Li et al. *Journal of Inequalities and Applications* 2013, 2013:480
http://www.journalofinequalitiesandapplications.com/content/2013/1/480

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Some new bounds for the minimum eigenvalue of the Hadamard product of an M-matrix and an inverse M-matrix

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Abstract

Let A and B be nonsingular M-matrices. Several new bounds on the minimum eigenvalue for the Hadamard product of B and the inverse matrix of A are given. These bounds can improve considerably some previous results.

MSC: 15A42; 15B34

Keywords: M-matrix; Hadamard product; minimum eigenvalue

1 Introduction

Let $\mathbb{C}^{n\times n}$ ($\mathbb{R}^{n\times n}$) denote the set of all $n \times n$ complex (real) matrices, $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $N = \{1, 2, \dots, n\}$. We write $A \geq 0$ if $a_{ij} \geq 0$ for any $i, j \in N$. If $A \geq 0$, A is called a nonnegative matrix. The spectral radius of A is denoted by $\rho(A)$.

We denote by Z_n the class of all $n \times n$ real matrices, whose off-diagonal entries are nonpositive. A matrix $A = (a_{ij}) \in Z_n$ is called a nonsingular M-matrix if there exist a nonnegative matrix B and a nonnegative real number s such that $A = sI - B$ with $s > \rho(B)$, where I is the identity matrix. M_n will be used to denote the set of all $n \times n$ nonsingular M-matrices. Let us denote $\tau(A) = \min\{\mathrm{Re}(\lambda) : \lambda \in \sigma(A)\}$, where $\sigma(A)$ denotes the spectrum of A.

The Hadamard product of two matrices $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ and $B = (b_{ij}) \in \mathbb{C}^{n\times n}$ is the matrix $A \circ B = (a_{ij}b_{ij}) \in \mathbb{C}^{n\times n}$. If $A, B \in M_n$, then $B \circ A^{-1}$ is also an M-matrix (see [1]).

Let $A = (a_{ij})$ be an $n \times n$ matrix with all diagonal entries being nonzero throughout. For $i, j, k \in N$, $i \neq j$, denote

$$R_i = \sum_{j\neq i} |a_{ij}|, \qquad d_i = \frac{R_i}{|a_{ii}|};$$

$$r_{ji} = \frac{|a_{ji}|}{|a_{jj}| - \sum_{k\neq j,i} |a_{jk}|}, \qquad r_i = \max_{j\neq i}\{r_{ji}\};$$

$$m_{ji} = \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| r_i}{|a_{jj}|}, \qquad m_i = \max_{j\neq i}\{m_{ij}\};$$

$$u_{ji} = \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| m_{ki}}{|a_{jj}|}, \qquad u_i = \max_{j\neq i}\{u_{ij}\}.$$

In 2013, Zhou *et al.* [2] obtained the following result: If $A = (a_{ij}) \in M_n$ is a strictly row diagonally dominant matrix, $B = (b_{ij}) \in M_n$ and $A^{-1} = (\alpha_{ij})$, then

$$\tau(B \circ A^{-1}) \geq \min_{i \in N} \left\{ \frac{b_{ii} - m_i \sum_{j \neq i} |b_{ji}|}{a_{ii}} \right\}. \tag{1}$$

In 2013, Cheng *et al.* [3] presented the following result: If $A = (a_{ij}) \in M_n$ and $A^{-1} = (\alpha_{ij})$ is a doubly stochastic matrix, then

$$\tau(A \circ A^{-1}) \geq \min_{1 \leq i \leq n} \left\{ \frac{a_{ii} - u_i \sum_{j \neq i} |a_{ji}|}{1 + \sum_{j \neq i} u_{ji}} \right\}. \tag{2}$$

In this paper, we present some new lower bounds of $\tau(B \circ A^{-1})$ and $\tau(A \circ A^{-1})$, which improve (1) and (2).

2 Main results

In this section, we present our main results. Firstly, we give some lemmas.

