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Half-discrete Hardy-Hilbert's inequality with two interval variables

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Abstract
By using the way of weight functions and the technique of real analysis, a half-discrete Hardy-Hilbert's inequality with two interval variables is derived. The equivalent forms, operator expressions, some reverses as well as a few particular cases are obtained.
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1 Introduction

Assuming that $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $f(\geq 0)\in L^p(\mathbf{R}_+)$, $g(\geq 0)\in L^q(\mathbf{R}_+)$,

$$\|f\|_p=\left\{\int_0^\infty f^p(x)\,dx\right\}^{\frac{1}{p}}>0,\qquad \|g\|_q=\left\{\int_0^\infty g^q(y)\,dy\right\}^{\frac{1}{q}}>0,$$

we obtain the following Hardy-Hilbert's integral inequality (*cf.* [1]):

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy<\frac{\pi}{\sin(\pi/p)}\|f\|_p\|g\|_q, \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is best possible. If $a_m,b_n\geq 0$, $a=\{a_m\}_{m=1}^\infty\in l^p$, $b=\{b_n\}_{n=1}^\infty\in l^q$,

$$\|a\|_p=\left\{\sum_{m=1}^\infty a_m^p\right\}^{\frac{1}{p}}>0,\qquad \|b\|_q=\left\{\sum_{n=1}^\infty b_n^q\right\}^{\frac{1}{q}}>0,$$

then we still have the following discrete Hardy-Hilbert's inequality with the same best constant factor $\frac{\pi}{\sin(\pi/p)}$:

$$\sum_{m=1}^\infty\sum_{n=1}^\infty \frac{a_m b_n}{m+n}<\frac{\pi}{\sin(\pi/p)}\|a\|_p\|b\|_q. \tag{2}$$

Inequalities (1) and (2) are important in mathematical analysis and its applications (*cf.* [2–4]). In 1998, Yang [5] proved an extension of (1) (for $p=q=2$) by introducing an independent parameter $\lambda\in(0,1]$. Recently, refining the results of [5], Yang [6] derived some extensions of (1) and (2) as follows: For $\lambda > 0$, $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $\phi(x) = x^{p(1-\frac{\lambda}{r})-1}$, $\psi(x) = x^{q(1-\frac{\lambda}{s})-1}$,

$$0 < \|f\|_{p,\phi} := \left\{ \int_0^\infty \phi(x)\left|f(x)\right|^p dx \right\}^{\frac{1}{p}} < \infty, \quad 0 < \|g\|_{q,\psi} < \infty,$$

$$0 < \|a\|_{p,\phi} := \left\{ \sum_{n=1}^\infty \phi(n)|a_n|^n \right\}^{\frac{1}{p}} < \infty, \quad 0 < \|b\|_{q,\psi} < \infty,$$

we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} \, dx \, dy < B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right) \|f\|_{p,\phi} \|g\|_{q,\psi}, \tag{3}$$

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{(m+n)^\lambda} < B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right) \|a\|_{p,\phi} \|b\|_{q,\psi} \tag{4}$$

($0 < \lambda \leq 2 \min\{r, s\}$), where

$$B(u,v) = \int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}} \, dt \quad (u, v > 0)$$

is the beta function. Some Hilbert-type inequalities about other measurable kernels are provided in [7–14].

Regarding the case of half-discrete Hilbert-type inequalities with non-homogeneous kernels, Hardy, Littlewood and Polya provided some results in Theorem 351 of [1]. However, they had not proved that the constant factors in the new inequalities were best possible. Yang [15] proved some results by introducing an interval variable and that the constant factors are best possible.

In this paper, by using the way of weight functions and the technique of real analysis, a half-discrete Hardy-Hilbert's inequality with the best constant factor is given as follows:

$$\int_0^\infty f(x) \sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda} \, dx < B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right) \|f\|_{p,\phi} \|a\|_{q,\psi} \quad (0 < \lambda \leq s). \tag{5}$$

The best extension of (5) with two interval variables, some equivalent forms, operator expressions, some reverses as well as a few particular cases are also considered.

2 Some lemmas

Lemma 1 *If* $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $\lambda > 0$, $u(x)$ ($x \in (b,c)$, $-\infty \leq b < c \leq \infty$) *and* $v(x)$ ($x \in (n_0 - 1, \infty)$, $n_0 \in \mathbf{N}$) *are strictly increasing differential functions, and* $[v(x)]^{\frac{\lambda}{s}-1} v'(x)$ *is decreasing in* $(n_0 - 1, \infty)$, $u(b^+) = v((n_0 - 1)^+) = 0$, $u(c^-) = v(\infty) = \infty$, $\mathbf{N}_{n_0} = \{n_0, n_0 + 1, \ldots\}$. *Define two weight functions as follows*:

$$\omega(n) := \left[v(n)\right]^{\frac{\lambda}{s}} \int_b^c \frac{[u(x)]^{\frac{\lambda}{r}-1} u'(x)}{(u(x)+v(n))^\lambda} \, dx, \quad n \in \mathbf{N}_{n_0}, \tag{6}$$

$$\varpi(x) := \left[u(x)\right]^{\frac{\lambda}{r}} \sum_{n=n_0}^\infty \frac{[v(n)]^{\frac{\lambda}{s}-1} v'(n)}{(u(x)+v(n))^\lambda}, \quad x \in (b,c). \tag{7}$$

Then the following inequality holds:

$$0 < B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right)\bigl(1 - \theta_\lambda(x)\bigr) < \varpi(x) < \omega(n) = B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right), \tag{8}$$

$$\theta_\lambda(x) := \frac{1}{B(\frac{\lambda}{r}, \frac{\lambda}{s})} \int_0^{\frac{v(n_0)}{u(x)}} \frac{t^{\frac{\lambda}{s}-1}\,dt}{(t+1)^\lambda} = O\left(\frac{1}{[u(x)]^{\lambda/s}}\right), \quad x \in (b,c). \tag{9}$$

