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On complete convergence for weighted sums of martingale-difference random fields

Mi Hwa Ko[*]

[*]Correspondence:
songhack@wonkwang.ac.kr
Division of Mathematics and
Informational Statistics, Wonkwang
University, Jeonbuk, 570-749, Korea

Abstract

Let $\{a_{\mathbf{n},\mathbf{i}}, \mathbf{n} \in Z_+^d, \mathbf{i} \le \mathbf{n}\}$ be an array of real numbers, and let $\{X_{\mathbf{i}}, \mathbf{i} \in Z_+^d\}$ be the martingale differences with respect to $\{\mathcal{G}_{\mathbf{n}}, \mathbf{n} \in Z_+^d\}$ satisfying $E(E(X|\mathcal{G}_k)|\mathcal{G}_{\mathbf{m}}) = E(X|\mathcal{G}_{\mathbf{k} \wedge \mathbf{m}})$ a.s., where $\mathbf{k} \wedge \mathbf{m}$ denotes componentwise minimum, $\{\mathcal{G}_{\mathbf{k}}, \mathbf{k} \in Z_+^d\}$ is a family of σ-algebras such that $\forall \mathbf{k} \le \mathbf{n}$, $\mathcal{G}_{\mathbf{k}} \subset \mathcal{G}_{\mathbf{n}} \subset \mathcal{G}$, and X is any integrable random variable defined on the initial probability space. The aim of this paper is to obtain some results concerning complete convergence of weighted sums $\sum_{\mathbf{i} \le \mathbf{n}} a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}$.

MSC: 60F05; 60F15

Keywords: complete convergence; weighted sums; martingale difference; maximal moment inequality; σ-algebra

1 Introduction

The concept of complete convergence for sums of independent and identically distributed random variables was introduced by Hsu and Robbins [1] as follows: A sequence of random variables $\{X_n\}$ is said to be completely to a constant c if

$$\sum_{n=1}^{\infty} P\big(|X_n - c| > \epsilon\big) < \infty \quad \text{for all } \epsilon > 0.$$

This result has been generalized and extended to the random fields $\{X_{\mathbf{n}}, \mathbf{n} \in Z_+^d\}$ of random variables. For example, Fazekas and Tómács [2] and Czerebak-Mrozowicz *et al.* [3] for fields of pairwise independent random variables, and Gut and Stadtmüller [4] for random fields of i.i.d. random variables.

Let Z_+ be the set of positive integers. For fixed $d \in Z_+$, set $Z_+^d = \{\mathbf{n} = (n_1, n_2, \ldots, n_d) : n_i \in Z_+, i = 1, 2, \ldots, d\}$ with coordinatewise partial order, $\le$, *i.e.*, for $\mathbf{m} = (m_1, m_2, \ldots, m_d)$, $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in Z_+^d$, $\mathbf{m} \le \mathbf{n}$ if and only if $m_i \le n_i$, $i = 1, 2, \ldots, d$. For $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in Z_+^d$, let $|\mathbf{n}| = \prod_{i=1}^{d} n_i$. For a field $\{a_{\mathbf{n}}, \mathbf{n} \in Z_+^d\}$ of real numbers, the limit superior is defined by $\inf_{r \ge 1} \sup_{|\mathbf{n}| \ge r} a_{\mathbf{n}}$ and is denoted by $\limsup_{|\mathbf{n}| \to \infty} a_{\mathbf{n}}$.

Note that $|\mathbf{n}| \to \infty$ is equivalent to $\max\{n_1, n_2, \ldots, n_d\} \to \infty$, which is weaker than the condition $\min\{n_1, n_2, \ldots, n_d\} \to \infty$ when $d \ge 2$.

Let $\{X_{\mathbf{n}}, \mathbf{n} \in Z_+^d\}$ be a field of random variables, and let $\{a_{\mathbf{n},\mathbf{k}}, \mathbf{n} \in Z_+^d, \mathbf{k} \le \mathbf{n}\}$ be an array of real numbers. The weighted sums $\sum_{\mathbf{k} \le \mathbf{n}} a_{\mathbf{n},\mathbf{k}} X_{\mathbf{k}}$ can play an important role in various applied and theoretical problems, such as those of the least squares estimators (see Kafles

and Bhaskara Rao [5]) and M-estimates (see Rao and Zhao [6]) in linear models, the non-parametric regression estimators (see Priestley and Chao [7]), *etc.* So, the study of the limiting behavior of the weighted sums is very important and significant (see Chen and Hao [8]).

