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An investigation on a new class of harmonic mappings

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Abstract

In the present paper, we give an extension of the idea which was introduced by Sakaguchi (J. Math. Soc. Jpn. 11:72-75, 1959), and we give some applications of this extended idea for the investigation of the class of harmonic mappings.

MSC: Primary 30C45; 30C55

Keywords: harmonic mappings; distortion theorem; grow theorem

1 Introduction

Let $\mathbb{D} = \{z \mid |z| < 1\}$ be the open unit disc in the complex plane \mathbb{C} . A complex valued harmonic function $f : \mathbb{D} \rightarrow \mathbb{C}$ has the representation

$$f = h(z) + \overline{g(z)}, \quad (1.1)$$

where $h(z)$ and $g(z)$ are analytic in \mathbb{D} and have the following power series expansions:

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n, \quad z \in \mathbb{D},$$

where $a_n, b_n \in \mathbb{C}$, $n = 0, 1, 2, \dots$. Choose $g(0) = 0$ (i.e., $b_0 = 0$), so the representation (1.1) is unique in \mathbb{D} and is called the canonical representation of f in \mathbb{D} . It is convenient to make further normalization (without loss of generality) $h(0) = 0$ (i.e., $a_0 = 0$) and $h'(0) = 1$ (i.e., $a_1 = 1$). The family of such functions f is denoted by $\mathcal{S}_{\mathcal{H}}$ [1]. It is known that if f is a sense-preserving harmonic mapping of \mathbb{D} onto some other region, then by Lewy's theorem its Jacobian is strictly positive, i.e.,

$$J_{f(z)} = |h'(z)|^2 - |g'(z)|^2 > 0.$$

Equivalently, the inequality $|g'(z)| < |h'(z)|$ holds for all $z \in \mathbb{D}$. The family of all functions $f \in \mathcal{S}_{\mathcal{H}}$ with the additional property that $g'(0) = 0$ (i.e., $b_1 = 0$) is denoted by $\mathcal{S}_{\mathcal{H}}^0$ [1]. Observe that the classical family of univalent functions \mathcal{S} consists of all functions $f \in \mathcal{S}_{\mathcal{H}}^0$ such that $g(z) = 0$ for all $z \in \mathbb{D}$. Thus, it is clear that $\mathcal{S} \subset \mathcal{S}_{\mathcal{H}}^0 \subset \mathcal{S}_{\mathcal{H}}$.

Let Ω be the family of functions $\phi(z)$ regular in the open unit disc \mathbb{D} and satisfy the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$.

Denote by \mathcal{P} the family of functions $p(z) = 1 + p_1z + p_2z^2 + \dots$ regular in \mathbb{D} such that $p(z)$ in \mathcal{P} if and only if

$$p(z) = \frac{1 + \phi(z)}{1 - \phi(z)} \tag{1.2}$$

for some $\phi(z) \in \Omega$ and every $z \in \mathbb{D}$.

If $s(z) = z + c_2z^2 + \dots$ is regular in the open unit disc \mathbb{D} and satisfies the condition

$$\operatorname{Re} \left(e^{i\alpha} z \frac{s'(z)}{s(z)} \right) > 0, \quad z \in \mathbb{D} \tag{1.3}$$

for some real α , $|\alpha| < \pi/2$, then $s(z)$ is said to be an α -spirallike function in \mathbb{D} [2, 3]. Such functions are known to be univalent [4]. The class of such functions is denoted by \mathcal{S}_α^* .

Let $F_1(z) = z + \alpha_2z^2 + \alpha_3z^3 + \dots$ and $F_2(z) = z + \beta_2z^2 + \beta_3z^3 + \dots$ be analytic functions in \mathbb{D} . If there exists a function $\phi(z) \in \Omega$ such that $F_1(z) = F_2(\phi(z))$ for every $z \in \mathbb{D}$, then we say that $F_1(z)$ is subordinate to $F_2(z)$ and we write $F_1 \prec F_2$. We also note that if $F_1 \prec F_2$, then $F_1(\mathbb{D}) \subset F_2(\mathbb{D})$ [2, 3]. We also note that $w(z) = g'(z)/h'(z)$ is the analytic second dilatation of f and $|w(z)| < 1$ for every $z \in \mathbb{D}$.

In this paper we investigate the class of harmonic mappings defined by

$$\mathcal{S}_{\mathcal{H}(\alpha)} = \left\{ f = h(z) + \overline{g(z)} \mid w(z) \prec \frac{1+z}{1-z}, h(z) \in \mathcal{S}_\alpha^* \right\}.$$

For this aim we need the following theorems and lemmas.

