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# Global well-posedness of 2D generalized MHD equations with fractional diffusion

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**Abstract**
In this paper we prove the uniqueness of weak solutions and the global-in-time existence of smooth solutions of the 2D generalized MHD system with fractional diffusion with $\frac{1}{2}$ power.

**MSC:** 35Q30; 76D03; 76D09

**Keywords:** MHD; fractional diffusion; uniqueness; smooth solution

## 1 Introduction

In this paper, we consider the following 2D generalized MHD system with $0 < \alpha \leq 1$ [1]:

$$\operatorname{div} u = \operatorname{div} b = 0, \tag{1.1}$$

$$\partial_t u + (u \cdot \nabla)u + \nabla\left(\pi + \frac{1}{2}|b|^2\right) + (-\Delta)^\alpha u = b \cdot \nabla b, \tag{1.2}$$

$$\partial_t b + u \cdot \nabla b - b \cdot \nabla u - \Delta b = 0, \tag{1.3}$$

$$(u, b)(t = 0) = (u_0, b_0). \tag{1.4}$$

Here, $u$ is the fluid velocity field, $\pi$ is the pressure and $b$ is the magnetic field.

Very recently, Ji [1] used the Fourier series analysis motivated in [2] to prove the global-in-time existence of smooth solutions of problem (1.1)-(1.4) when $\frac{1}{2} < \alpha \leq 1$, and Ji [1] pointed out that his result did not seem to come directly from the method like energy estimates. In this paper, we use the standard energy method to deal with the case $\alpha = \frac{1}{2}$; of course, our method also works when $\alpha > \frac{1}{2}$. We will prove the following.

**Theorem 1.1** *Let* $\alpha = \frac{1}{2}$. *Let* $u_0, b_0 \in H^1$ *with* $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$ *in* $\mathbb{R}^2$. *Then problem* (1.1)-(1.4) *has a unique global-in-time weak solution* $(u, b)$ *satisfying*

$$(u, b) \in L^\infty(0, T; H^1), \qquad u \in L^2(0, T; H^{3/2}), \qquad b \in L^2(0, T; H^2) \tag{1.5}$$

*for any* $T > 0$.

**Theorem 1.2** *Let* $\alpha = \frac{1}{2}$. *Let* $u_0, b_0 \in H^s$ *with* $s > 1$ *and* $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$ *in* $\mathbb{R}^2$. *Then problem* (1.1)-(1.4) *has a unique global-in-time smooth solution* $(u, b)$ *satisfy-*

*ing*

$$u, b \in L^\infty(0,T;H^s), \qquad u \in L^2(0,T;H^{s+\frac{1}{2}}), \qquad b \in L^2(0,T;H^{s+1}) \tag{1.6}$$

*for any* $T > 0$.

For 3D case and other related problems, we refer to [3, 4].

Our proof will use the following commutator estimates due to Kato and Ponce [5]:

$$\left\| \Lambda^s(fg) - f\Lambda^s g \right\|_{L^p} \le C\left( \|\nabla f\|_{L^{p_1}} \left\| \Lambda^{s-1} g \right\|_{L^{q_1}} + \left\| \Lambda^s f \right\|_{L^{p_2}} \|g\|_{L^{q_2}} \right), \tag{1.7}$$

with $s \ge 1$, $\Lambda := (-\Delta)^{1/2}$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$.

## 2 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. The global-in-time existence of weak solutions satisfying (1.5) was proved in [1, 6], we only need to show the uniqueness. Let $(u_i, \pi_i, b_i)$ $(i = 1, 2)$ be two weak solutions of problem (1.1)-(1.4). We define

$$\delta u := u_1 - u_2, \qquad \delta\pi := \pi_1 - \pi_2, \qquad \delta b := b_1 - b_2.$$

Then it follows from (1.1)-(1.3) that

$$\operatorname{div} \delta u = 0, \qquad \operatorname{div} \delta b = 0, \tag{2.1}$$

$$\partial_t \delta u + u_1 \cdot \nabla \delta u + \delta u \cdot \nabla u_2 + \nabla\left( \pi + \frac{1}{2}\left(b_1^2 - b_2^2\right) \right) + (-\Delta)^{1/2}\delta u$$
$$= b_1 \cdot \nabla \delta b + \delta b \cdot \nabla b_2, \tag{2.2}$$

$$\partial_t \delta b + u_1 \cdot \nabla \delta b + \delta u \cdot \nabla b_2 - b_1 \cdot \nabla \delta u - \delta b \cdot \nabla u_2 - \Delta \delta b = 0. \tag{2.3}$$

Testing (2.2) by $\delta u$ and using (1.1) and (2.1), we see that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int |\delta u|^2\,dx + \int \left|\Lambda^{1/2}\delta u\right|^2 dx &= -\int \delta u \cdot \nabla u_2 \cdot \delta u\,dx \\ &\quad + \int b_1 \cdot \nabla \delta b \cdot \delta u\,dx + \int \delta b \cdot \nabla b_2 \cdot \delta u\,dx \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{2.4}$$

