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Some inequalities related to (i,j)-type L_p-mixed affine surface area and L_p-mixed curvature image

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Abstract

In this article, we introduce two concepts: the $(i,0)$-type L_p-mixed affine surface area and (i,j)-type L_p-mixed affine surface area in the set of convex bodies such that L_p-affine surface area by Lutwak *et al.* is proposed in its special cases. Besides, applying these concepts, we establish the extension results of the well-known L_p-Petty affine projection inequality, L_p-Busemann centroid inequality and its dual inequality.

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1 Introduction

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space $\mathbb{R}^n$. For the set of convex bodies, which contain the origin in their interiors, and the set of origin-symmetric convex bodies in $\mathcal{K}^n$, we write $\mathcal{K}^n_o$ and $\mathcal{K}^n_e$, respectively. Let S^n_o denote the set of star bodies (about the origin) in $\mathbb{R}^n$, and let S^n_e denote the set of origin-symmetric star bodies in $\mathcal{S}^n_o$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, and let $V(K)$ denote the n-dimensional volume of body K. If K is the standard unit ball B in $\mathbb{R}^n$, then it is denoted by $\omega_n = V(B)$. Note that $\omega_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)}$ defines ω_n for all non-negative real n (not just the positive integers).

A body $K \in \mathcal{K}^n$ is said to have a continuous ith curvature function $f_i(K,\cdot) : S^{n-1} \to [0,\infty)$ if and only if $S_i(K,\cdot)$ is absolutely continuous with respect to S and has the Radon-Nikodym derivative (see [1])

$$\frac{dS_i(K,\cdot)}{dS} = f_i(K,\cdot). \tag{1.1}$$

Let $\mathcal{F}^n_i$ denote the subset of all bodies $\mathcal{K}^n$ which have a positive continuous ith curvature function. Let $\mathcal{F}^n_{i,o}$, $\mathcal{F}^n_{i,e}$ denote the set of all bodies in $\mathcal{K}^n_o$, $\mathcal{K}^n_e$, respectively, and both of them have a positive continuous ith curvature function.

A convex body $K \in \mathcal{K}^n_o$ is said to have an L_p-curvature function $f_p(K,\cdot) : S^{n-1} \to \mathbb{R}$ if its L_p-surface area measure $S_p(K,\cdot)$ is absolutely continuous with respect to the spherical Lebesgue measure S, and it has the Radon-Nikodym derivative (see [2])

$$\frac{dS_p(K,\cdot)}{dS} = f_p(K,\cdot). \tag{1.2}$$

For $p \geq 1$, $K \in \mathcal{F}_o^n$, then L_p-affine surface area $\Omega_p(K)$ of K by (see [3–5])

$$\Omega_p(K) = \int_{S^{n-1}} f_p(K,u)^{\frac{n}{n+p}}\, dS(u). \tag{1.3}$$

For each $K \in \mathcal{K}^n$ and $p \geq 1$, the L_p-projection body, $\Pi_p K$, of K is an origin-symmetric convex body whose support function is given by (see [6])

$$h^p_{\Pi_p K}(u) = \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} \left|\langle u,v\rangle\right|^p dS_p(K,v), \quad \text{for all } u \in S^{n-1}, \tag{1.4}$$

where $c_{n,p} = \frac{\omega_{n+p}}{\omega_2\omega_n\omega_{p-1}}$. When $p = 1$, (1.4) is the notion of projection body (see [7]).

It is easy to show that if E is an ellipsoid which is centered at the origin, then (see [8, p.105])

$$\Pi_p^* E = \left(\frac{\omega_n}{V(E)}\right)^{\frac{1}{p}} E. \tag{1.5}$$

The well-known L_p-Petty affine projection inequality is expressed as follows (see [8–10]).

Theorem A (L_p-Petty affine projection inequality) *If $K, L \in \mathcal{F}_o^n$, $p \geq 1$, then*

$$n\omega_n^{\frac{n}{n+p}} V(\Pi_p K)^{\frac{p}{n+p}} \geq \Omega_p(K), \tag{1.6}$$

with equality if and only if K is an ellipsoid which is centered at the origin.