Theorem 1.1 [5] *Let $s(z) \in \mathcal{S}_\alpha^*$, then*

$$rF(\cos \alpha, -r) \leq |s(z)| \leq rF(\cos \alpha, r) \tag{1.4}$$

and

$$\begin{aligned} & [(1-r)\cos \alpha - (1+r)\sin \alpha]F(\cos \alpha, -r) \\ & \leq |s'(z)| \leq [(1+r)\cos \alpha + (1-r)\sin \alpha]F(\cos \alpha, r), \end{aligned} \tag{1.5}$$

where

$$F(\cos \alpha, r) = \frac{1}{(1+r)^{\cos \alpha(\cos \alpha - 1)}(1-r)^{\cos \alpha(\cos \alpha + 1)}} \tag{1.6}$$

for all $|z| = r < 1$ and $|\alpha| < \pi/2$.

Lemma 1.2 [6] *Let $\phi(z)$ be regular in the unit disc \mathbb{D} with $\phi(0) = 0$, then if $|\phi(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_1 , one has $z_1\phi'(z_1) = k\phi(z_1)$ for some $k \geq 1$.*

Lemma 1.3 [7] *If $s_1(z)$ and $s_2(z)$ are regular in \mathbb{D} , $s_1(0) = s_2(0)$, $s_2(z)$ maps \mathbb{D} onto a many-sheeted region which is starlike with respect to the origin, and $s_1'(z)/s_2'(z) \in \mathcal{P}$, then $s_1(z)/s_2(z) \in \mathcal{P}$.*

Lemma 1.4 [8] *Let $f = h(z) + \overline{g(z)}$ be an element of $\mathcal{S}_{\mathcal{H}\mathcal{P}ST(\alpha)^*}$, then*

$$\frac{|b_1| - r}{1 - |b_1|r} \leq \left| \frac{g'(z)}{h'(z)} \right| \leq \frac{|b_1| + r}{1 + |b_1|r} \tag{1.7}$$

for all $|z| = r < 1$. This inequality is sharp because the extremal function is

$$e^{i\alpha} \frac{g'(z)}{h'(z)} = \frac{z + b}{1 + \overline{b}z},$$

where $b = e^{i\alpha} b_1$.

2 Main results

Theorem 2.1 $s(z) \in \mathcal{S}_\alpha^*$ if and only if

$$z \frac{s'(z)}{s(z)} - 1 < \frac{2 \cos \alpha e^{-i\alpha} z}{1 - z} \tag{2.1}$$

for all $z \in \mathbb{D}$.

Proof Let $s(z)$ in \mathcal{S}_α^* , then we have

$$e^{i\alpha} z \frac{s'(z)}{s(z)} = \cos \alpha \frac{1 + \phi(z)}{1 - \phi(z)} + i \sin \alpha$$

or

$$e^{i\alpha} z \frac{s'(z)}{s(z)} = e^{i\alpha} \frac{1 + \phi(z)}{1 - \phi(z)}$$

for some $\phi(z) \in \Omega$ and all $z \in \mathbb{D}$. Thus

$$\begin{aligned} z \frac{s'(z)}{s(z)} - 1 &= \frac{1 + e^{-2i\alpha} \phi(z)}{1 - \phi(z)} - 1 \\ &= \frac{1 + (\cos 2\alpha - i \sin 2\alpha) \phi(z) - 1 + \phi(z)}{1 - \phi(z)} \\ &= \frac{2 \cos \alpha e^{-i\alpha} \phi(z)}{1 - \phi(z)} \end{aligned}$$

for some $\phi(z) \in \Omega$ and all $z \in \mathbb{D}$. Since $\phi(z) \in \Omega$, we have that (2.1) is true. The sufficient part of the proof can be seen by following the above steps in the opposite direction by considering the subordination principle. \square

Theorem 2.2 *Let $f = h(z) + \overline{g(z)}$ be an element of $\mathcal{S}_{\mathcal{H}(\alpha)}$. If $w(z) = \frac{g'(z)}{h'(z)} \in \mathcal{P}$, then $\frac{g(z)}{h(z)} \in \mathcal{P}$ for all $z \in \mathbb{D}$.*

Proof A version of this theorem was proved by Sakaguchi for a univalent starlike function [7, 9]. Since $f = h(z) + \overline{g(z)} \in \mathcal{S}_{\mathcal{H}(\alpha)}$, then $h(z)$ and $g(z)$ are regular in \mathbb{D} and $h(0) = g(0) = 0$. On the other hand, we have

$$w(z) = \frac{g'(z)}{h'(z)} \in \mathcal{P} \quad \text{if and only if} \quad \frac{g'(z)}{h'(z)} < \frac{1+z}{1+z} \tag{2.2}$$

for all $z \in \mathbb{D}$. Geometrically, this means that $\frac{g'(z)}{h'(z)}$ maps \mathbb{D} inside the open disc centered on the real axis with diameter end points $\frac{1-r}{1+r}$ and $\frac{1+r}{1-r}$. Now we define a function $\phi(z)$ by

$$\frac{g(z)}{h(z)} = \frac{1 + \phi(z)}{1 - \phi(z)} \quad (z \in \mathbb{D}). \quad (2.3)$$