Lemma 1 [4] *Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$. If A is a strictly row diagonally dominant matrix, then $A^{-1} = (\alpha_{ij})$ satisfies*

$$|\alpha_{ji}| \leq d_j |\alpha_{ii}|, \quad j, i \in N, j \neq i.$$

Lemma 2 *Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$. If A is a strictly row diagonally dominant M-matrix, then $A^{-1} = (\alpha_{ij})$ satisfies*

$$\alpha_{ji} \leq w_{ji} \alpha_{ii}, \quad j, i \in N, j \neq i,$$

where

$$w_{ji} = \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| m_{ki} h_i}{|a_{jj}|}, \qquad h_i = \max_{j \neq i} \left\{ \frac{|a_{ji}|}{|a_{jj}| m_{ji} - \sum_{k \neq j,i} |a_{jk}| m_{ki}} \right\}.$$

Proof This proof is similar to the one of Lemma 2.2 in [3]. □

Lemma 3 *If $A = (a_{ij}) \in M_n$ and $A^{-1} = (\alpha_{ij})$ is a doubly stochastic matrix, then*

$$\alpha_{ii} \geq \frac{1}{1 + \sum_{j \neq i} w_{ji}}, \quad i \in N,$$

where w_{ji} is defined as in Lemma 2.

Proof This proof is similar to the one of Lemma 3.1 in [3]. □

Lemma 4 [4] *If $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is a strictly row diagonally dominant M-matrix, then, for $A^{-1} = (\alpha_{ij})$,*

$$\alpha_{ii} \geq \frac{1}{a_{ii}}, \quad i \in N.$$

Lemma 5 [5] *If $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ and $x_1, x_2, \ldots, x_n$ are positive real numbers, then all the eigenvalues of A lie in the region*

$$\bigcup_{i\neq j}\left\{z \in \mathbb{C} : |z - a_{ii}| \le x_i \sum_{k\neq i} \frac{1}{x_k}|a_{ki}|, i \in N\right\}.$$

Lemma 6 [6] *If $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ and $x_1, x_2, \ldots, x_n$ are positive real numbers, then all the eigenvalues of A lie in the region*

$$\bigcup_{i\neq j}\left\{z \in \mathbb{C} : |z - a_{ii}||z - a_{jj}| \le \left(x_i \sum_{k\neq i} \frac{1}{x_k}|a_{ki}|\right)\left(x_j \sum_{k\neq j} \frac{1}{x_k}|a_{kj}|\right), i, j \in N\right\}.$$

Theorem 1 *If $A = (a_{ij})$, $B = (b_{ij}) \in M_n$ and $A^{-1} = (\alpha_{ij})$, then*

$$\tau\left(B \circ A^{-1}\right) \ge \min_{1\le i\le n}\left\{\frac{b_{ii} - w_i \sum_{j\neq i}|b_{ji}|}{a_{ii}}\right\}, \tag{3}$$

where $w_i = \max_{j\neq i}\{w_{ij}\}$ and w_{ij} is defined as in Lemma 2.

Proof It is evident that the result holds with equality for $n = 1$.

We next assume that $n \ge 2$.

Since A is an M-matrix, there exists a positive diagonal matrix D such that $D^{-1}AD$ is a strictly row diagonally dominant M-matrix, and

$$\tau\left(B \circ A^{-1}\right) = \tau\left(D^{-1}\left(B \circ A^{-1}\right)D\right) = \tau\left(B \circ \left(D^{-1}AD\right)^{-1}\right).$$

Therefore, for convenience and without loss of generality, we assume that A is a strictly row diagonally dominant matrix.

(i) First, we assume that A and B are irreducible matrices. Then, for any $i \in N$, we have $0 < w_i < 1$. Since $\tau(B \circ A^{-1})$ is an eigenvalue of $B \circ A^{-1}$, then by Lemma 2 and Lemma 5, there exists an i such that

$$\begin{aligned}\left|\tau\left(B \circ A^{-1}\right) - b_{ii}\alpha_{ii}\right| &\le w_i \sum_{j\neq i}\frac{1}{w_j}|b_{ji}\alpha_{ji}| \le w_i \sum_{j\neq i}\frac{1}{w_j}|b_{ji}||w_{ji}||\alpha_{ii}| \\ &\le w_i \sum_{j\neq i}\frac{1}{w_j}|b_{ji}||w_j||\alpha_{ii}| = w_i|\alpha_{ii}|\sum_{j\neq i}|b_{ji}|.\end{aligned}$$