Let $K \in \mathcal{S}_o^n$, and let $p \geq 1$, then the L_p-centroid body, $\Gamma_p K$, of K is the origin-symmetric convex body whose support function is given by (see [6, 11])

$$h^p_{\Gamma_p K}(u) = \frac{1}{c_{n,p} V(K)} \int_K \left|\langle u,x\rangle\right|^p dx \quad \text{for all } u \in S^{n-1}. \tag{1.7}$$

If E is an ellipsoid which is centered at the origin, then $\Gamma_p E = E$. In particular, $\Gamma_p B = B$.

The well-known L_p-Busemann-Petty centroid inequality is as follows (see [6]).

Theorem B (L_p-Busemann-Petty centroid inequality) *If $K \in S_o^n$ and $p \geq 1$, then*

$$V(\Gamma_p K) \geq V(K), \tag{1.8}$$

with the equality if and only if K is an ellipsoid which is centered at the origin.

Lutwak *et al.* introduced the concept of dual L_p-centroid bodies (see [12]). We give the concept of the unusual normalization of dual L_p-centroid bodies such that $\Gamma_{-p} B = B$: Let $K \in \mathcal{K}_o^n$ and real $p > 0$, then radial function of dual L_p-centroid body, $\Gamma_{-p} K$, of K is defined by

$$\rho^{-p}_{\Gamma_{-p} K}(u) = \frac{1}{(n+p)c_{n,p} V(K)} \int_{S^{n-1}} \left|\langle u,v\rangle\right|^p dS_p(K,v) \quad \text{for all } u \in S^{n-1}. \tag{1.9}$$

It is easy to show that if E is an ellipsoid which is centered at the origin, then

$$\Gamma_{-p}E = \left(\frac{V(E)}{\omega_n}\right)^{\frac{1}{p}} \Pi_p^* E. \tag{1.10}$$

Combined with (1.5) and (1.10), we have that $\Gamma_{-p}E = E$. In particular, $\Gamma_{-p}B = B$.

Si Lin gives the following dual inequality of inequality (1.8) (see [13, p.9, Theorem 5.4]).

Theorem C (Dual L_p-Busemann-Petty centroid inequality) *If $K \in \mathcal{K}_o^n$ and $p \geq 1$, then*

$$V(\Gamma_{-p}K) \leq V(K), \tag{1.11}$$

with equality if and only if K is an ellipsoid which is centered at the origin.

Liu *et al.* [14], Lu and Wang [15], Ma and Liu [16, 17] independently proposed the notion of L_p-mixed curvature function: Let $p \geq 1$, $i = 0, 1, \ldots, n-1$, a convex body $K \in \mathcal{K}_o^n$ is said to have an L_p-mixed curvature function $f_{p,i}(K,\cdot) : S^{n-1} \to \mathbb{R}$, if its L_p-mixed surface area measure $S_{p,i}(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S and has the Radon-Nikodym derivative

$$\frac{dS_{p,i}(K,\cdot)}{dS} = f_{p,i}(K,\cdot). \tag{1.12}$$

If the mixed surface area measure $S_i(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S, we have

$$f_{p,i}(K,u) = f_i(K,u)h(K,u)^{1-p}. \tag{1.13}$$

According to the concept of L_p-mixed curvature function of convex body, Lu and Wang [15] and Ma introduce the concept of L_p-mixed curvature image of convex body as follows: For each $K \in \mathcal{F}_{i,o}^n$ $(i = 0, 1, \ldots, n-1)$ and real $p \geq 1$, define $\Lambda_{p,i}K \in \mathcal{S}_o^n$, the L_p-mixed curvature image of K, by

$$\rho(\Lambda_{p,i}K,\cdot)^{n+p-i} = \frac{\widetilde{W}_i(\Lambda_{p,i}K)}{\omega_n} f_{p,i}(K,\cdot). \tag{1.14}$$