Now, we consider the notion of martingale differences. Let $\{\mathcal{G}_{\mathbf{k}}, \mathbf{k} \in Z_+^d\}$ be a family of σ-algebras such that

$$\mathcal{G}_{\mathbf{k}} \subset \mathcal{G}_{\mathbf{n}} \subset \mathcal{G}, \quad \forall \mathbf{k} \leq \mathbf{n},$$

and for any integrable random variable X defined on the initial probability space,

$$E\big(E(X|\mathcal{G}_{\mathbf{k}})|\mathcal{G}_{\mathbf{m}}\big) = E(X|\mathcal{G}_{\mathbf{k} \wedge \mathbf{m}}) \quad \text{a.s.,} \tag{1.1}$$

where $\mathbf{k} \wedge \mathbf{m}$ denotes the componentwise minimum.

An $\{\mathcal{G}_{\mathbf{k}}, \mathbf{k} \in Z_+^d\}$-adapted, integrable process $\{Y_{\mathbf{k}}, \mathbf{k} \in Z_+^d\}$ is called a martingale if and only if

$$E(Y_{\mathbf{n}}|\mathcal{G}_{\mathbf{m}}) = Y_{\mathbf{m} \wedge \mathbf{n}} \quad \text{a.s.}$$

Let us observe that for martingale $\{(Y_{\mathbf{n}}, \mathcal{G}_{\mathbf{n}}), \mathbf{n} \in Z_+^d\}$, the random variables

$$X_{\mathbf{n}} = \sum_{\mathbf{a} \in \{0,1\}^d} (-1)^{\sum_{i=1}^d a_i} Y_{\mathbf{n}-\mathbf{a}},$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_d)$ and $\mathbf{n} \in Z_+^d$, are martingale differences with respect to $\{\mathcal{G}_{\mathbf{n}}, \mathbf{n} \in Z_+^d\}$ (see Kuczmaszewska and Lagodowski [9]).

For the results concerning complete convergence for martingale arrays obtained in the one-dimensional case, we refer to Lagodowski and Rychlik [10], Elton [11], Lesigne and Volny [12], Stoica [13] and Ghosal and Chandra [14]. Recently, complete convergence for martingale difference random fields was proved by Kuczmaszewska and Lagodowski [9].

The aim of this paper is to obtain some results concerning complete convergence of weighted sums $\sum_{\mathbf{i} \leq \mathbf{n}} a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}$, where $\{a_{\mathbf{n},\mathbf{i}}, \mathbf{n} \in Z_+^d, \mathbf{i} \leq \mathbf{n}\}$ is an array of real numbers, and $\{X_{\mathbf{i}}, \mathbf{i} \in Z_+^d\}$ is the martingale differences with respect to $\{\mathcal{G}_{\mathbf{n}}, \mathbf{n} \in Z_+^d\}$ satisfying (1.1).

2 Results

The following moment maximal inequality provides us a useful tool to prove the main results of this section (see Kuczmaszewska and Lagodowski [9]).

Lemma 2.1 *Let* $\{(Y_{\mathbf{n}}, \mathcal{G}_{\mathbf{n}}), \mathbf{n} \in Z_+^d\}$ *be a martingale, and let* $\{(X_{\mathbf{n}}, \mathcal{G}_{\mathbf{n}}), \mathbf{n} \in Z_+^d\}$ *be the martingale differences corresponding to it. Let* $q > 1$. *There exists a finite and positive constant* C *depending only on* q *and* d *such that*

$$E\Big(\max_{\mathbf{k} \leq \mathbf{n}} |Y_{\mathbf{k}}|^q\Big) \leq CE\Big(\sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}}^2\Big)^{q/2}. \tag{2.1}$$

Let us denote $\mathcal{G}_{\mathbf{i}}^* = \sigma\{\mathcal{G}_{\mathbf{j}} : \mathbf{j} < \mathbf{i}\}$. Now, we are ready to formulate the next result.