Proof Setting $t = \frac{u(x)}{v(n)}$ in (6), we find $dt = \frac{1}{v(n)}u'(x)\,dx$, and

$$\omega(n) = \int_0^\infty \frac{1}{(t+1)^\lambda} t^{\frac{\lambda}{r}-1}\,dt = B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right).$$

For any $x \in (a,b)$, in view of the fact that

$$\frac{1}{(u(x)+v(y))^\lambda}\bigl[v(y)\bigr]^{\frac{\lambda}{s}-1}v'(y) \quad \bigl(y \in (n_0 - 1, \infty)\bigr)$$

is strictly decreasing, we find

$$\begin{aligned}
\varpi(x) &< \bigl[u(x)\bigr]^{\frac{\lambda}{r}} \int_{n_0-1}^\infty \frac{1}{(u(x)+v(y))^\lambda}\bigl[v(y)\bigr]^{\frac{\lambda}{s}-1}v'(y)\,dy \\
&\overset{t=v(y)/u(x)}{=} \int_0^\infty \frac{t^{\frac{\lambda}{s}-1}\,dt}{(t+1)^\lambda} = B\left(\frac{\lambda}{s}, \frac{\lambda}{r}\right) = \omega(n), \\
\varpi(x) &> \bigl[u(x)\bigr]^{\frac{\lambda}{r}} \int_{n_0}^\infty \frac{1}{(u(x)+v(y))^\lambda}\bigl[v(y)\bigr]^{\frac{\lambda}{s}-1}v'(y)\,dy \\
&\overset{t=v(y)/u(x)}{=} \int_{\frac{v(n_0)}{u(x)}}^\infty \frac{1}{(t+1)^\lambda} t^{\frac{\lambda}{s}-1}\,dt \\
&= B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right)\bigl[1 - \theta_\lambda(x)\bigr], \quad x \in (b,c), \\
0 < \theta_\lambda(x) &= \frac{1}{B(\frac{\lambda}{r}, \frac{\lambda}{s})} \int_0^{\frac{v(n_0)}{u(x)}} \frac{1}{(t+1)^\lambda} t^{\frac{\lambda}{s}-1}\,dt \\
&< \frac{1}{B(\frac{\lambda}{r}, \frac{\lambda}{s})} \int_0^{\frac{v(n_0)}{u(x)}} t^{\frac{\lambda}{s}-1}\,dt = \frac{s}{\lambda B(\frac{\lambda}{r}, \frac{\lambda}{s})}\left(\frac{v(n_0)}{u(x)}\right)^{\frac{\lambda}{s}}.
\end{aligned}$$

Hence, we have (8) and (9). □

Lemma 2 *Let the assumptions of Lemma* 1 *be fulfilled and, additionally,* $p > 0$ $(p \neq 1)$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n \geq 0$, $n \geq n_0$ $(n \in \mathbf{N})$, $f(x)$ *is a non-negative measurable function in* (b,c). *Then*

(i) *For* $p > 1$, *we have the following inequalities*:

$$\begin{aligned}
J_1 &:= \left\{\sum_{n=n_0}^\infty \frac{v'(n)}{[v(n)]^{1-\frac{p\lambda}{s}}}\left[\int_b^c \frac{f(x)}{(u(x)+v(n))^\lambda}\,dx\right]^p\right\}^{\frac{1}{p}} \\
&\leq \left[B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right)\right]^{\frac{1}{q}} \left\{\int_b^c \varpi(x)\frac{[u(x)]^{p(1-\frac{\lambda}{r})-1}}{[u'(x)]^{p-1}} f^p(x)\,dx\right\}^{\frac{1}{p}},
\end{aligned} \tag{10}$$

$$L_1 := \left\{ \int_b^c \frac{[\varpi(x)]^{1-q} u'(x)}{[u(x)]^{1-\frac{q\lambda}{r}}} \left[\sum_{n=n_0}^{\infty} \frac{a_n}{(u(x)+v(n))^{\lambda}} \right]^q dx \right\}^{\frac{1}{q}}$$
$$\le \left\{ B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right) \sum_{n=n_0}^{\infty} \frac{[v(n)]^{q(1-\frac{\lambda}{s})-1}}{[v'(n)]^{q-1}} a_n^q \right\}^{\frac{1}{q}}. \tag{11}$$

(ii) *For* $0 < p < 1$, *we have the reverses of* (10) *and* (11).

Proof (i) By Hölder's inequality with weight (*cf.* [16]) and (8), it follows that

$$\left[\int_b^c \frac{f(x)}{(u(x)+v(n))^{\lambda}} dx \right]^p$$
$$= \left\{ \int_b^c \frac{1}{(u(x)+v(n))^{\lambda}} \left[\frac{[u(x)]^{(1-\frac{\lambda}{r})/q}}{[v(n)]^{(1-\frac{\lambda}{s})/p}} \frac{[v'(n)]^{1/p}}{[u'(x)]^{1/q}} f(x) \right] \left[\frac{[v(n)]^{(1-\frac{\lambda}{s})/p}}{[u(x)]^{(1-\frac{\lambda}{r})/q}} \frac{[u'(x)]^{1/q}}{[v'(n)]^{1/p}} \right] dx \right\}^p$$
$$\le \int_b^c \frac{v'(n)}{(u(x)+v(n))^{\lambda}} \frac{[u(x)]^{(1-\frac{\lambda}{r})(p-1)}}{[v(n)]^{1-\frac{\lambda}{s}} [u'(x)]^{p-1}} f^p(x)\, dx$$
$$\times \left\{ \int_b^c \frac{u'(x)}{(u(x)+v(n))^{\lambda}} \frac{[v(n)]^{(1-\frac{\lambda}{s})(q-1)}}{[u(x)]^{1-\frac{\lambda}{r}} [v'(n)]^{q-1}} dx \right\}^{p-1}$$
$$= \left\{ \frac{\omega(n)[v(n)]^{q(1-\frac{\lambda}{s})-1}}{[v'(n)]^{q-1}} \right\}^{p-1} \int_b^c \frac{v'(n) f^p(x)}{(u(x)+v(n))^{\lambda}} \frac{[u(x)]^{(1-\frac{\lambda}{r})(p-1)}\, dx}{[v(n)]^{1-\frac{\lambda}{s}} [u'(x)]^{p-1}}$$
$$= \frac{[B(\frac{\lambda}{r}, \frac{\lambda}{s})]^{p-1}}{[v(n)]^{\frac{p\lambda}{s}-1} v'(n)} \int_b^c \frac{v'(n) f^p(x)}{(u(x)+v(n))^{\lambda}} \frac{[u(x)]^{(1-\frac{\lambda}{r})(p-1)}\, dx}{[v(n)]^{1-\frac{\lambda}{s}} [u'(x)]^{p-1}}.$$