If $i = 0$ in (1.14), then $\Lambda_{p,0}K = \Lambda_p K$. The unusual normalization of definition (1.14) is chosen so that for the unit ball B, we have $\Lambda_{p,i}B = B$. For $K \in \mathcal{F}_{i,o}^n$, $n - i \neq p \geq 1$, $\lambda > 0$,

$$\Lambda_{p,i}\lambda K = \lambda^{\frac{n-p-i}{p}} \Lambda_{p,i}K. \tag{1.15}$$

Let $\mathcal{C}_{i,o}^n$ denote the set of L_p-mixed curvature images of convex bodies in $\mathcal{F}_{i,o}^n$. That is,

$$\mathcal{C}_{i,o}^n = \left\{Q = \Lambda_{p,i}L : L \in \mathcal{F}_{i,o}^n\right\}.$$

Because the L_p-mixed curvature image belongs to star bodies, thus, $\mathcal{C}_{i,o}^n \subseteq \mathcal{S}_o^n$.

For each $K \in \mathcal{K}^n$, real $p \geq 1$ and $i = 0, 1, \ldots, n-1$, the L_p-mixed projection body, $\Pi_{p,i}K$, of K is an origin-symmetric convex body whose support function is given by (see [18])

$$h^p_{\Pi_{p,i}K}(u) = \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} \left|\langle u, v\rangle\right|^p \, dS_{p,i}(K, v), \quad \text{for all } u \in S^{n-1}, \tag{1.16}$$

where $S_{p,i}(K, \cdot)$ ($i = 0, 1, \ldots, n-1$) is L_p-mixed surface area measure. Obviously, $\Pi_{p,0}K = \Pi_p K$, and for the standard unit ball B, we have $\Pi_{p,i}B = B$. For $K \in \mathcal{K}^n$, $\lambda > 0$, $p \geq 1$ and $0 \leq i < n$, then

$$\Pi_{p,i}(\lambda K) = \lambda^{\frac{n-i-p}{p}} \Pi_{p,i}K. \tag{1.17}$$

Let $K \in \mathcal{S}^n_o$, real $p \geq 1$ and i be arbitrary real numbers, then the L_p-mixed centroid body, $\Gamma_{p,i}K$, of K is the origin-symmetric convex body whose support function is given by (see [19])

$$h^p_{\Gamma_{p,i}K}(u) = \frac{1}{(n+p)c_{n,p}\widetilde{W}_i(K)} \int_{S^{n-1}} \left|\langle u, v\rangle\right|^p \rho_K^{n+p-i}(v) \, dS(v) \quad \text{for all } u \in S^{n-1}. \tag{1.18}$$

Obviously, $\Gamma_{p,0}K = \Gamma_p K$, and for the standard unit ball B, we have $\Gamma_{p,i}B = B$.

Ma introduced the concept of dual L_p-mixed centroid body (see [19]). Further, we introduced the concept of the unusual normalization of dual L_p-mixed centroid body $\Gamma_{-p,i}K$ as follows: Let $K \in \mathcal{K}^n_o$, $p > 0$, $i = 0, 1, \ldots, n-1$, then the dual L_p-mixed centroid bodies, $\Gamma_{-p,i}K$, of K are defined by:

$$\rho^{-p}_{\Gamma_{-p,i}K}(u) = \frac{1}{(n+p)c_{n,p}W_i(K)} \int_{S^{n-1}} \left|\langle u, v\rangle\right|^p \, dS_{p,i}(K, v) \quad \text{for all } u \in S^{n-1}. \tag{1.19}$$

Obviously, $\Gamma_{-p,0}K = \Gamma_{-p}K$, and for the standard unit ball B, we have $\Gamma_{-p,i}B = B$.

In this article, we will first introduce the concept of $(i, 0)$-type L_p-mixed affine surface area of convex body as follows.

Definition 1.1 For $K \in \mathcal{F}^n_{i,o}$, $i = 0, 1, \ldots, n-1$ and $p \geq 1$, the $(i, 0)$-type L_p-mixed affine surface area, $\Omega^{(i)}_p(K)$, of K is defined by

$$\Omega^{(i)}_p(K) = \int_{S^{n-1}} f_{p,i}(K, u)^{\frac{n-i}{n+p-i}} \, dS(u). \tag{1.20}$$

Next, we have established an extension of L_p-Petty affine projection inequality (1.6) as follows.