Theorem 2.2 *Let* $\{a_{\mathbf{n}}, \mathbf{i}, \mathbf{n} \in Z_+^d, \mathbf{i} \le \mathbf{n}\}$ *be an array of real numbers, and let* $\{X_{\mathbf{n}}, \mathbf{n} \in Z_+^d\}$ *be the martingale differences with respect to* $\{\mathcal{G}_{\mathbf{n}}, \mathbf{n} \in Z_+^d\}$ *satisfying* (1.1). *For* $\alpha p > 1$, $p > 1$ *and* $\alpha > \frac{1}{2}$, *we assume that*

(i) $\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p - 2} \sum_{\mathbf{i} \le \mathbf{n}} P\{|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| > |\mathbf{n}|^\alpha\} < \infty$,

(ii) $\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha(p-q)-3+q/2} \sum_{\mathbf{i} \le \mathbf{n}} |a_{\mathbf{n},\mathbf{i}}|^q E(|X_{\mathbf{i}}|^q I[|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| \le |\mathbf{n}|^\alpha]) < \infty$ *for* $q \ge 2$,

(ii)′ $\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha(p-q)-2} \sum_{\mathbf{i} \le \mathbf{n}} |a_{\mathbf{n},\mathbf{i}}|^q E(|X_{\mathbf{i}}|^q I[|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| \le |\mathbf{n}|^\alpha]) < \infty$ *for* $1 < q < 2$ *and*

(iii) $\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p - 2} P\{\max_{\mathbf{j} \le \mathbf{n}} |\sum_{\mathbf{i} \le \mathbf{j}} E(a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}} I[|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| \le |\mathbf{n}|^\alpha] |\mathcal{G}_{\mathbf{i}}^*)| > \epsilon |\mathbf{n}|^\alpha\} < \infty$ *for all* $\epsilon > 0$.

Then we have

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p - 2} P\left\{ \max_{\mathbf{j} \le \mathbf{n}} |S_{\mathbf{j}}| > \epsilon |\mathbf{n}|^\alpha \right\} < \infty \quad \text{for all } \epsilon > 0, \tag{2.2}$$

where $S_{\mathbf{n}} = \sum_{1 \le \mathbf{i} \le \mathbf{n}} a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}$.

Proof Let us notice that the series $\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p - 2}$ is finite, then (2.2) always holds. Therefore, we consider only the case such that $\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p - 2}$ is divergent. Let $X_{\mathbf{n},\mathbf{i}} = X_{\mathbf{i}} I[|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| \le |\mathbf{n}|^\alpha]$, $X_{\mathbf{n},\mathbf{i}}^* = X_{\mathbf{n},\mathbf{i}} - E(X_{\mathbf{n},\mathbf{i}} |\mathcal{G}_{\mathbf{i}}^*)$ and $S_{\mathbf{n},\mathbf{j}}^* = \sum_{\mathbf{i} \le \mathbf{j}} a_{\mathbf{n},\mathbf{i}} X_{\mathbf{n},\mathbf{i}}^*$.

Then

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p - 2} P\left\{ \max_{\mathbf{j} \le \mathbf{n}} |S_{\mathbf{j}}| > \epsilon |\mathbf{n}|^\alpha \right\}$$

$$\le \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p - 2} P\left\{ |a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| > |\mathbf{n}|^\alpha \right\}$$

$$+ \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p - 2} P\left\{ \max_{\mathbf{j} \le \mathbf{n}} \left| \sum_{\mathbf{i} \le \mathbf{j}} a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}} I[|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| \le |\mathbf{n}|^\alpha] \right| > \epsilon |\mathbf{n}|^\alpha \right\}$$

$$\le \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p - 2} \sum_{\mathbf{i} \le \mathbf{n}} P\left\{ |a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| > |\mathbf{n}|^\alpha \right\} + \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p - 2} P\left\{ \max_{\mathbf{j} \le \mathbf{n}} \left| \sum_{\mathbf{i} \le \mathbf{j}} (a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}} I[|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| \le |\mathbf{n}|^\alpha] \right.\right.$$

$$\left.\left. - E(a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}} I[|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| \le |\mathbf{n}|^\alpha] |\mathcal{G}_{\mathbf{i}}^*)) \right| > \frac{\epsilon}{2} |\mathbf{n}|^\alpha \right\}$$

$$+ \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p - 2} P\left\{ \max_{\mathbf{j} \le \mathbf{n}} \left| \sum_{\mathbf{i} \le \mathbf{j}} E(a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}} I[|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| \le |\mathbf{n}|^\alpha] |\mathcal{G}_{\mathbf{i}}^*) \right| > \frac{\epsilon}{2} |\mathbf{n}|^\alpha \right\}$$

$$= I_1 + I_2 + I_3.$$

Clearly, $I_1 < \infty$ by (i), and $I_3 < \infty$ by (iii). It remains to prove that $I_2 < \infty$. Thus, the proof will be completed by proving that