Then $\phi(z)$ is analytic in \mathbb{D} , and $\phi(0) = 0$. On the other hand,

$$w(z) = \frac{g'(z)}{h'(z)} = \frac{2z\phi'(z)}{1 - \phi(z)} \frac{1}{e^{i\alpha}(1 + e^{-2i\alpha})\phi(z)} + \frac{1 + \phi(z)}{1 - \phi(z)} \quad (z \in \mathbb{D}). \quad (2.4)$$

Now, it is easy to realize that the subordination (2.2) is equivalent to $|\phi(z)| < 1$ in (2.3) for all $z \in \mathbb{D}$. Indeed, assume to the contrary that there exists $z_1 \in \mathbb{D}$ such that $|\phi(z_1)| = 1$. Then by Jack's lemma (Lemma 1.2), $z_1\phi'(z_1) = k\phi(z_1)$, $k \geq 1$, for such z_1 we have

$$w(z_1) = \frac{g'(z_1)}{h'(z_1)} = \frac{2k\phi(z_1)}{1 - \phi(z_1)} \frac{1}{e^{i\alpha}(1 + e^{-2i\alpha})\phi(z_1)} + \frac{1 + \phi(z_1)}{1 - \phi(z_1)} = w(\phi(z_1)) \notin w(\mathbb{D}),$$

since $|\phi(z_1)| = 1$ and $k \geq 1$. But this is a contradiction to the condition $w(z) = \frac{g'(z)}{h'(z)} < \frac{1+z}{1-z}$, and so the assumption is wrong, i.e., $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. \square

Remark 2.3 Theorem 2.2 is an extension of Lemma 1.3 to the harmonic mappings.

Corollary 2.4 Let $f = h(z) + \overline{g(z)}$ be an element of $\mathcal{S}_{\mathcal{H}(\alpha)}$, then

$$\frac{r(1-r)}{1+r} F(\cos \alpha, -r) \leq |g(z)| \leq \frac{r(1+r)}{1-r} F(\cos \alpha, r) \quad (2.5)$$

and

$$\begin{aligned} & \left[(1-r)\cos \alpha - (1+r)\sin \alpha \right] \frac{1-r}{1+r} F(\cos \alpha, -r) \\ & \leq |g'(z)| \leq \left[(1+r)\cos \alpha + (1-r)\sin \alpha \right] \frac{1+r}{1-r} F(\cos \alpha, r), \end{aligned} \quad (2.6)$$

where $F(\cos \alpha, r)$ is given by (1.6) for all $|z| = r < 1$ and $|\alpha| < \pi/2$.

Proof The proof of this theorem is a simple consequence of Theorem 1.1 and Theorem 2.2 since

$$\operatorname{Re} w(z) = \operatorname{Re} \frac{g'(z)}{h'(z)} \Rightarrow \frac{g'(z)}{h'(z)} < \frac{1+z}{1-z} \Rightarrow \frac{1-r}{1+r} \leq \left| \frac{g'(z)}{h'(z)} \right| \leq \frac{1+r}{1-r},$$

then

$$|h'(z)| \frac{1-r}{1+r} \leq |g'(z)| \leq |h'(z)| \frac{1+r}{1-r}$$

and

$$\operatorname{Re} \frac{g(z)}{h(z)} > 0 \Rightarrow \frac{g(z)}{h(z)} < \frac{1+z}{1-z} \Rightarrow \frac{1-r}{1+r} \leq \left| \frac{g(z)}{h(z)} \right| \leq \frac{1+r}{1-r},$$

then

$$|h(z)| \frac{1-r}{1+r} \leq |g(z)| \leq |h(z)| \frac{1+r}{1-r}. \quad \square$$

Corollary 2.5 Let $f = h(z) + \overline{g(z)}$ be an element of $\mathcal{S}_{\mathcal{H}(\alpha)}$, then

$$\begin{aligned} & \frac{[(1-r)\cos\alpha - (1+r)\sin\alpha]^2 (F(\cos\alpha, -r))^2 (1-r^2)(1-|b_1|^2)}{(1+|b_1|r)^2} \\ & \leq J_{f(z)} \leq \frac{[(1+r)\cos\alpha + (1-r)\sin\alpha]^2 (F(\cos\alpha, r))^2 (1-r^2)(1-|b_1|^2)}{(1-|b_1|r)^2}, \end{aligned}$$

where $F(\cos\alpha, r)$ is given by (1.6) for all $|z| = r < 1$, $|\alpha| < \pi/2$ and $J_{f(z)}$ is the Jacobian of f defined by $J_{f(z)} = |h'(z)|^2 - |g'(z)|^2$ for all $z \in \mathbb{D}$.

Proof The proof of this corollary is a simple consequence of Lemma 1.1 and Lemma 1.4. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors jointly worked on the results and they read and approved the final manuscript.

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Received: 28 December 2012 Accepted: 28 August 2013 Published: 07 Nov 2013

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10.1186/1029-242X-2013-478

Cite this article as: Yavuz Duman et al.: An investigation on a new class of harmonic mappings. *Journal of Inequalities and Applications* 2013, **2013**:478