Theorem 1.1 *Let $K \in \mathcal{F}^n_{i,o}$, $i = 0, 1, \ldots, n-1$ and $p \geq 1$, then*

$$n\omega_n^{\frac{n-i}{n+p-i}} W_i(\Pi_{p,i}K)^{\frac{p}{n+p-i}} \geq \Omega^{(i)}_p(K), \tag{1.21}$$

with equality in inequality (1.21) *for* $0 < i < n-1$ *if and only if* K *is a ball which is centered at the origin*; *for* $i = 0$ *if and only if* K *is an ellipsoid which is centered at the origin.*

Further, we obtain the following generalized L_p-Busemann-Petty centroid inequality.

Theorem 1.2 *Suppose that* $K \in C^n_{i,o} \subseteq \mathcal{S}^n_o$, $i = 0, 1, \ldots, n-1$ *and* $p \geq 1$, *then*

$$W_i(\Gamma_{p,i}K) \geq \widetilde{W}_i(K), \tag{1.22}$$

with equality in inequality (1.22) *for* $0 < i < n-1$ *if and only if* K *is a ball which is centered at the origin*; *for* $i = 0$ *if and only if* K *is an ellipsoid which is centered at the origin*.

Finally, we get the following dual inequality of the inequality (1.22).

Theorem 1.3 *Suppose that* $K \in \mathcal{K}^n_o$. *If* $\Gamma_{-p,i}K \in C^n_{i,o} \subseteq \mathcal{S}^n_o$, $i = 0, 1, \ldots, n-1$ *and* $p \geq 1$, *then*

$$\widetilde{W}_i(\Gamma_{-p,i}K) \leq W_i(K), \tag{1.23}$$

with equality in inequality (1.23) *for* $0 < i < n-1$ *if and only if* K *is a ball which is centered at the origin*; *for* $i = 0$ *if and only if* K *is an ellipsoid which is centered at the origin*.

2 Preliminaries

2.1 Support function, radial function and polar of convex body

If $K \in \mathcal{K}^n$, then its support function, $h_K = h(K, \cdot) : \mathbb{R}^n \to (0, \infty)$, is defined by (see [20, 21])

$$h(K, x) = \max\{\langle x, y\rangle : y \in K\}, \quad x \in \mathbb{R}^n.$$

Obviously, if $K \in \mathcal{K}^n$, λ is a positive constant, $x \in \mathbb{R}^n$, then $h(\lambda K, x) = \lambda h(K, x)$.

If K is a compact star-shaped (about the origin) in $\mathbb{R}^n$, its radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \to [0, \infty)$, is defined by (see [20, 21])

$$\rho(K, x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

When ρ_K is positive and continuous, K is called a star body (about the origin). Obviously, if $K \in \mathcal{S}^n_o$, $\alpha > 0$, $x \in \mathbb{R}^n$, then $\rho(K, \alpha x) = \alpha^{-1}\rho(K, x)$ and $\rho(\alpha K, x) = \alpha\rho(K, x)$. Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent on $u \in S^{n-1}$.

For $K \in \mathcal{K}^n_o$, the polar body, K^*, of K is defined by (see [20, 21])

$$K^* = \{x \in \mathbb{R}^n : \langle x, y\rangle \leq 1, y \in K\}.$$

Obviously, we have $(K^*)^* = K$. If $\lambda > 0$, then

$$(\lambda K)^* = \lambda^{-1}K^*. \tag{2.1}$$

If $K \in \mathcal{K}^n_o$, then the support and radial function of the polar body K^*, of K are defined respectively by (see [20, 21])

$$h_{K^*}(u) = \frac{1}{\rho_K(u)} \quad \text{and} \quad \rho_{K^*}(u) = \frac{1}{h_K(u)} \tag{2.2}$$

for all $u \in S^{n-1}$.