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p - 2} P\left\{ \max_{\mathbf{j} \le \mathbf{n}} |S_{\mathbf{n},\mathbf{j}}^*| > \epsilon |\mathbf{n}|^\alpha \right\} < \infty.$$

To prove it, we first observe that $\{(S_{\mathbf{n},\mathbf{j}}^*, \mathcal{G}_{\mathbf{j}}), \mathbf{j} \le \mathbf{n}\}$ is a martingale. In fact, if $\mathbf{i} > \mathbf{j}$, then $\mathcal{G}_{\mathbf{i} \wedge \mathbf{j}} \subset \mathcal{G}_{\mathbf{i}}^*$ and by (1.1), we have

$$E(a_{\mathbf{n},\mathbf{i}} X_{\mathbf{n},\mathbf{i}}^* |\mathcal{G}_{\mathbf{j}}) = E(a_{\mathbf{n},\mathbf{i}} X_{\mathbf{n},\mathbf{i}} - E(a_{\mathbf{n},\mathbf{i}} X_{\mathbf{n},\mathbf{i}} |\mathcal{G}_{\mathbf{i}}^*) |\mathcal{G}_{\mathbf{i}})$$

$$= E(E(a_{\mathbf{n},\mathbf{i}} X_{\mathbf{n},\mathbf{i}} - E(a_{\mathbf{n},\mathbf{i}} X_{\mathbf{n},\mathbf{i}} |\mathcal{G}_{\mathbf{i}}^*) |\mathcal{G}_{\mathbf{i}}) |\mathcal{G}_{\mathbf{j}})$$

$$= E\big(a_{\mathbf{n},\mathbf{i}}X_{\mathbf{n},\mathbf{i}} - E\big(a_{\mathbf{n},\mathbf{i}}X_{\mathbf{n},\mathbf{i}}|\mathcal{G}_{\mathbf{i}}^*\big)|\mathcal{G}_{\mathbf{i}\wedge\mathbf{j}}\big)$$

$$= 0.$$

Then, by the Markov inequality and Lemma 2.1, there exists some constant C such that

$$P\Big\{\max_{\mathbf{j}\leq\mathbf{n}}\big|S_{\mathbf{n},\mathbf{j}}^*\big| > \epsilon|\mathbf{n}|^\alpha\Big\} \leq C\frac{E(\max_{\mathbf{j}\leq\mathbf{n}}|S_{\mathbf{n},\mathbf{j}}^*|^q)}{|\mathbf{n}|^{\alpha q}}$$

$$\leq \frac{C}{|\mathbf{n}|^{\alpha q}}E\bigg(\sum_{\mathbf{i}\leq\mathbf{n}}a_{\mathbf{n},\mathbf{i}}^2X_{\mathbf{n},\mathbf{i}}^{*2}\bigg)^{q/2} = I_4.$$

Case $q \geq 2$; we get

$$I_4 \leq \frac{C}{|\mathbf{n}|^{\alpha q}}|\mathbf{n}|^{q/2-1}\sum_{\mathbf{i}\leq\mathbf{n}}E\big|a_{\mathbf{n},\mathbf{i}}X_{\mathbf{n},\mathbf{i}}^*\big|^q$$

$$\leq C|\mathbf{n}|^{q/2-1-\alpha q}\sum_{\mathbf{i}\leq\mathbf{n}}E\big(|a_{\mathbf{n},\mathbf{i}}X_{\mathbf{i}}|^q I\big[|a_{\mathbf{n},\mathbf{i}}X_{\mathbf{i}}| \leq |\mathbf{n}|^\alpha\big]\big).$$

Note that the last estimation follows from the Jensen inequality. Thus, we have

$$\sum_{\mathbf{n}}|\mathbf{n}|^{\alpha p-2}P\Big\{\max_{\mathbf{j}\leq\mathbf{n}}\big|S_{\mathbf{n},\mathbf{j}}^*\big| > \epsilon|\mathbf{n}|^\alpha\Big\}$$

$$\leq C\sum_{\mathbf{n}}|\mathbf{n}|^{\alpha p-3-q(\alpha-1/2)}\sum_{\mathbf{i}\leq\mathbf{n}}E\big(|a_{\mathbf{n},\mathbf{i}}X_{\mathbf{i}}|^q I\big[|a_{\mathbf{n},\mathbf{i}}X_{\mathbf{i}}| \leq |\mathbf{n}|^\alpha\big]\big) < \infty$$

by assumption (ii).