Then, by the Lebesgue term-by-term integration theorem (*cf.* [17]), we obtain

$$J_1 \le \left[B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right) \right]^{\frac{1}{q}} \left\{ \sum_{n=n_0}^{\infty} \int_b^c \frac{v'(n) f^p(x)}{(u(x)+v(n))^{\lambda}} \frac{[u(x)]^{(1-\frac{\lambda}{r})(p-1)}\, dx}{[v(n)]^{1-\frac{\lambda}{s}} [u'(x)]^{p-1}} \right\}^{\frac{1}{p}}$$
$$= \left[B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right) \right]^{\frac{1}{q}} \left\{ \int_b^c \sum_{n=n_0}^{\infty} \frac{v'(n) f^p(x)}{(u(x)+v(n))^{\lambda}} \frac{[u(x)]^{(1-\frac{\lambda}{r})(p-1)}\, dx}{[v(n)]^{1-\frac{\lambda}{s}} [u'(x)]^{p-1}} \right\}^{\frac{1}{p}}$$
$$= \left[B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right) \right]^{\frac{1}{q}} \left\{ \int_b^c \varpi(x) \frac{[u(x)]^{p(1-\frac{\lambda}{r})-1}}{[u'(x)]^{p-1}} f^p(x)\, dx \right\}^{\frac{1}{p}},$$

and (10) follows.

Still, by Hölder's inequality, we have

$$\left[\sum_{n=n_0}^{\infty} \frac{a_n}{(u(x)+v(n))^{\lambda}} \right]^q$$
$$= \left\{ \sum_{n=n_0}^{\infty} \frac{1}{(u(x)+v(n))^{\lambda}} \left[\frac{[u(x)]^{(1-\frac{\lambda}{r})/q}}{[v(n)]^{(1-\frac{\lambda}{s})/p}} \frac{[v'(n)]^{1/p}}{[u'(x)]^{1/q}} \right] \left[\frac{[v(n)]^{(1-\frac{\lambda}{s})/p}}{[u(x)]^{(1-\frac{\lambda}{r})/q}} \frac{[u'(x)]^{1/q}}{[v'(n)]^{1/p}} a_n \right] \right\}^q$$
$$\le \left\{ \sum_{n=n_0}^{\infty} \frac{1}{(u(x)+v(n))^{\lambda}} \frac{[u(x)]^{(1-\frac{\lambda}{r})(p-1)}}{[v(n)]^{1-\frac{\lambda}{s}}} \frac{v'(n)}{[u'(x)]^{p-1}} \right\}^{q-1}$$

$$\times \sum_{n=n_0}^{\infty} \frac{1}{(u(x)+v(n))^{\lambda}} \frac{[v(n)]^{(1-\frac{\lambda}{s})(q-1)}}{[u(x)]^{1-\frac{\lambda}{r}}} \frac{u'(x)}{[v'(n)]^{q-1}} a_n^q$$

$$= \frac{[u(x)]^{1-\frac{q\lambda}{r}}}{[\varpi(x)]^{1-q} u'(x)} \sum_{n=n_0}^{\infty} \frac{[u(x)]^{\frac{\lambda}{r}-1} u'(x) [v(n)]^{\frac{\lambda}{s}}}{(u(x)+v(n))^{\lambda}} \frac{[v(n)]^{q(1-\frac{\lambda}{s})-1}}{[v'(n)]^{q-1}} a_n^q.$$

Then, by the Lebesgue term-by-term integration theorem, we have

$$\begin{aligned} L_1 &\le \left\{ \int_b^c \sum_{n=n_0}^{\infty} \frac{[u(x)]^{\frac{\lambda}{r}-1} u'(x) [v(n)]^{\frac{\lambda}{s}}}{(u(x)+v(n))^{\lambda}} \frac{[v(n)]^{q(1-\frac{\lambda}{s})-1}}{[v'(n)]^{q-1}} a_n^q \, dx \right\}^{\frac{1}{q}} \\ &= \left\{ \sum_{n=n_0}^{\infty} \left[\bigl[v(n)\bigr]^{\frac{\lambda}{s}} \int_b^c \frac{[u(x)]^{\frac{\lambda}{r}-1} u'(x)}{(u(x)+v(n))^{\lambda}} \, dx \right] \frac{[v(n)]^{q(1-\frac{\lambda}{s})-1}}{[v'(n)]^{q-1}} a_n^q \right\}^{\frac{1}{q}} \\ &= \left\{ \sum_{n=n_0}^{\infty} \omega(n) \frac{[v(n)]^{q(1-\frac{\lambda}{s})-1}}{[v'(n)]^{q-1}} a_n^q \right\}^{\frac{1}{q}}, \end{aligned}$$

and then, in view of (8), inequality (11) follows.