2.2 The quermassintegrals, L_p-mixed quermassintegrals and L_p-mixed volume

For $K \in \mathcal{K}^n$ and $i = 0, 1, \ldots, n-1$, the quermassintegrals, $W_i(K)$, of K are defined by (see [20, 21])

$$W_i(K) = \frac{1}{n} \int_{S^{n-1}} h(K,u)\, dS_i(K,u). \tag{2.3}$$

From (2.3), we easily see that

$$W_0(K) = V(K). \tag{2.4}$$

For $p \geq 1$, $K, L \in \mathcal{K}_o^n$ and $\varepsilon > 0$, the Firey L_p-combination $K +_p \varepsilon \cdot L \in \mathcal{K}_o^n$ is defined by (see [22])

$$h(K +_p \varepsilon \cdot L, \cdot)^p = h(K, \cdot)^p + \varepsilon h(L, \cdot)^p,$$

where '$\cdot$' in $\varepsilon \cdot L$ denotes the Firey scalar multiplication.

For $K, L \in \mathcal{K}_o^n$, $\varepsilon > 0$ and real $p \geq 1$, the L_p-mixed quermassintegrals, $W_{p,i}(K,L)$, of K and L ($i = 0, 1, \ldots, n-1$) are defined by (see [1])

$$\frac{n-i}{p} W_{p,i}(K,L) = \lim_{\varepsilon \to 0^+} \frac{W_i(K +_p \varepsilon \cdot L) - W_i(K)}{\varepsilon}. \tag{2.5}$$

Obviously, for $p = 1$, $W_{1,i}(K,L)$ is just the classical mixed quermassintegral $W_i(K,L)$. For $i = 0$, the L_p-mixed quermassintegral $W_{p,0}(K,L)$ is just the L_p-mixed volume $V_p(K,L)$, namely,

$$W_{p,0}(K,L) = V_p(K,L). \tag{2.6}$$

For $p \geq 1$, $i = 0, 1, \ldots, n-1$ and each $K \in \mathcal{K}_o^n$, there exists a positive Borel measure $S_{p,i}(K, \cdot)$ on S^{n-1} such that the L_p-mixed quermassintegral $W_{p,i}(K,L)$ has the following integral representation (see [1]):

$$W_{p,i}(K,L) = \frac{1}{n} \int_{S^{n-1}} h_L^p(v)\, dS_{p,i}(K,v) \tag{2.7}$$

for all $L \in \mathcal{K}_o^n$. It turns out that the measure $S_{p,i}(K, \cdot)$ ($i = 0, 1, \ldots, n-1$) on S^{n-1} is absolutely continuous with respect to $S_i(K, \cdot)$ and has the Radon-Nikodym derivative

$$\frac{dS_{p,i}(K, \cdot)}{dS_i(K, \cdot)} = h^{1-p}(K, \cdot). \tag{2.8}$$

From (2.3) and (2.7), it follows immediately that for each $K \in \mathcal{K}_o^n$,

$$W_{p,i}(K,K) = W_i(K). \tag{2.9}$$

If $K, L \in \mathcal{K}_o^n$, $p \geq 1$, by definition (1.12), then formula (2.7) of the L_p-mixed quermassintegral can be rewritten as follows:

$$W_{p,i}(K,L) = \frac{1}{n} \int_{S^{n-1}} h(L,u)^p f_{p,i}(K,u)\, dS(u). \tag{2.10}$$

We shall require the Minkowski inequality for the L_p-mixed quermassintegrals $W_{p,i}$ as follows (see [1]): For $K, L \in \mathcal{K}_o^n$, and $p \geq 1$, $0 \leq i < n$, then

$$W_{p,i}(K,L)^{n-i} \geq W_i(K)^{n-i-p} W_i(L)^p, \tag{2.11}$$

with equality for $p = 1$ and $0 \leq i < n-1$ if and only if K and L are homothetic; for $p > 1$ and $0 \leq i \leq n-1$ if and only if K and L are dilates. For $p = 1$ and $i = n-1$, inequality (2.11) is identical.