Case $1 < q < 2$; we get

$$I_4 \leq \frac{C}{|\mathbf{n}|^{\alpha q}}\sum_{\mathbf{i}\leq\mathbf{n}}E\big|a_{\mathbf{n},\mathbf{i}}X_{\mathbf{n},\mathbf{i}}^*\big|^q$$

$$\leq C|\mathbf{n}|^{-\alpha q}\sum_{\mathbf{i}\leq\mathbf{n}}E\big(|a_{\mathbf{n},\mathbf{i}}X_{\mathbf{i}}|^q I\big[|a_{\mathbf{n},\mathbf{i}}X_{\mathbf{i}}| \leq |\mathbf{n}|^\alpha\big]\big).$$

Therefore, for $1 < q < 2$, we obtain

$$\sum_{\mathbf{n}}|\mathbf{n}|^{\alpha p-2}P\Big\{\max_{\mathbf{j}\leq\mathbf{n}}\big|S_{\mathbf{n},\mathbf{j}}^*\big| > \epsilon|\mathbf{n}|^\alpha\Big\}$$

$$\leq C\sum_{\mathbf{n}}|\mathbf{n}|^{\alpha(p-q)-2}\sum_{\mathbf{i}\leq\mathbf{n}}E\big(|a_{\mathbf{n},\mathbf{i}}X_{\mathbf{i}}|^q I\big[|a_{\mathbf{n},\mathbf{i}}X_{\mathbf{i}}| \leq |\mathbf{n}|^\alpha\big]\big) < \infty$$

by assumption (ii)$'$. Thus, $I_2 < \infty$ for all $q > 1$, and the proof of Theorem 2.2 is complete.
□

Corollary 2.3 *Let $\{a_{\mathbf{n}}, \mathbf{i}, \mathbf{n} \in Z_+^d, \mathbf{i} \leq \mathbf{n}\}$ be an array of real numbers. Let $\{X_{\mathbf{n}}, \mathbf{n} \in Z_+^d\}$ be martingale differences with respect to $\{\mathcal{G}_{\mathbf{n}}, \mathbf{n} \in Z_+^d\}$ satisfying (1.1), and $EX_{\mathbf{n}} = 0$ for $\mathbf{n} \in Z_+^d$. Let $p \geq 1, \alpha > \frac{1}{2}$ and $\alpha p > 1$. Assume that* (i) *and for some $q > 1$,* (ii) *or* (ii)$'$ *hold respectively. If*

$$\max_{\mathbf{j}\leq\mathbf{n}}\sum_{\mathbf{i}\leq\mathbf{j}}E\big(a_{\mathbf{n},\mathbf{i}}X_{\mathbf{i}}I\big[|a_{\mathbf{n},\mathbf{i}}X_{\mathbf{i}}| \leq |\mathbf{n}|^\alpha\big]|\mathcal{G}_{\mathbf{i}}^*\big) = o\big(|\mathbf{n}|^\alpha\big), \tag{2.3}$$

then (2.2) *holds.*

Proof It is easy to see that (2.3) implies (iii). We omit details that prove it. □

The following corollary shows that assumption (iii) in Theorem 2.2 is natural, and in the case of independent random fields, it reduces to the known one.