(ii) By reverse Hölder's inequality (*cf.* [16]) and in the same way, for $q < 0$, we can obtain the reverses of (10) and (11). □

3 Main results

We set $\Phi(x) := \frac{[u(x)]^{p(1-\frac{\lambda}{r})-1}}{[u'(x)]^{p-1}}$, $\widetilde{\Phi}(x) := (1-\theta_\lambda(x))\Phi(x)$ $(x \in (b,c))$, and

$$\Psi(n) := \frac{[v(n)]^{q(1-\frac{\lambda}{s})-1}}{[v'(n)]^{q-1}} \quad (n \ge n_0, n \in \mathbf{N})$$

($\theta_\lambda(x)$ is indicated by (9)), wherefrom

$$\bigl[\Phi(x)\bigr]^{1-q} = \frac{u'(x)}{[u(x)]^{1-\frac{q\lambda}{r}}}, \qquad \bigl[\Psi(n)\bigr]^{1-p} = \frac{v'(n)}{[v(n)]^{1-\frac{p\lambda}{s}}}.$$

Theorem 3 *Let the assumptions of Lemma* 1 *be fulfilled and, additionally,* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x) \ge 0$ $(x \in (b,c))$, $a_n \ge 0$ $(n \ge n_0,\ n \in \mathbf{N})$, $f \in L_{p,\Phi}(b,c)$, $a = \{a_n\}_{n=n_0}^{\infty} \in l_{q,\Psi}$,

$$0 < \|f\|_{p,\Phi} = \left\{ \int_b^c \Phi(x) f^p(x) \, dx \right\}^{\frac{1}{p}} < \infty$$

and $0 < \|a\|_{q,\Psi} = \{\sum_{n=n_0}^{\infty} \Psi(n) a_n^q\}^{\frac{1}{q}} < \infty$. *Then the following equivalent inequalities hold*:

$$\begin{aligned} I &:= \sum_{n=n_0}^{\infty} \int_b^c \frac{a_n f(x) \, dx}{(u(x)+v(n))^{\lambda}} = \int_b^c \sum_{n=n_0}^{\infty} \frac{a_n f(x) \, dx}{(u(x)+v(n))^{\lambda}} \\ &< B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right) \|f\|_{p,\Phi} \|a\|_{q,\Psi}, \end{aligned} \tag{12}$$

$$J := \left\{ \sum_{n=n_0}^{\infty} \left[\Psi(n)\right]^{1-p} \left[\int_b^c \frac{f(x)}{(u(x)+v(n))^{\lambda}}\,dx \right]^p \right\}^{\frac{1}{p}} < B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right) \|f\|_{p,\Phi}, \tag{13}$$

$$L := \left\{ \int_b^c \left[\Phi(x)\right]^{1-q} \left[\sum_{n=n_0}^{\infty} \frac{a_n}{(u(x)+v(n))^{\lambda}} \right]^q dx \right\}^{\frac{1}{q}} < B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right) \|a\|_{q,\Psi}, \tag{14}$$

where the constant factor $B(\frac{\lambda}{r}, \frac{\lambda}{s})$ *is best possible.*

Proof By the Lebesgue term-by-term integration theorem, there are two expressions for I in (12). In view of (8) and (10), we obtain (13).

By Hölder's inequality, we have

$$I = \sum_{n=n_0}^{\infty} \left[\Psi^{\frac{-1}{q}}(n) \int_b^c \frac{f(x)\,dx}{(u(x)+v(n))^{\lambda}} \right] \left[\Psi^{\frac{1}{q}}(n) a_n \right] \le J \|a\|_{q,\Psi}. \tag{15}$$

Then, by (13), we have (12). On the other hand, assuming that (12) is valid, we set

$$a_n := \left[\Psi(n)\right]^{1-p} \left[\int_b^c \frac{f(x)}{(u(x)+v(n))^{\lambda}}\,dx \right]^{p-1}, \quad n \ge n_0,$$

then it follows that $J^{p-1} = \|a\|_{q,\Psi}$. By (10), we find $J < \infty$. If $J = 0$, then (13) is trivially valid; if $J > 0$, then, by (12), we have

$$\|a\|_{q,\Psi}^q = J^p = I < B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right) \|f\|_{p,\Phi} \|a\|_{q,\Psi},$$

$$\|a\|_{q,\Psi}^{q-1} = J < B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right) \|f\|_{p,\Phi},$$

and thus we get (13), which is equivalent to (12).

In view of (8) and (11), we have (14).

By Hölder's inequality, we find

$$I = \int_b^c \left[\Phi^{\frac{1}{p}}(x) f(x) \right] \left[\Phi^{\frac{-1}{p}}(x) \sum_{n=n_0}^{\infty} \frac{a_n}{(u(x)+v(n))^{\lambda}} \right] dx \le \|f\|_{p,\Phi} L. \tag{16}$$

Then, by (14), we have (12). On the other hand, assuming that (12) is valid, we set

$$f(x) := \left[\Phi(x)\right]^{1-q} \left[\sum_{n=n_0}^{\infty} \frac{a_n}{(u(x)+v(n))^{\lambda}} \right]^{q-1}, \quad x \in (b,c),$$

then it follows that $L^{q-1} = \|f\|_{p,\Phi}$. By (11), we find $L < \infty$. If $L = 0$, then (14) is trivially valid; if $L > 0$, then, by (12), we have

$$\|f\|_{p,\Phi}^{p} = L^{q} = I < B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right)\|f\|_{p,\Phi}\|a\|_{q,\Psi},$$

$$\|f\|_{p,\Phi}^{p-1} = L < B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right)\|a\|_{q,\Psi},$$

and we have (14), which is equivalent to (12).

Hence inequalities (12), (13) and (14) are equivalent.