By Lemma 4, the above inequality and $0 \le \tau(B \circ A^{-1}) \le b_{ii}\alpha_{ii}$, for any $i \in N$, we obtain

$$\left|\tau\left(B \circ A^{-1}\right)\right| \ge b_{ii}\alpha_{ii} - w_i|\alpha_{ii}|\sum_{j\neq i}|b_{ji}| \ge \frac{b_{ii} - w_i \sum_{j\neq i}|b_{ji}|}{a_{ii}} \ge \min_{1\le i\le n}\left\{\frac{b_{ii} - w_i \sum_{j\neq i}|b_{ji}|}{a_{ii}}\right\}.$$

(ii) Now, assume that one of A and B is reducible. It is well known that a matrix in Z_n is a nonsingular M-matrix if and only if all its leading principal minors are positive (see [7]). If we denote by $T = (t_{ij})$ the $n \times n$ permutation matrix with $t_{12} = t_{23} = \cdots = t_{n-1,n} = t_{n1} = 1$, the remaining t_{ij} zero, then both $A - \epsilon T$ and $B - \epsilon T$ are irreducible nonsingular M-matrices for any chosen positive real number ϵ sufficiently small such that all the leading principal minors of both $A - \epsilon T$ and $B - \epsilon T$ are positive. Now, we substitute $A - \epsilon T$ and $B - \epsilon T$ for A and B, respectively, in the previous case, and then letting $\epsilon \to 0$, the result follows by continuity. □

From Lemma 3 and Theorem 1, we can easily obtain the following corollaries.

Corollary 1 *If $A = (a_{ij}), B = (b_{ij}) \in M_n$ and $A^{-1} = (\alpha_{ij})$ is a doubly stochastic matrix, then*

$$\tau\left(B \circ A^{-1}\right) \geq \min_{1 \leq i \leq n} \left\{ \frac{b_{ii} - w_i \sum_{j \neq i} |b_{ji}|}{1 + \sum_{j \neq i} w_{ji}} \right\}.$$

Corollary 2 *If $A = (a_{ij}) \in M_n$ and $A^{-1} = (\alpha_{ij})$ is a doubly stochastic matrix, then*

$$\tau\left(A \circ A^{-1}\right) \geq \min_{1 \leq i \leq n} \left\{ \frac{a_{ii} - w_i \sum_{j \neq i} |a_{ji}|}{1 + \sum_{j \neq i} w_{ji}} \right\}. \tag{4}$$

Remark 1 We next give a simple comparison between (3) and (1), (4) and (2), respectively. Since $m_{ji} h_i \leq r_i$, $0 \leq h_i \leq 1$, $j, i \in N$, $j \neq i$, then $w_{ji} \leq m_{ji}$, $w_i \leq m_i$ and $w_{ji} \leq u_{ji}$, $w_i \leq u_i$ for any $j, i \in N$, $j \neq i$. Therefore,

$$\tau\left(B \circ A^{-1}\right) \geq \min_{1 \leq i \leq n} \left\{ \frac{b_{ii} - w_i \sum_{j \neq i} |b_{ji}|}{a_{ii}} \right\} \geq \min_{1 \leq i \leq n} \left\{ \frac{b_{ii} - m_i \sum_{j \neq i} |b_{ji}|}{a_{ii}} \right\},$$

$$\tau\left(A \circ A^{-1}\right) \geq \min_{1 \leq i \leq n} \left\{ \frac{a_{ii} - w_i \sum_{j \neq i} |a_{ji}|}{1 + \sum_{j \neq i} w_{ji}} \right\} \geq \min_{1 \leq i \leq n} \left\{ \frac{a_{ii} - u_i \sum_{j \neq i} |a_{ji}|}{1 + \sum_{j \neq i} u_{ji}} \right\}.$$

So, the bound in (3) is bigger than the bound in (1) and the bound in (4) is bigger than the bound in (2).

Theorem 2 *If $A = (a_{ij}), B = (b_{ij}) \in M_n$ and $A^{-1} = (\alpha_{ij})$, then*

$$\tau\left(B \circ A^{-1}\right) \geq \min_{i \neq j} \frac{1}{2} \left\{ \alpha_{ii} b_{ii} + \alpha_{jj} b_{jj} - \left[(\alpha_{ii} b_{ii} - \alpha_{jj} b_{jj})^2 + 4 \left(w_i \sum_{k \neq i} |b_{ki}| \alpha_{ii} \right) \left(w_j \sum_{k \neq j} |b_{kj}| \alpha_{jj} \right) \right]^{\frac{1}{2}} \right\},$$

where w_i $(i \in N)$ is defined as in Theorem 1.