Corollary 2.4 *Let* $\{a_{\mathbf{n}}, \mathbf{i}, \mathbf{n} \in Z_+^d, \mathbf{i} \le \mathbf{n}\}$ *be an array of real numbers. Let* $\{X_{\mathbf{n}}, \mathbf{n} \in Z_+^d\}$ *be a field of independent random variables such that* $EX_{\mathbf{n}} = 0$ *for* $\mathbf{n} \in Z_+^d$. *Let* $p \ge 1$, $\alpha > \frac{1}{2}$ *and* $\alpha p > 1$. *Assume that* (i) *and for some* $q > 1$, (ii) *or* (ii)′ *hold respectively. If*

$$\frac{1}{|\mathbf{n}|^\alpha} \max_{\mathbf{j} \le \mathbf{n}} \sum_{\mathbf{i} \le \mathbf{j}} E\big(a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}} I\big[|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| \le |\mathbf{n}|^\alpha\big]\big) \to 0 \quad as \ |\mathbf{n}| \to \infty, \tag{2.4}$$

then (2.2) *holds.*

Proof Since $\{X_{\mathbf{n}}, \mathbf{n} \in Z_+^d\}$ is a field of independent random variables, we have

$$\frac{1}{|\mathbf{n}|^\alpha} \max_{\mathbf{j} \le \mathbf{n}} \sum_{\mathbf{i} \le \mathbf{j}} E\big(a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}} I\big[|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| \le |\mathbf{n}|^\alpha\big] | \mathcal{G}_{\mathbf{i}}^*\big) = \frac{1}{|\mathbf{n}|^\alpha} \max_{\mathbf{j} \le \mathbf{n}} \sum_{\mathbf{i} \le \mathbf{j}} E\big(a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}} I\big[|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| \le |\mathbf{n}|^\alpha\big]\big).$$

Now, it is easy to see that (2.4) implies (iii) of Theorem 2.2. Thus, by Theorem 2.2, result (2.2) follows. □

Remark Theorem 2.2 and Corollary 2.4 are extensions of Theorem 4.1 and Corollary 4.1 in Kuczmaszewska and Lagodowski [9] to the weighted sums case, respectively.

Corollary 2.5 *Let* $\{a_{\mathbf{n}}, \mathbf{i}, \mathbf{n} \in Z_+^d, \mathbf{1} \le \mathbf{i} \le \mathbf{n}\}$ *be an array of real numbers. Let* $\{X_{\mathbf{n}}, \mathbf{n} \in Z_+^d\}$ *be the martingale differences with respect to* $\{\mathcal{G}_{\mathbf{n}}, \mathbf{n} \in Z_+^d\}$ *satisfying* (1.1) *and* $EX_{\mathbf{n}} = 0$. *Let* $p \ge 1$, $\alpha > \frac{1}{2}$ *and* $\alpha p > 1$ *and* $E|X_{\mathbf{n}}|^{1+\lambda_{\mathbf{n}}} < \infty$ *for* $\lambda_{\mathbf{n}}$ *with* $0 < \lambda_{\mathbf{n}} < 1$ *for* $\mathbf{n} \in Z_+^d$. *If*

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p - 2} |\mathbf{n}|^{-\alpha(1+\lambda_{\mathbf{n}})} \sum_{\mathbf{i} \le \mathbf{n}} |a_{\mathbf{n},\mathbf{i}}|^{1+\lambda_{\mathbf{n}}} E|X_{\mathbf{i}}|^{1+\lambda_{\mathbf{n}}} < \infty, \tag{2.5}$$

$$\max_{\mathbf{1} \le \mathbf{j} \le \mathbf{n}} \sum_{\mathbf{i} \le \mathbf{j}} E\big(|a_{\mathbf{n},\mathbf{i}} X_i| I\big[|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| \le |\mathbf{n}|^\alpha\big] | \mathcal{G}_{\mathbf{i}}^*\big) = o\big(|\mathbf{n}|^\alpha\big), \tag{2.6}$$

then (2.2) *holds.*

Proof If the series $\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p - 2} < \infty$, then (2.2) always holds. Hence, we only consider the case $\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p - 2} = \infty$. It follows from (2.5) that

$$|\mathbf{n}|^{-\alpha(1+\lambda_{\mathbf{n}})} \sum_{\mathbf{i} \le \mathbf{n}} |a_{\mathbf{n},\mathbf{i}}|^{1+\lambda_{\mathbf{n}}} E|X_i|^{1+\lambda_{\mathbf{n}}} < 1.$$

By (2.5) and the Markov inequality,

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p - 2} P\big(|a_{\mathbf{n},\mathbf{i}} X_{\mathbf{i}}| > |\mathbf{n}|^\alpha\big)$$

$$\le \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha p - 2} |\mathbf{n}|^{-\alpha(1+\lambda_{\mathbf{n}})} \sum_{\mathbf{i} \le \mathbf{n}} |a_{\mathbf{n},\mathbf{i}}|^{1+\lambda_{\mathbf{n}}} E|X_{\mathbf{i}}|^{1+\lambda_{\mathbf{n}}} < \infty, \tag{2.7}$$

which satisfies (i) of Theorem 2.2.

