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A new note on absolute matrix summability

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Abstract

In the present paper, we have proved theorems dealing with matrix summability factors by using quasi β-power increasing sequences. Some new results have also been obtained.

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1 Introduction

A positive sequence (γ_n) is said to be quasi β-power increasing sequence if there exists a constant $K = K(\beta,\gamma) \geq 1$ such that $Kn^\beta \gamma_n \geq m^\beta \gamma_m$ holds for all $n \geq m \geq 1$ [1]. A sequence (λ_n) is said to be of bounded variation, denote by $(\lambda_n) \in \mathcal{BV}$, if $\sum_{n=1}^{\infty} |\Delta\lambda_n| = \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$. Let $\sum a_n$ be a given infinite series with the partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as } n \to \infty \ (P_{-i} = p_{-i} = 0, i \geq 1). \tag{1}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{2}$$

defines the sequence (σ_n) of the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) [2].

The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$ if [3]

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\sigma_n - \sigma_{n-1}|^k < \infty. \tag{3}$$

Let $A = (a_{nv})$ be a normal matrix, *i.e.*, a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, \ldots. \tag{4}$$

The series $\sum a_n$ is said to be summable $|A, p_n|_k$, $k \geq 1$ if [4]

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left|\bar{\Delta} A_n(s)\right|^k < \infty, \tag{5}$$

where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

Before stating the main theorem, we must first introduce some further notations.

Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, \ldots \tag{6}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \qquad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, \ldots. \tag{7}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{8}$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{9}$$

2 Known result

Recently, many authors have come up with theorems dealing with the applications of power increasing sequences [1, 5–7]. Among them, Bor and Özarslan have proved two theorems for $|\bar{N}, p_n|_k$ summability method by using quasi β-power increasing sequence [5]. Their theorems are as follows.

Theorem A *Let* (X_n) *be a quasi* β*-power increasing sequence for some* $0 < \beta < 1$, *and let there be sequences* (β_n) *and* (λ_n) *such that*

$$|\Delta \lambda_n| \leq \beta_n, \tag{10}$$

$$\beta_n \to 0 \quad \textit{as } n \to 0, \tag{11}$$

$$\sum_{n=1}^{\infty} n |\Delta \beta_n| X_n < \infty, \tag{12}$$

$$|\lambda_n| X_n = O(1) \quad \textit{as } n \to \infty. \tag{13}$$

If

$$\sum_{v=1}^{n} \frac{|s_v|^k}{v} = O(X_n), \tag{14}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |s_n|^k = O(X_m), \quad m \to \infty, \tag{15}$$

then $\sum a_n \lambda_n$ *is summable* $|\bar{N}, p_n|_k$, $k \geq 1$.

Theorem B *Let* (X_n) *be a quasi* β*-power increasing sequence for some* $0 < \beta < 1$, *and let sequences* (β_n) *and* (λ_n) *satisfy conditions* (10)-(13) *and* (15). *If*

$$\sum_{n=1}^{\infty} P_n |\Delta\beta_n| X_n < \infty, \tag{16}$$

$$\sum_{n=1}^{m} \frac{|s_n|^k}{P_n} = O(X_m), \tag{17}$$

then $\sum a_n \lambda_n$ *is summable* $|\bar{N}, p_n|_k$, $k \geq 1$.

3 The main result

The aim of this paper is to generalize Theorem A and Theorem B to $|A, p_n|_k$ summability. Now, we shall prove the following two theorems.

Theorem 1 *Let* $A = (a_{nv})$ *be a positive normal matrix such that*

$$\bar{a}_{n0} = 1, \quad n = 0, 1, \ldots, \tag{18}$$

$$a_{n-1,v} \geq a_{nv}, \quad \text{for } n \geq v + 1, \tag{19}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{20}$$

and (X_n) *is a quasi* β*-power increasing sequence for some* $0 < \beta < 1$. *If all the conditions of Theorem* A *and*

$$(\lambda_n) \in \mathcal{BV} \tag{21}$$

are satisfied, then the series $\sum a_n \lambda_n$ *is summable* $|A, p_n|_k$, $k \geq 1$.

In the special case of $a_{nv} = \frac{p_v}{P_n}$, this theorem reduces to Theorem A.

Theorem 2 *Let* $A = (a_{nv})$ *be a positive normal matrix as in Theorem* 1, *and let* (X_n) *is a quasi* β*-power increasing sequence for some* $0 < \beta < 1$. *If all the conditions of Theorem* B *and* (21) *are satisfied, then the series* $\sum a_n \lambda_n$ *is summable* $|A, p_n|_k$, $k \geq 1$.

We need following lemmas for the proof of our theorems.

Lemma 1 [1] *Let (X_n) be a quasi β-power increasing sequence for some $0 < \beta < 1$. If conditions* (11) *and* (12) *satisfied, then*

$$nX_n\beta_n = O(1) \quad \text{as } n \to \infty, \tag{22}$$

$$\sum_{n=1}^{\infty} X_n\beta_n < \infty. \tag{23}$$

Lemma 2 *Let (X_n) be a quasi β-power increasing sequence for some $0 < \beta < 1$. If conditions* (11) *and* (16) *are satisfied, then*

$$P_n\beta_nX_n = O(1), \tag{24}$$

$$\sum_{n=1}^{\infty} p_n\beta_nX_n < \infty. \tag{25}$$

The proof of Lemma 2 is similar to that of Bor in [8] and hence is omitted.