Proof It is evident that the result holds with equality for $n = 1$.

We next assume that $n \geq 2$. For convenience and without loss of generality, we assume that A is a strictly row diagonally dominant matrix.

(i) First, we assume that A and B are irreducible matrices. Let $R_j^\sigma = \sum_{k \neq j} |a_{jk}| m_{ki} h_i$, $j, i \in N$, $j \neq i$. Then, for any $j, i \in N$, $j \neq i$, we have

$$R_j^\sigma = \sum_{k \neq j} |a_{jk}| m_{ki} h_i \leq |a_{ji}| + \sum_{k \neq j, i} |a_{jk}| m_{ki} h_i \leq R_j < a_{jj}.$$

Therefore, there exists a real number z_{ji} $(0 \le z_{ji} \le 1)$ such that

$$|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| m_{ki} h_i = z_{ji} R_j + (1 - z_{ji}) R_j^{\sigma}, \quad j, i \in N, j \neq i.$$

Hence,

$$w_{ji} = \frac{z_{ji} R_j + (1 - z_{ji}) R_j^{\sigma}}{a_{jj}}, \quad j \in N.$$

Let $z_j = \max_{i\neq j} z_{ji}$. Obviously, $0 < z_j \le 1$ (if $z_j = 0$, then A is reducible, which is a contradiction). Let

$$w_j = \max_{i\neq j}\{w_{ji}\} = \frac{z_j R_j + (1 - z_j) R_j^{\sigma}}{a_{jj}}, \quad j \in N.$$

Since A is irreducible, then $R_j > 0$, $R_j^{\sigma} \ge 0$, and $0 < w_j < 1$. Let $\tau(B \circ A^{-1}) = \lambda$. By Lemma 6, there exist $i_0, j_0 \in N$, $i_0 \neq j_0$ such that

$$|\lambda - \alpha_{i_0 i_0} b_{i_0 i_0}||\lambda - \alpha_{j_0 j_0} b_{j_0 j_0}| \le \left(w_{i_0} \sum_{k\neq i_0} \frac{1}{w_k} |\alpha_{k i_0} b_{k i_0}| \right)\left(w_{j_0} \sum_{k\neq j_0} \frac{1}{w_k} |\alpha_{k j_0} b_{k j_0}| \right).$$

And by Lemma 2, we have

$$\begin{aligned}
&\left(w_{i_0} \sum_{k\neq i_0} \frac{1}{w_k} |\alpha_{k i_0} b_{k i_0}| \right)\left(w_{j_0} \sum_{k\neq j_0} \frac{1}{w_k} |\alpha_{k j_0} b_{k j_0}| \right) \\
&\quad \le \left(w_{i_0} \sum_{k\neq i_0} |b_{k i_0}| \alpha_{i_0 i_0} \right)\left(w_{j_0} \sum_{k\neq j_0} |b_{k j_0}| \alpha_{j_0 j_0} \right).
\end{aligned}$$

Therefore,

$$|\lambda - \alpha_{i_0 i_0} b_{i_0 i_0}||\lambda - \alpha_{j_0 j_0} b_{j_0 j_0}| \le \left(w_{i_0} \sum_{k\neq i_0} |b_{k i_0}| \alpha_{i_0 i_0} \right)\left(w_{j_0} \sum_{k\neq j_0} |b_{k j_0}| \alpha_{j_0 j_0} \right).$$

Furthermore, we obtain

$$\begin{aligned}
\lambda \ge \frac{1}{2}\Bigg\{ &\alpha_{i_0 i_0} b_{i_0 i_0} + \alpha_{j_0 j_0} b_{j_0 j_0} - \Bigg[(\alpha_{i_0 i_0} b_{i_0 i_0} - \alpha_{j_0 j_0} b_{j_0 j_0})^2 \\
&+ 4\left(w_{i_0} \sum_{k\neq i_0} |b_{k i_0}| \alpha_{i_0 i_0} \right)\left(w_{j_0} \sum_{k\neq j_0} |b_{k j_0}| \alpha_{j_0 j_0} \right) \Bigg]^{\frac{1}{2}} \Bigg\},
\end{aligned}$$