As the proof of Corollary 2.3, (2.6) implies (iii) of Theorem 2.2.

It remains to show that Theorem 2.2(ii) or (ii)$'$ is satisfied.

For $1 < q < 2$, take $1 + \lambda_\mathbf{n} < q$. Then we have

$$\sum_\mathbf{n} |\mathbf{n}|^{\alpha(p-q)-2} \sum_{\mathbf{i} \leq \mathbf{n}} |a_{\mathbf{n},\mathbf{i}}|^q E\big(|X_\mathbf{i}|^q I\big[|a_{\mathbf{n},\mathbf{i}} X_\mathbf{i}| \leq |\mathbf{n}|^\alpha\big]\big)$$

$$\leq \sum_\mathbf{n} |\mathbf{n}|^{\alpha p - 2} |\mathbf{n}|^{-\alpha(1+\lambda_\mathbf{n})} |\mathbf{n}|^{-\alpha q + \alpha(1+\lambda_\mathbf{n})} |\mathbf{n}|^{\alpha q - \alpha(1+\lambda_\mathbf{n})} \sum_{\mathbf{i} \leq \mathbf{n}} |a_{\mathbf{n},\mathbf{i}}|^{1+\lambda_\mathbf{n}} E|X_\mathbf{i}|^{1+\lambda_\mathbf{n}}$$

$$= \sum_\mathbf{n} |\mathbf{n}|^{\alpha p - 2} |\mathbf{n}|^{-\alpha(1+\lambda_\mathbf{n})} \sum_{\mathbf{i} \leq \mathbf{n}} |a_{\mathbf{n},\mathbf{i}}|^{1+\lambda_\mathbf{n}} E|X_i|^{1+\lambda_\mathbf{n}} < \infty \quad \text{by (2.5),}$$

which satisfies Theorem 2.2(ii)$'$. Hence, the proof is complete. □

Corollary 2.6 *Let $\{a_\mathbf{n}, \mathbf{i}, \mathbf{n} \in Z_+^d, \mathbf{1} \leq \mathbf{i} \leq \mathbf{n}\}$ be an array of real numbers, and let $\{X_\mathbf{n}, \mathbf{n} \in Z_+^d\}$ be the martingale differences with respect to $\{\mathcal{G}_\mathbf{n}, \mathbf{n} \in Z_+^d\}$ satisfying (1.1), $EX_\mathbf{n} = 0$ and $E|X_\mathbf{n}|^p < \infty$ for $1 < p < 2$. Let $\alpha > \frac{1}{2}$, $\alpha p > 1$ and $1 < p < 2$. If*

$$\sum_{\mathbf{1} \leq \mathbf{i} \leq \mathbf{n}} |a_{\mathbf{n},\mathbf{i}}|^p E|X_\mathbf{i}|^p = O\big(|\mathbf{n}|^\delta\big) \quad \text{for } 0 < \delta < 1, \tag{2.8}$$

and Theorem 2.2(iii) hold, then (2.2) holds.

Proof By (2.8) and the Markov inequality,

$$\sum_\mathbf{n} |\mathbf{n}|^{\alpha p - 2} \sum_{\mathbf{i} \leq \mathbf{n}} P\big(|a_{\mathbf{n},\mathbf{i}} X_\mathbf{i}| > |\mathbf{n}|^\alpha\big)$$

$$\leq \sum_\mathbf{n} |\mathbf{n}|^{\alpha p - 2} \sum_{\mathbf{i} \leq \mathbf{n}} \frac{|a_{\mathbf{n},\mathbf{i}}|^p E|X_\mathbf{i}|^p}{|\mathbf{n}|^{\alpha p}}$$