There exists a unified constant $d \in (b,c)$ satisfying $u(d) = 1$. For $0 < \varepsilon < \frac{q\lambda}{s}$, setting

$$\widetilde{f}(x) := 0, \quad x \in (b,d); \qquad \widetilde{f}(x) := \left[u(x)\right]^{\frac{\lambda}{r}-\frac{\varepsilon}{p}-1}u'(x), \quad x \in [d,c),$$

$\widetilde{a}_n := [v(n)]^{\frac{\lambda}{s}-\frac{\varepsilon}{q}-1}v'(n)$, $n \geq n_0$, if there exists a positive number k ($\leq B(\frac{\lambda}{r}, \frac{\lambda}{s})$) such that (12) is valid when replacing $B(\frac{\lambda}{r}, \frac{\lambda}{s})$ by k, then, in particular, we have

$$\begin{aligned}
\widetilde{I} &:= \int_b^c \sum_{n=n_0}^{\infty} \frac{\widetilde{a}_n}{(u(x)+v(n))^{\lambda}}\widetilde{f}(x)\,dx < k\|\widetilde{f}\|_{p,\Phi}\|\widetilde{a}\|_{q,\Psi} \\
&= k\left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}}\left\{\left[v(n_0)\right]^{-\varepsilon-1}v'(n_0) + \sum_{n=n_0+1}^{\infty}\left[v(n)\right]^{-\varepsilon-1}v'(n)\right\}^{\frac{1}{q}} \\
&< k\left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}}\left\{\left[v(n_0)\right]^{-\varepsilon-1}v'(n_0) + \int_{n_0}^{\infty}\left[v(y)\right]^{-\varepsilon-1}v'(y)\,dy\right\}^{\frac{1}{q}} \\
&= \frac{k}{\varepsilon}\left\{\varepsilon\left[v(n_0)\right]^{-\varepsilon-1}v'(n_0) + \left[v(n_0)\right]^{-\varepsilon}\right\}^{\frac{1}{q}}.
\end{aligned} \tag{17}$$

In view of the decreasing property of $\frac{[v(y)]^{\frac{\lambda}{s}-\frac{\varepsilon}{q}-1}v'(y)}{(u(x)+v(y))^{\lambda}}$, we find

$$\begin{aligned}
\widetilde{I} &= \int_d^c \left[u(x)\right]^{\frac{\lambda}{r}-\frac{\varepsilon}{p}-1}u'(x)\sum_{n=n_0}^{\infty}\frac{[v(n)]^{\frac{\lambda}{s}-\frac{\varepsilon}{q}-1}v'(n)}{(u(x)+v(n))^{\lambda}}\,dx \\
&\geq \int_d^c \left[u(x)\right]^{\frac{\lambda}{r}-\frac{\varepsilon}{p}-1}u'(x)\left[\int_{n_0}^{\infty}\frac{[v(y)]^{\frac{\lambda}{s}-\frac{\varepsilon}{q}-1}v'(y)}{(u(x)+v(y))^{\lambda}}\,dy\right]dx \\
&\overset{t=v(y)/u(x)}{=} \int_d^c \left[u(x)\right]^{-\varepsilon-1}u'(x)\left[\int_{\frac{v(n_0)}{u(x)}}^{\infty}\frac{t^{\frac{\lambda}{s}-\frac{\varepsilon}{q}-1}}{(1+t)^{\lambda}}\,dt\right]dx \\
&= \int_d^c \left[u(x)\right]^{-\varepsilon-1}u'(x)\left[\int_0^{\infty}\frac{t^{\frac{\lambda}{s}-\frac{\varepsilon}{q}-1}\,dt}{(1+t)^{\lambda}} - \int_0^{\frac{v(n_0)}{u(x)}}\frac{t^{\frac{\lambda}{s}-\frac{\varepsilon}{q}-1}\,dt}{(1+t)^{\lambda}}\right]dx \\
&= \frac{1}{\varepsilon}B\left(\frac{\lambda}{s}-\frac{\varepsilon}{q}, \frac{\lambda}{r}+\frac{\varepsilon}{q}\right) - A(x),
\end{aligned} \tag{18}$$

$$A(x) := \int_d^c \left[u(x)\right]^{-\varepsilon-1}u'(x)\left[\int_0^{\frac{v(n_0)}{u(x)}}\frac{t^{\frac{\lambda}{s}-\frac{\varepsilon}{q}-1}}{(1+t)^{\lambda}}\,dt\right]dx.$$

Since we find

$$\begin{aligned}0 < A(x) &< \int_d^c \bigl[u(x)\bigr]^{-\varepsilon-1} u'(x) \biggl[\int_0^{\frac{v(n_0)}{u(x)}} t^{\frac{\lambda}{s}-\frac{\varepsilon}{q}-1}\,dt\biggr]\,dx \\ &= \frac{[v(n_0)]^{\frac{\lambda}{s}-\frac{\varepsilon}{q}}}{(\frac{\lambda}{s}-\frac{\varepsilon}{q})(\frac{\lambda}{s}+\frac{\varepsilon}{p})},\end{aligned}$$

then it follows that

$$A(x) = O(1) \quad \bigl(\varepsilon \to 0^+\bigr).$$

By (17) and (18), we have

$$\begin{aligned}&B\biggl(\frac{\lambda}{s}-\frac{\varepsilon}{q}, \frac{\lambda}{r}+\frac{\varepsilon}{q}\biggr) - \varepsilon O(1) \\ &\quad < k\bigl\{\varepsilon\bigl[v(n_0)\bigr]^{-\varepsilon-1} v'(n_0) + \bigl[v(n_0)\bigr]^{-\varepsilon}\bigr\}^{\frac{1}{q}},\end{aligned}$$

and then $B(\frac{\lambda}{r}, \frac{\lambda}{s}) \leq k$ ($\varepsilon \to 0^+$). Hence $k = B(\frac{\lambda}{r}, \frac{\lambda}{s})$ is the best possible constant factor of (12).

We conform that the constant factor $B(\frac{\lambda}{r}, \frac{\lambda}{s})$ in (13) ((14)) is best possible. Otherwise, we would reach a contradiction by (15) ((16)) that the constant factor in (12) is not best possible. □

Remark 1 We set two weight normed spaces as follows:

$$L_{p,\Phi}(b,c) = \bigl\{f; \|f\|_{p,\Phi} < \infty\bigr\}, \qquad l_{q,\Psi} = \bigl\{a = \{a_n\}_{n=n_0}^{\infty}; \|a\|_{q,\Psi} < \infty\bigr\}.$$

(i) Define a half-discrete Hilbert's operator as follows: $T : L_{p,\Phi}(b,c) \to l_{p,\Psi^{1-p}}$, for $f \in L_{p,\Phi}(b,c)$, there exists a unified representation $Tf \in l_{p,\Psi^{1-p}}$ satisfying

$$Tf(n) = \int_b^c \frac{f(x)}{(u(x)+v(n))^{\lambda}}\,dx, \quad n \geq n_0.$$

Then, by (12), it follows that

$$\|Tf\|_{p,\Psi^{1-p}} < B\biggl(\frac{\lambda}{r}, \frac{\lambda}{s}\biggr)\|f\|_{p,\Phi},$$

and then T is bounded with

$$\|T\| \leq B\biggl(\frac{\lambda}{r}, \frac{\lambda}{s}\biggr).$$

Since the constant factor in (13) is best possible, we have $\|T\| = B(\frac{\lambda}{r}, \frac{\lambda}{s})$.