that is,

$$\begin{aligned}
&\tau\left(B \circ A^{-1}\right) \\
&\quad \ge \frac{1}{2}\Bigg\{ \alpha_{i_0 i_0} b_{i_0 i_0} + \alpha_{j_0 j_0} b_{j_0 j_0} - \Bigg[(\alpha_{i_0 i_0} b_{i_0 i_0} - \alpha_{j_0 j_0} b_{j_0 j_0})^2
\end{aligned}$$

$$+ 4\left(w_{i_0}\sum_{k\neq i_0}|b_{ki_0}|\alpha_{i_0i_0}\right)\left(w_{j_0}\sum_{k\neq j_0}|b_{kj_0}|\alpha_{j_0j_0}\right)\Bigg]^{\frac{1}{2}}\Bigg\}$$

$$\geq \min_{i\neq j}\frac{1}{2}\Bigg\{\alpha_{ii}b_{ii}+\alpha_{jj}b_{jj}-\Bigg[(\alpha_{ii}b_{ii}-\alpha_{jj}b_{jj})^2+4\left(w_i\sum_{k\neq i}|b_{ki}|\alpha_{ii}\right)\left(w_j\sum_{k\neq j}|b_{kj}|\alpha_{jj}\right)\Bigg]^{\frac{1}{2}}\Bigg\}.$$

(ii) Now, assume that one of A and B is reducible. We substitute $A - \epsilon T$ and $B - \epsilon T$ for A and B, respectively, in the previous case, and then letting $\epsilon \to 0$, the result follows by continuity. □

Corollary 3 *If $A = (a_{ij}) \in M_n$ and $A^{-1} = (\alpha_{ij})$, then*

$$\tau\left(A\circ A^{-1}\right) \geq \min_{i\neq j}\frac{1}{2}\Bigg\{\alpha_{ii}a_{ii}+\alpha_{jj}a_{jj}-\Bigg[(\alpha_{ii}a_{ii}-\alpha_{jj}a_{jj})^2 + 4\left(w_i\sum_{k\neq i}|a_{ki}|\alpha_{ii}\right)\left(w_j\sum_{k\neq j}|a_{kj}|\alpha_{jj}\right)\Bigg]^{\frac{1}{2}}\Bigg\}.$$

Example 1 Let

$$A = \begin{pmatrix} 39 & -16 & -2 & -3 & -2 & -5 & -2 & -3 & -5 & 0 \\ -26 & 44 & -2 & -4 & -2 & -1 & 0 & -2 & -3 & -3 \\ -1 & -9 & 29 & -3 & -4 & 0 & -5 & -4 & -1 & -1 \\ -2 & -3 & -10 & 36 & -12 & 0 & -5 & -1 & -2 & 0 \\ 0 & -3 & -1 & -9 & 44 & -16 & -3 & -4 & -4 & -3 \\ -3 & -4 & -3 & -4 & -12 & 48 & -18 & -1 & 0 & -2 \\ -2 & -1 & -4 & -3 & -4 & -16 & 45 & -9 & -4 & -1 \\ -1 & -2 & -2 & -2 & -3 & -1 & -5 & 38 & -20 & -1 \\ -2 & -1 & 0 & -3 & -4 & -5 & -2 & -10 & 47 & -19 \\ -1 & -4 & -4 & -4 & 0 & -3 & -4 & -3 & -7 & 31 \end{pmatrix},$$

$$B = \begin{pmatrix} 90 & -3 & -2 & -7 & -4 & -7 & -6 & -3 & -9 & -3 \\ -4 & 100 & -5 & -4 & -8 & -7 & -1 & -9 & -8 & -8 \\ -5 & -9 & 62 & -4 & -7 & -9 & -9 & -1 & -4 & -8 \\ -8 & -8 & -10 & 99 & 0 & -6 & -8 & -9 & -3 & -6 \\ -3 & -8 & -10 & -6 & 62 & -3 & -6 & -7 & -5 & -1 \\ -2 & -3 & -5 & -10 & -6 & 55 & -5 & -1 & -3 & -10 \\ -8 & -5 & -8 & -8 & -3 & -3 & 52 & -6 & -1 & -4 \\ -4 & -5 & -8 & -4 & -1 & -1 & -6 & 57 & -7 & -7 \\ -2 & -1 & -6 & -10 & -2 & -6 & -5 & -9 & 86 & -5 \\ -5 & -7 & -3 & -9 & -5 & -7 & -9 & -5 & -9 & 72 \end{pmatrix}.$$

It is easily proved that A and B are nonsingular M-matrices and A is a doubly stochastic matrix.

