(\frac{\kappa}{2}, \frac{3}{4}\gamma)$.

(ii) If $r = |x - x_0| \geq 1$, then

$$\left|u(x) - u(x_0)\right| \leq 2\sup_\Omega |u| \leq 2C_1|x - x_0|^{\gamma_1}. \tag{4.12}$$

Combining (4.11) and (4.12), we have

$$\left|u(x) - u(x_0)\right| \leq M_1|x - x_0|^{\gamma_1}$$

with

$$M_1 = \max\big(CC_1, C\|f\|_{L^{\frac{q}{2}}(\Omega)}, CM, 2C_1\big)$$

and

$$\gamma_1 = \min\left(\frac{\kappa}{2}, \frac{3}{4}\gamma\right).$$

Therefore the theorem follows from Theorem 3.5. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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