(ii) Define a half-discrete Hilbert's operator as follows:

$$\widetilde{T} : l_{q,\Psi} \to L_{q,\Phi^{1-q}}(b,c),$$

for $a \in l_{q,\Psi}$, there exists a unified representation $\widetilde{T}a \in L_{q,\Phi^{1-q}}(b,c)$ satisfying

$$(\widetilde{T}a)(x) = \sum_{n=n_0}^{\infty} \frac{a_n}{(u(x)+v(n))^{\lambda}}, \quad x \in (b,c).$$

Then, by (13) it follows that

$$\|\widetilde{T}a\|_{q\cdot\Phi^{1-q}} < B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right)\|a\|_{q,\Psi},$$

and then $\widetilde{T}$ is bounded with

$$\|\widetilde{T}\| \le B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right).$$

Since the constant factor in (14) is best possible, we have $\|\widetilde{T}\| = B(\frac{\lambda}{r}, \frac{\lambda}{s})$.

In the following theorem, for $0 < p < 1$, we still use the formal symbols of $\|f\|_{p,\widetilde{\Phi}}$ and $\|a\|_{q,\Psi}$ *et al.*

Theorem 4 *Let the assumptions of Lemma* 1 *be fulfilled and, additionally,* $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1, f(x) \ge 0$ $(x \in (b,c))$, $a_n \ge 0$ $(n \ge n_0, n \in \mathbf{N})$,

$$0 < \|f\|_{p,\widetilde{\Phi}} = \left\{\int_b^c \bigl(1-\theta_\lambda(x)\bigr)\Phi(x)f^p(x)\,dx\right\}^{\frac{1}{p}} < \infty$$

and $0 < \|a\|_{q,\Psi} = \{\sum_{n=n_0}^{\infty} \Psi(n)a_n^q\}^{\frac{1}{q}} < \infty$. *Other conditions are similar to those in Theorem* 3, *then we have the following equivalent inequalities*:

$$I := \sum_{n=n_0}^{\infty} \int_b^c \frac{a_n f(x)\,dx}{(u(x)+v(n))^{\lambda}} = \int_b^c \sum_{n=n_0}^{\infty} \frac{a_n f(x)\,dx}{(u(x)+v(n))^{\lambda}}$$
$$> B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right)\|f\|_{p,\widetilde{\Phi}}\|a\|_{q,\Psi}, \tag{19}$$

$$J := \left\{\sum_{n=n_0}^{\infty} \bigl[\Psi(n)\bigr]^{1-p}\left[\int_b^c \frac{f(x)}{(u(x)+v(n))^{\lambda}}\,dx\right]^p\right\}^{\frac{1}{p}}$$
$$> B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right)\|f\|_{p,\widetilde{\Phi}}, \tag{20}$$

$$\widetilde{L} := \left\{\int_b^c \bigl[\widetilde{\Phi}(x)\bigr]^{1-q}\left[\sum_{n=n_0}^{\infty} \frac{a_n}{(u(x)+v(n))^{\lambda}}\right]^q dx\right\}^{\frac{1}{q}}$$
$$> B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right)\|a\|_{q,\Psi}. \tag{21}$$

Moreover, if there exists a constant $\delta_0 > 0$ *such that for any* $\delta \in [0,\delta_0)$, $[v(y)]^{\frac{\lambda}{s}+\delta-1}v'(y)$ *is decreasing in* (n_0-1,∞), *then the constant factor* $B(\frac{\lambda}{r}, \frac{\lambda}{s})$ *in the above inequalities is best possible.*

Proof In view of (8) and the reverse of (10), for

$$\varpi(x) > B\left(\frac{\lambda}{r},\frac{\lambda}{s}\right)\bigl(1-\theta_\lambda(x)\bigr),$$

we have (20).

By reverse Hölder's inequality, we obtain

$$\begin{aligned} I &= \sum_{n=n_0}^{\infty}\left[\Psi^{\frac{-1}{q}}(n)\int_b^c \frac{f(x)\,dx}{(u(x)+v(n))^\lambda}\right]\left[\Psi^{\frac{1}{q}}(n)a_n\right] \\ &\geq J\|a\|_{q,\Psi}. \end{aligned} \tag{22}$$

Then, by (20), we have (19). On the other hand, assuming that (19) is valid, we set a_n as in Theorem 3, then it follows that $J^{p-1} = \|a\|_{q,\Psi}$. By the reverse of (10), we find $J > 0$. If $J = \infty$, then (20) is trivially valid; if $J < \infty$, then, by (19), we have

$$\|a\|_{q,\Psi}^q = J^p = I > B\left(\frac{\lambda}{r},\frac{\lambda}{s}\right)\|f\|_{p,\widetilde{\Phi}}\|a\|_{q,\Psi},$$

$$\|a\|_{q,\Psi}^{q-1} = J > B\left(\frac{\lambda}{r},\frac{\lambda}{s}\right)\|f\|_{p,\widetilde{\Phi}},$$

and we have (20), which is equivalent to (19).

In view of (8) and the reverse of (11), for

$$\bigl[\varpi(x)\bigr]^{1-q} > \left[B\left(\frac{\lambda}{r},\frac{\lambda}{s}\right)\bigl(1-\theta_\lambda(x)\bigr)\right]^{1-q} \quad (q<0),$$

we have (21).