Testing (2.3) by $\delta b$ and using (1.1) and (2.1), we find that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int |\delta b|^2\,dx + \int |\nabla \delta b|^2\,dx &= -\int \delta u \cdot \nabla b_2 \cdot \delta b\,dx \\ &\quad + \int b_1 \cdot \nabla \delta u \cdot \delta b\,dx + \int \delta b \cdot \nabla u_2 \cdot \delta b\,dx \\ &=: I_4 + I_5 + I_6. \end{aligned} \tag{2.5}$$

In the following calculations, we use the Sobolev embedding $\dot{H}^{1/2} \subset L^4$ and the Gagliardo-Nirenberg inequalities

$$\|w\|_{L^{8/3}}^2 \le C\|w\|_{L^2}\left\|\Lambda^{1/2}w\right\|_{L^2}, \tag{2.6}$$

$$\|w\|_{L^4}^2 \le C\|w\|_{L^2}\|\nabla w\|_{L^2}. \tag{2.7}$$

Using (1.1), (2.1), (1.5), (2.6) and (2.7), we bound $I_1$, $I_2 + I_5$, $I_3 + I_4$ and $I_6$ as follows:

$$\begin{aligned}
I_1 &\le \|\nabla u_2\|_{L^4}\|\delta u\|_{L^{8/3}}^2 \le C\|u_2\|_{\dot{H}^{3/2}}\|\delta u\|_{L^2}\left\|\Lambda^{1/2}\delta u\right\|_{L^2} \\
&\le \frac{1}{16}\left\|\Lambda^{1/2}\delta u\right\|_{L^2}^2 + C\|u_2\|_{\dot{H}^{3/2}}^2\|\delta u\|_{L^2}^2, \\
I_2 + I_5 &= 0, \\
I_3 + I_4 &\le C\|\nabla b_2\|_{L^4}\|\delta u\|_{L^2}\|\delta b\|_{L^4} \\
&\le C\|\nabla b_2\|_{L^4}\|\delta u\|_{L^2}\|\delta b\|_{L^2}^{1/2}\|\nabla\delta b\|_{L^2}^{1/2} \\
&\le \frac{1}{16}\|\nabla\delta b\|_{L^2}^2 + C\|\delta u\|_{L^2}^2 + C\|\nabla b_2\|_{L^4}^2\|\delta b\|_{L^2}^2, \\
I_6 &\le \|\nabla u_2\|_{L^2}\|\delta b\|_{L^4}^2 \le C\|\delta b\|_{L^4}^2 \le C\|\delta b\|_{L^2}\|\nabla\delta b\|_{L^2} \\
&\le \frac{1}{16}\|\nabla\delta b\|_{L^2}^2 + C\|\delta b\|_{L^2}^2.
\end{aligned}$$

Adding up (2.4) and (2.5) and using the above estimates, we conclude that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\left(|\delta u|^2 + |\delta b|^2\right)dx \\
&\quad \le C\|u_2\|_{\dot{H}^{3/2}}^2\|\delta u\|_{L^2}^2 + C\|\delta u\|_{L^2}^2 + C\|\nabla b_2\|_{L^4}^2\|\delta b\|_{L^2}^2 + C\|\delta b\|_{L^2}^2,
\end{aligned}$$

which gives

$$\delta u = \delta b = 0.$$

This completes the proof.

## 3 Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. We only need to prove *a priori* estimates (1.6) for simplicity.

First, we have (1.5).

Applying $\Lambda^s$ to (1.2), testing by $\Lambda^s u$ and using (1.1), we see that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\left|\Lambda^s u\right|^2 dx + \int\left|\Lambda^{s+\frac{1}{2}}u\right|^2 dx \\
&\quad = -\int\left(\Lambda^s(u\cdot\nabla u) - u\nabla\Lambda^s u\right)\Lambda^s u\,dx \\
&\qquad + \int\left(\Lambda^s(b\cdot\nabla b) - b\cdot\nabla\Lambda^s b\right)\Lambda^s u\,dx + \int b\cdot\nabla\Lambda^s b\cdot\Lambda^s u\,dx \\
&\quad =: J_1 + J_2 + J_3.
\end{aligned} \tag{3.1}$$

Applying $\Lambda^s$ to (1.3), testing by $\Lambda^s b$ and using (1.1), we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \left|\Lambda^s b\right|^2 dx + \int \left|\Lambda^{s+1} b\right|^2 dx \\
&\quad = -\int \left(\Lambda^s(u\cdot\nabla b) - u\cdot\nabla\Lambda^s b\right)\Lambda^s b\,dx \\
&\qquad + \int \left(\Lambda^s(b\cdot\nabla u) - b\cdot\nabla\Lambda^s u\right)\Lambda^s b\,dx + \int b\cdot\nabla\Lambda^s u\cdot\Lambda^s b\,dx \\
&\quad =: J_4 + J_5 + J_6.
\end{aligned} \tag{3.2}$$