4 Proof of Theorem 1

Let (T_n) denote A-transform of the series $\sum a_n\lambda_n$. Then by (8), (9) and applying Abel's transformation, we have

$$\begin{aligned}
\bar{\Delta} T_n &= \sum_{v=1}^{n} \hat{a}_{nv} a_v \lambda_v \\
&= \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv}\lambda_v) \sum_{k=1}^{v} a_k + \hat{a}_{nn}\lambda_n \sum_{v=1}^{n} a_v \\
&= \sum_{v=1}^{n-1} (\hat{a}_{nv}\lambda_v - \hat{a}_{n,v+1}\lambda_{v+1}) s_v + a_{nn}\lambda_n s_n \\
&= \sum_{v=1}^{n-1} (\hat{a}_{nv}\lambda_v - \hat{a}_{n,v+1}\lambda_{v+1} - \hat{a}_{n,v+1}\lambda_v + \hat{a}_{n,v+1}\lambda_v) s_v + a_{nn}\lambda_n s_n \\
&= \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv})\lambda_v s_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1}\Delta\lambda_v s_v + a_{nn}\lambda_n s_n \\
&= T_{n,1} + T_{n,2} + T_{n,3} \quad \text{say.}
\end{aligned}$$

Since

$$|T_{n,1} + T_{n,2} + T_{n,3}|^k \le 3^k \bigl(|T_{n,1}|^k + T_{n,2}|^k + T_{n,3}|^k\bigr),$$

to complete the proof of the Theorem 1, it is sufficient to show that

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |T_{n,r}|^k < \infty, \quad \text{for } r = 1, 2, 3. \tag{26}$$

First, applying Hölder's inequality with indices k and k', where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, we get that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,1}|^k &\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\Delta_v\hat{a}_{nv}||\lambda_v||s_v|\right)^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\Delta_v\hat{a}_{nv}||\lambda_v|^k|s_v|^k\right)\times\left(\sum_{v=1}^{n-1}|\Delta_v\hat{a}_{nv}|\right)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}a_{nn}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\Delta_v\hat{a}_{nv}||\lambda_v|^k|s_v|^k\right) \\
&= O(1)\sum_{v=1}^{m}|\lambda_v|^k|s_v|^k\sum_{n=v+1}^{m+1}|\Delta_v\hat{a}_{nv}| \\
&= O(1)\sum_{v=1}^{m}\frac{p_v}{P_v}|\lambda_v|^{k-1}|\lambda_v||s_v|^k = O(1)\sum_{v=1}^{m}\frac{p_v}{P_v}|\lambda_v||s_v|^k \\
&= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{i=1}^{v}\frac{p_i}{P_i}|s_i|^k + O(1)|\lambda_m|\sum_{v=1}^{m}\frac{p_v}{P_v}|s_v|^k \\
&= O(1)\sum_{v=1}^{m-1}\beta_v X_v + O(1)|\lambda_m|X_m \\
&= O(1) \quad \text{as } m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 1 and Lemma 1.

Since $(\lambda_n) \in \mathcal{BV}$ by (21), applying Hölder's inequality with the same indices as those above, we have

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_n(2)\right|^k &\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\Delta\lambda_v||\hat{a}_{n,v+1}||s_v|\right)^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\Delta\lambda_v||\hat{a}_{n,v+1}||s_v|^k\right) \\
&\quad \times\left(\sum_{v=1}^{n-1}|\Delta\lambda_v||\hat{a}_{n,v+1}|\right)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}a_{nn}\right)^{k-1}\left(\sum_{v=1}^{n-1}\beta_v|\hat{a}_{n,v+1}||s_v|^k\right)\times\left(\sum_{v=1}^{n-1}|\Delta\lambda_v|\right)^{k-1} \\
&= O(1)\sum_{v=1}^{m}\beta_v|s_v|^k\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}| \\
&= O(1)\sum_{v=1}^{m}\beta_v|s_v|^k \\
&= O(1)\sum_{v=1}^{m}(v\beta_v)\frac{|s_v|^k}{v}
\end{aligned}$$

$$\begin{aligned}
&= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{i=1}^{v}\frac{|s_i|^k}{i} + O(1)m\beta_m\sum_{v=1}^{m}\frac{|s_v|^k}{v} \\
&= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)X_v + O(1)m\beta_m X_m \\
&= O(1)\sum_{v=1}^{m-1}v|\Delta\beta_v|X_v + O(1)\sum_{v=1}^{m-1}\beta_{v+1}X_{v+1} + O(1)m\beta_m X_m \\
&= O(1) \quad \text{as } m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 1 and Lemma 1.

Finally, by following the similar process as in $T_{n,1}$, we have that

$$\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_n(3)\right|^k &\le \sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|a_{nn}|^k|\lambda_n|^k|s_n|^k \\
&= O(1)\sum_{n=1}^{m}\frac{p_n}{P_n}|\lambda_n||s_n|^k \\
&= O(1) \quad \text{as } m\to\infty.
\end{aligned}$$

So, we get

$$\sum_{n=1}^{\infty}(P_n/p_n)^{k-1}|T_{n,r}|^k < \infty, \quad \text{for } r = 1,2,3.$$

This completes the proof of Theorem 1.

5 Proof of Theorem 2

Using Lemma 2 and proceeding as in the proof of Theorem 1, replacing $\sum_{v=1}^{m}\beta_v|s_v|^k$ by $\sum_{v=1}^{m}\beta_v P_v(\frac{|s_v|^k}{P_v})$, we can easily prove Theorem 2.

If we take $p_n = 1$ in these theorems, then we have two new results dealing with $|A|_k$ summability factors of infinite series. Also, if we take $k = 1$, then we obtain another two new results concerning $|A|$ summability. Finally, by taking (X_n) as almost increasing sequence in the theorems, we get new results dealing with $|A,p_n|_k$ summability factors of infinite series.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript and read and approved the final manuscript.

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