By reverse Hölder's inequality, we have

$$\begin{aligned} I &= \int_b^c \left[\widetilde{\Phi}^{\frac{1}{p}}(x)f(x)\right]\left[\widetilde{\Phi}^{\frac{-1}{p}}(x)\sum_{n=n_0}^{\infty}\frac{a_n}{(u(x)+v(n))^\lambda}\right]dx \\ &\geq \|f\|_{p,\widetilde{\Phi}}\widetilde{L}. \end{aligned} \tag{23}$$

Then, by (21), we have (19). On the other hand, assuming that (19) is valid, setting

$$f(x) := \bigl[\widetilde{\Phi}(x)\bigr]^{1-q}\left[\sum_{n=n_0}^{\infty}\frac{a_n}{(u(x)+v(n))^\lambda}\right]^{q-1}, \quad x\in(b,c),$$

then $\widetilde{L}^{q-1} = \|f\|_{p,\widetilde{\Phi}}$. By the reverse of (11), we find $\widetilde{L} > 0$. If $\widetilde{L} = \infty$, then (21) is trivially valid; if $\widetilde{L} < \infty$, then, by (19), we have

$$\|f\|_{p,\widetilde{\Phi}}^p = \widetilde{L}^q = I > B\left(\frac{\lambda}{r},\frac{\lambda}{s}\right)\|f\|_{p,\widetilde{\Phi}}\|a\|_{q,\Psi},$$

$$\|f\|_{p,\widetilde{\Phi}}^{p-1} = \widetilde{L} > B\left(\frac{\lambda}{r},\frac{\lambda}{s}\right)\|a\|_{q,\Psi},$$

and we have (21), which is equivalent to (19).

Hence inequalities (19), (20) and (21) are equivalent.

For $0 < \varepsilon < \min\{\frac{|q|\lambda}{r}, |q|\delta_0\}$, setting $\widetilde{f}(x) = 0$, $x \in (b,d)$;

$$\widetilde{f}(x) = \left[u(x)\right]^{\frac{\lambda}{r} - \frac{\varepsilon}{p} - 1} u'(x), \quad x \in [d,c),$$

$\widetilde{a}_n = [\nu(n)]^{\frac{\lambda}{s} - \frac{\varepsilon}{q} - 1} \nu'(n)$, $n \geq n_0$, if there exists a positive number k ($\geq B(\frac{\lambda}{r}, \frac{\lambda}{s})$) such that (19) is still valid when replacing $B(\frac{\lambda}{r}, \frac{\lambda}{s})$ by k, then, in particular, for $q < 0$, in view of (9), we have

$$\begin{aligned}
\widetilde{I} &:= \int_b^c \sum_{n=n_0}^{\infty} \frac{\widetilde{a}_n \widetilde{f}(x)\,dx}{(u(x)+\nu(n))^{\lambda}} > k \|\widetilde{f}\|_{p,\widetilde{\Phi}} \|\widetilde{a}\|_{q,\Psi} \\
&= k\left\{\int_d^c \left(1 - O\left(\frac{1}{[u(x)]^{\lambda/s}}\right)\right)\left[u(x)\right]^{-\varepsilon-1} u'(x)\,dx\right\}^{\frac{1}{p}} \left\{\sum_{n=n_0}^{\infty}\left[\nu(n)\right]^{-\varepsilon-1}\nu'(n)\right\}^{\frac{1}{q}} \\
&= k\left\{\frac{1}{\varepsilon} - O(1)\right\}^{\frac{1}{p}} \left\{\left[\nu(n_0)\right]^{-\varepsilon-1}\nu'(n_0) + \sum_{n=n_0+1}^{\infty}\left[\nu(n)\right]^{-\varepsilon-1}\nu'(n)\right\}^{\frac{1}{q}} \\
&> k\left\{\frac{1}{\varepsilon} - O(1)\right\}^{\frac{1}{p}} \left\{\left[\nu(n_0)\right]^{-\varepsilon-1}\nu'(n_0) + \int_{n_0}^{\infty}\left[\nu(y)\right]^{-\varepsilon-1}\nu'(y)\,dy\right\}^{\frac{1}{q}} \\
&= \frac{k}{\varepsilon}\left\{1 - \varepsilon O(1)\right\}^{\frac{1}{p}} \left\{\varepsilon\left[\nu(n_0)\right]^{-\varepsilon-1}\nu'(n_0) + \left[\nu(n_0)\right]^{-\varepsilon}\right\}^{\frac{1}{q}}.
\end{aligned} \tag{24}$$

In view of the decreasing property of $\frac{[\nu(y)]^{\frac{\lambda}{s} - \frac{\varepsilon}{q} - 1}\nu'(y)}{(u(x)+\nu(y))^{\lambda}}$, we find

$$\begin{aligned}
\widetilde{I} &= \int_d^c \left[u(x)\right]^{\frac{\lambda}{r} - \frac{\varepsilon}{p} - 1} u'(x) \sum_{n=n_0}^{\infty} \frac{[\nu(n)]^{\frac{\lambda}{s} - \frac{\varepsilon}{q} - 1}\nu'(n)}{(u(x)+\nu(n))^{\lambda}}\,dx \\
&\leq \int_d^c \left[u(x)\right]^{\frac{\lambda}{r} - \frac{\varepsilon}{p} - 1} u'(x) \left[\int_{n_0-1}^{\infty} \frac{[\nu(y)]^{\frac{\lambda}{s} - \frac{\varepsilon}{q} - 1}\nu'(y)}{(u(x)+\nu(y))^{\lambda}}\,dy\right] dx \\
&\overset{t=\nu(y)/u(x)}{=} \int_d^c \left[u(x)\right]^{-\varepsilon-1} u'(x) \left[\int_0^{\infty} \frac{t^{\frac{\lambda}{s} - \frac{\varepsilon}{q} - 1}}{(1+t)^{\lambda}}\,dt\right] dx \\
&= \frac{1}{\varepsilon} B\left(\frac{\lambda}{s} - \frac{\varepsilon}{q}, \frac{\lambda}{r} + \frac{\varepsilon}{q}\right).
\end{aligned} \tag{25}$$

By (24) and (25), we have

$$\begin{aligned}
&B\left(\frac{\lambda}{s} - \frac{\varepsilon}{q}, \frac{\lambda}{r} + \frac{\varepsilon}{q}\right) \\
&\quad > k\left\{1 - \varepsilon O(1)\right\}^{\frac{1}{p}} \left\{\varepsilon\left[\nu(n_0)\right]^{-\varepsilon-1}\nu'(n_0) + \left[\nu(n_0)\right]^{-\varepsilon}\right\}^{\frac{1}{q}},
\end{aligned}$$

and then

$$B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right) \geq k \quad \left(\varepsilon \to 0^+\right).$$

Hence $k = B(\frac{\lambda}{r}, \frac{\lambda}{s})$ is the best possible constant factor of (19).