Using (1.7), (2.6), (2.7) and (1.5), we bound $J_1$, $J_2$, $J_3 + J_6$, $J_4$ and $J_5$ as follows:

$$\begin{aligned}
J_1 &\le C\|\nabla u\|_{L^4}\left\|\Lambda^s u\right\|_{L^{8/3}}^2 \\
&\le C\|u\|_{\dot H^{3/2}}\left\|\Lambda^s u\right\|_{L^2}\left\|\Lambda^{s+\frac{1}{2}} u\right\|_{L^2} \\
&\le \frac{1}{8}\left\|\Lambda^{s+\frac{1}{2}} u\right\|_{L^2}^2 + C\|u\|_{\dot H^{3/2}}^2\left\|\Lambda^s u\right\|_{L^2}^2, \\
J_2 &\le C\|\nabla b\|_{L^4}\left\|\Lambda^s b\right\|_{L^2}\left\|\Lambda^s u\right\|_{L^4} \\
&\le C\|\nabla b\|_{L^4}\left\|\Lambda^s b\right\|_{L^2}\left\|\Lambda^{s+\frac{1}{2}} u\right\|_{L^2} \\
&\le \frac{1}{8}\left\|\Lambda^{s+\frac{1}{2}} u\right\|_{L^2}^2 + C\|\nabla b\|_{L^4}^2\left\|\Lambda^s b\right\|_{L^2}^2, \\
J_3 + J_6 &= 0, \\
J_4, J_5 &\le C\|\nabla u\|_{L^2}\left\|\Lambda^s b\right\|_{L^4}^2 + C\|\nabla b\|_{L^{8/3}}\left\|\Lambda^s u\right\|_{L^{8/3}}\left\|\Lambda^s b\right\|_{L^4} \\
&\le C\left\|\Lambda^s b\right\|_{L^4}^2 + C\|\nabla b\|_{L^{8/3}}^2\left\|\Lambda^s u\right\|_{L^{8/3}}^2 \\
&\le C\left\|\Lambda^s b\right\|_{L^2}\left\|\Lambda^{s+1} b\right\|_{L^2} + C\|\nabla b\|_{L^{8/3}}^2\left\|\Lambda^s u\right\|_{L^2}\left\|\Lambda^{s+\frac{1}{2}} u\right\|_{L^2} \\
&\le \frac{1}{8}\left\|\Lambda^{s+1} b\right\|_{L^2}^2 + \frac{1}{8}\left\|\Lambda^{s+\frac{1}{2}} u\right\|_{L^2}^2 + C\left\|\Lambda^s b\right\|_{L^2}^2 + C\|\nabla b\|_{L^{8/3}}^4\left\|\Lambda^s u\right\|_{L^2}^2.
\end{aligned}$$

Adding up (3.1) and (3.2) and using the above estimates, we arrive at

$$\begin{aligned}
&\frac{d}{dt}\int \left(\left|\Lambda^s u\right|^2 + \left|\Lambda^s b\right|^2\right) dx + \int \left(\left|\Lambda^{s+\frac{1}{2}} u\right|^2 + \left|\Lambda^{s+1} b\right|^2\right) dx \\
&\quad \le C\|u\|_{\dot H^{3/2}}^2\left\|\Lambda^s u\right\|_{L^2}^2 + C\|\nabla b\|_{L^4}^2\left\|\Lambda^s b\right\|_{L^2}^2 + C\left\|\Lambda^s b\right\|_{L^2}^2 + C\|\nabla b\|_{L^{8/3}}^4\left\|\Lambda^s u\right\|_{L^2}^2,
\end{aligned}$$

which yields (1.6).

This completes the proof.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
ZW proposed the problems and finished the whole manuscript. WZ modified the proofs. All authors read and approved the final manuscript.

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**References**

1. Ji, E: On two-dimensional magnetohydrodynamic equations with fractional diffusion. Nonlinear Anal. **80**, 55-65 (2013)
2. Mattingly, JC, Sinai, YG: An elementary proof of the existence and uniqueness theorem for the Navier-Stokes equations. Commun. Contemp. Math. **1**, 497-516 (1999)
3. Zhou, Y: Regularity criteria for the generalized viscous MHD equations. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **24**(3), 491-505 (2007)
4. Zhou, Y, Fan, J: A regularity criterion for the 2D MHD system with zero magnetic diffusivity. J. Math. Anal. Appl. **378**(1), 169-172 (2011)
5. Kato, T, Ponce, G: Commutator estimates and the Euler and Navier-Stokes equations. Commun. Pure Appl. Math. **41**, 891-907 (1988)
6. Cao, C, Wu, J: Global regularity for the 2D MHD equations with mixed partial dissipation and magneto diffusion. Adv. Math. **226**, 1803-1822 (2011)

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