$$\leq C \sum_\mathbf{n} |\mathbf{n}|^{-2+\delta} < \infty. \tag{2.9}$$

By taking $q < p$, we have

$$\sum_\mathbf{n} |\mathbf{n}|^{\alpha(p-q)-2} \sum_{\mathbf{i} \leq \mathbf{n}} |a_{\mathbf{n},\mathbf{i}}|^q E\big(|X_\mathbf{i}|^q I\big[|a_{\mathbf{n},\mathbf{i}} X_\mathbf{i}| \leq \epsilon |\mathbf{n}|^\alpha\big]\big)$$

$$\leq \sum_\mathbf{n} |\mathbf{n}|^{-2} \sum_{\mathbf{i} \leq \mathbf{n}} |a_{\mathbf{n},\mathbf{i}}|^p E|X_\mathbf{i}|^p$$

$$\leq C \sum_\mathbf{n} |\mathbf{n}|^{-2+\delta} < \infty. \tag{2.10}$$

Hence, by (2.9) and (2.10), conditions (i) and (ii)$'$ in Theorem 2.2 are satisfied, respectively.

To complete the proof, it is enough to note that by $EX_\mathbf{n} = 0$ for $\mathbf{n} \in Z_+^d$ and by (2.8), we get for $\mathbf{j} \leq \mathbf{n}$

$$|\mathbf{n}|^{-\alpha} \sum_{\mathbf{i} \leq \mathbf{j}} |a_{\mathbf{n},\mathbf{i}}| E|X_\mathbf{i}| I\big[|a_{\mathbf{n},\mathbf{i}} X_\mathbf{i}| \leq \epsilon |\mathbf{n}|^\alpha\big] \to 0 \quad \text{as } |\mathbf{n}| \to \infty. \tag{2.11}$$

Hence, the proof is complete. □

Corollary 2.7 *Let $\{X_\mathbf{n}, \mathbf{n} \in Z_+^d\}$ be the martingale differences with respect to $\{\mathcal{G}_\mathbf{n}, \mathbf{n} \in Z_+^d\}$ satisfying (1.1), let $EX_\mathbf{n} = 0$ and $E|X_\mathbf{n}|^p < \infty$ for $1 < p < 2$ and be stochastically dominated by a random variable X, i.e., there is a constant D such that $P(|X_\mathbf{n}| > x) \leq DP(|X| > x)$ for all $x \geq 0$ and $\mathbf{n} \in Z_+^d$. Let $\{a_\mathbf{n,i}, \mathbf{n} \in Z_+^d, \mathbf{i} \leq \mathbf{n}\}$ be an array of real numbers satisfying*

$$\sum_{\mathbf{i} \leq \mathbf{n}} |a_\mathbf{n,i}|^p = O(|\mathbf{n}|^\delta) \quad \text{for } 0 < \delta < 1. \tag{2.12}$$

If Theorem 2.2(iii) *holds, then* (2.2) *holds.*

Proof From (2.12), (2.8) follows. Hence, by Corollary 2.6, we obtain (2.2). □

Remark Linear random fields are of great importance in time series analysis. They arise in a wide variety of context. Applications to economics, engineering, and physical science are extremely broad (see Kim *et al.* [15]).

Let $Y_\mathbf{k} = \sum_{\mathbf{i} \geq 1} a_\mathbf{k+i} X_\mathbf{i}$, where $\{a_\mathbf{i}, \mathbf{i} \in Z_+^d\}$ is a field of real numbers with $\sum_{\mathbf{i} \geq \mathbf{1}} |a_\mathbf{i}| < \infty$, and $\{X_\mathbf{i}, \mathbf{i} \in Z_+^d\}$ is a field of the martingale difference random variables.

Define $a_\mathbf{n,i} = \sum_{\mathbf{1} \leq \mathbf{k} \leq \mathbf{n}} a_\mathbf{i+k}$. Then we have

$$\sum_{\mathbf{1} \leq \mathbf{k} \leq \mathbf{n}} Y_\mathbf{k} = \sum_{\mathbf{1} \leq \mathbf{k} \leq \mathbf{n}} \sum_{\mathbf{i} \geq 1} a_\mathbf{i+k} X_\mathbf{i} = \sum_{\mathbf{i} \geq 1} \sum_{\mathbf{1} \leq \mathbf{k} \leq \mathbf{n}} a_\mathbf{i+k} X_\mathbf{i} = \sum_{\mathbf{i} \geq 1} a_\mathbf{n,i} X_\mathbf{i}.$$

Hence, we can use the above results to investigate the complete convergence for linear random fields.

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