We conform that the constant factor $B(\frac{\lambda}{r},\frac{\lambda}{s})$ in (20) ((21)) is best possible. Otherwise, we would reach a contradiction by (22) ((23)) that the constant factor in (19) is not best possible. □

Remark 2 (i) If $\alpha > 0$, $u(x) = x^{\alpha}$, $b = 0$, $c = \infty$, $v(n) = n^{\alpha}$, $n_0 = 1$, in view of $[v(x)]^{\frac{\lambda}{s}-1}v'(x) = \alpha x^{\frac{\alpha\lambda}{s}-1}$ is decreasing, then we have $0 < \alpha\lambda \le s$ and for $\alpha = 1$, $0 < \lambda \le s$, $u(x) = x$ ($x \in (0,\infty)$), $v(n) = n$ ($n \in \mathbf{N}$) in (12), (13) and (14), we have (5) and the following equivalent inequalities:

$$\left\{\sum_{n=1}^{\infty} n^{\frac{p\lambda}{s}-1}\left[\int_0^{\infty}\frac{f(x)}{(x+n)^{\lambda}}\,dx\right]^p\right\}^{\frac{1}{p}} < B\left(\frac{\lambda}{r},\frac{\lambda}{s}\right)\|f\|_{p,\phi}, \tag{26}$$

$$\left\{\int_0^{\infty} x^{\frac{q\lambda}{r}-1}\left[\sum_{n=1}^{\infty}\frac{a_n}{(x+n)^{\lambda}}\right]^q dx\right\}^{\frac{1}{q}} < B\left(\frac{\lambda}{r},\frac{\lambda}{s}\right)\|a\|_{q,\psi}. \tag{27}$$

(ii) For $u(x) = \ln x$, $b = 1$, $c = \infty$, $v(n) = \ln n$, $n_0 = 2$, $0 < \lambda \le s$ in (12), (13) and (14), we have the following half-discrete Mulholland's inequality and its equivalent forms:

$$\int_1^{\infty} f(x)\sum_{n=2}^{\infty}\frac{a_n}{(\ln nx)^{\lambda}}\,dx < B\left(\frac{\lambda}{r},\frac{\lambda}{s}\right)\|f\|_{p,\widehat{\phi}}\|a\|_{q,\widehat{\psi}}, \tag{28}$$

$$\left\{\sum_{n=2}^{\infty}\frac{(\ln n)^{\frac{p\lambda}{s}-1}}{n}\left[\int_1^{\infty}\frac{f(x)}{(\ln nx)^{\lambda}}\,dx\right]^p\right\}^{\frac{1}{p}} < B\left(\frac{\lambda}{r},\frac{\lambda}{s}\right)\|f\|_{p,\widehat{\phi}}, \tag{29}$$

$$\left\{\int_1^{\infty}\frac{(\ln x)^{\frac{q\lambda}{r}-1}}{x}\left[\sum_{n=2}^{\infty}\frac{a_n}{(\ln nx)^{\lambda}}\right]^q dx\right\}^{\frac{1}{q}} < B\left(\frac{\lambda}{r},\frac{\lambda}{s}\right)\|a\|_{q,\widehat{\psi}}, \tag{30}$$

where $\widehat{\phi}(x) = (\ln x)^{p(1-\frac{\lambda}{r})-1}x^{p-1}$ and $\widehat{\psi}(n) = (\ln n)^{q(1-\frac{\lambda}{s})-1}n^{q-1}$.

(iii) For $x = \frac{1}{t}$, $g(t) = t^{\lambda-2}f(\frac{1}{t})$ in (5), (26) and (27), we can obtain the following equivalent inequalities with non-homogeneous kernel and the best constant factor $B(\frac{\lambda}{r},\frac{\lambda}{s})$:

$$\int_0^{\infty} g(t)\sum_{n=1}^{\infty}\frac{a_n}{(1+tn)^{\lambda}}\,dt < B\left(\frac{\lambda}{r},\frac{\lambda}{s}\right)\|g\|_{p,\psi}\|a\|_{q,\psi}, \tag{31}$$

$$\left\{\sum_{n=1}^{\infty} n^{\frac{p\lambda}{s}-1}\left[\int_0^{\infty}\frac{g(t)}{(1+tn)^{\lambda}}\,dt\right]^p\right\}^{\frac{1}{p}} < B\left(\frac{\lambda}{r},\frac{\lambda}{s}\right)\|g\|_{p,\psi}, \tag{32}$$

$$\left\{\int_0^{\infty} t^{\frac{q\lambda}{s}-1}\left[\sum_{n=1}^{\infty}\frac{a_n}{(1+tn)^{\lambda}}\right]^q dt\right\}^{\frac{1}{q}} < B\left(\frac{\lambda}{r},\frac{\lambda}{s}\right)\|a\|_{q,\psi}. \tag{33}$$

In fact, we can show that (31), (32) and (33) are respectively equivalent to (5), (26) and (27), and then it follows that (31), (32) and (33) are equivalent with the same best constant factor $B(\frac{\lambda}{r},\frac{\lambda}{s})$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

QC conceived of the study, and participated in its design and coordination. BY wrote and reformed the article. All authors read and approved the final manuscript.

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