(i) If we apply Theorem 4.8 of [2], we have

$$\tau\left(B\circ A^{-1}\right) \geq \min_{1\leq i\leq n}\left\{\frac{b_{ii} - m_i\sum_{j\neq i}|b_{ji}|}{a_{ii}}\right\} = 0.0027.$$

If we apply Theorem 2.4 of [8], we have

$$\tau\left(B \circ A^{-1}\right) \geq \left(1 - \rho(J_A)\rho(J_B)\right) \min_{1 \leq i \leq n} \frac{a_{ii}}{b_{ii}} = 0.3485.$$

But, if we apply Theorem 1, we have

$$\tau\left(B \circ A^{-1}\right) \geq \min_{1 \leq i \leq n} \left\{ \frac{b_{ii} - w_i \sum_{j \neq i} |b_{ji}|}{a_{ii}} \right\} = 0.0435.$$

If we apply Corollary 1, we have

$$\tau\left(B \circ A^{-1}\right) \geq \min_{1 \leq i \leq n} \left\{ \frac{b_{ii} - w_i \sum_{j \neq i} |b_{ji}|}{1 + \sum_{j \neq i} w_{ji}} \right\} = 0.2172.$$

If we apply Theorem 2, we have

$$\begin{aligned} \tau\left(B \circ A^{-1}\right) \geq{} & \min_{i \neq j} \frac{1}{2} \left\{ \alpha_{ii} b_{ii} + \alpha_{jj} b_{jj} - \left[(\alpha_{ii} b_{ii} - \alpha_{jj} b_{jj})^2 \right. \right. \\ & \left. \left. + 4 \left(w_i \sum_{k \neq i} |b_{ki}| \alpha_{ii} \right) \left(w_j \sum_{k \neq j} |b_{kj}| \alpha_{jj} \right) \right]^{\frac{1}{2}} \right\} \\ ={} & 0.7212. \end{aligned}$$

(ii) If we apply Theorem 3.2 of [3], we get

$$\tau\left(A \circ A^{-1}\right) \geq \min_{1 \leq i \leq n} \left\{ \frac{a_{ii} - u_i \sum_{j \neq i} |a_{ji}|}{1 + \sum_{j \neq i} u_{ji}} \right\} = 0.3269.$$

But, if we apply Corollary 2, we get

$$\tau\left(A \circ A^{-1}\right) \geq \min_{1 \leq i \leq n} \left\{ \frac{a_{ii} - w_i \sum_{j \neq i} |a_{ji}|}{1 + \sum_{j \neq i} w_{ji}} \right\} = 0.3605.$$

If we apply Corollary 3, we get

$$\begin{aligned} \tau\left(A \circ A^{-1}\right) \geq{} & \min_{i \neq j} \frac{1}{2} \left\{ \alpha_{ii} a_{ii} + \alpha_{jj} a_{jj} - \left[(\alpha_{ii} a_{ii} - \alpha_{jj} a_{jj})^2 \right. \right. \\ & \left. \left. + 4 \left(w_i \sum_{k \neq i} |b_{ki}| \alpha_{ii} \right) \left(w_j \sum_{k \neq j} |b_{kj}| \alpha_{jj} \right) \right]^{\frac{1}{2}} \right\} \\ ={} & 0.4072. \end{aligned}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

Acknowledgements
The authors are very indebted to the referees for their valuable comments and corrections, which improved the original manuscript of this paper. This work was supported by the National Natural Science Foundation of China (11361074), IRTSTYN and Foundation of Yunnan University (2012CG017).

Received: 27 July 2013 Accepted: 27 September 2013 Published: 07 Nov 2013

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10.1186/1029-242X-2013-480
Cite this article as: Li et al.: Some new bounds for the minimum eigenvalue of the Hadamard product of an *M*-matrix and an inverse *M*-matrix. *Journal of Inequalities and Applications* 2013, 2013:480