2.3 Dual quermassintegrals and L_p-dual mixed quermassintegrals

For $K \in S_o^n$ and any real i, the dual quermassintegrals, $\widetilde{W}_i(K)$, of K are defined by (see [20, 21])

$$\widetilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n-i}(u)\, dS(u). \tag{2.12}$$

Obviously,

$$\widetilde{W}_0(K) = V(K). \tag{2.13}$$

For $K, L \in S_o^n$, $p \geq 1$ and $\varepsilon > 0$, the L_p-harmonic radial combination $K \mathbin{+_{-p}} \varepsilon \cdot L \in S_o^n$ is defined by (see [2, 23, 24])

$$\rho(K \mathbin{+_{-p}} \varepsilon \cdot L, \cdot)^{-p} = \rho(K, \cdot)^{-p} + \varepsilon \rho(L, \cdot)^{-p}.$$

Note that here '$\varepsilon \cdot L$' is different from '$\varepsilon \cdot L$' in the Firey L_p-combination.

For $K, L \in S_o^n$, $\varepsilon > 0$, $p \geq 1$ and real $i \neq n$, the L_p-dual mixed quermassintegrals, $\widetilde{W}_{-p,i}(K,L)$, of K and L are defined by [25]

$$\frac{n-i}{-p} \widetilde{W}_{-p,i}(K,L) = \lim_{\varepsilon \to 0^+} \frac{\widetilde{W}_i(K \mathbin{+_{-p}} \varepsilon \cdot L) - \widetilde{W}_i(K)}{\varepsilon}. \tag{2.14}$$

If $i = 0$, we easily see that definition (2.14) is just the definition of L_p-dual mixed volume, namely,

$$\widetilde{W}_{-p,0}(K,L) = \widetilde{V}_{-p}(K,L). \tag{2.15}$$

From definition (2.14), the integral representation of the L_p-dual mixed quermassintegrals is given by (see [25]): If $K, L \in S_o^n$, $p \geq 1$, and real $i \neq n$, $i \neq n+p$, then

$$\widetilde{W}_{-p,i}(K,L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n+p-i}(u) \rho_L^{-p}(u)\, dS(u). \tag{2.16}$$

Together with (2.12) and (2.16), for $K \in S_o^n$, $p \geq 1$, and $i \neq n, n+p$, we get

$$\widetilde{W}_{-p,i}(K,K) = \widetilde{W}_i(K). \tag{2.17}$$

Analog of the Minkowski inequality for L_p-dual mixed quermassintegrals is as follows (see [25]): If $K, L \in S_o^n$, $p \geq 1$, then for $i < n$ or $i > n+p$,

$$\widetilde{W}_{-p,i}(K,L)^{n-i} \geq \widetilde{W}_i(K)^{n+p-i}\widetilde{W}_i(L)^{-p}. \tag{2.18}$$

For $n < i < n+p$, inequality (2.18) is reversed. With equality in every inequality if and only if K and L are dilates.

3 The (i,j)-type L_p-mixed affine surface area

In this section, we further propose the concept of (i,j)-type L_p-mixed affine surface area as follows.

Definition 3.1 For $K \in \mathcal{F}_{i,o}^n$, $i = 0, 1, \ldots, n-1$, $j \in \mathbb{R}$ and $p \geq 1$, the (i,j)-type L_p-mixed affine surface area, $\Omega_{p,j}^{(i)}(K)$, of K is defined by

$$\Omega_{p,j}^{(i)}(K) = \int_{S^{n-1}} f_{p,i}(K,u)^{\frac{n-i-j}{n+p-i}}\, dS(u). \tag{3.1}$$

Obviously, $\Omega_p^{(0)}(K) = \Omega_p(K)$ and $\Omega_{p,0}^{(i)}(K) = \Omega_p^{(i)}(K)$.

Next, we introduce the concept of $(i,0)$-type L_p-mixed affine surface area of the convex bodies $K_1, K_2, \ldots, K_{n-i}$ as follows.

Definition 3.2 For $p \geq 1$, $i = 0, 1, \ldots, n-1$, the $(i,0)$-type L_p-mixed affine surface area, $\Omega_p^{(i)}(K_1, \ldots, K_{n-i})$, of $K_1, \ldots, K_{n-i} \in \mathcal{F}_{i,o}^n$ is defined by

$$\Omega_p^{(i)}(K_1, \ldots, K_{n-i}) = \int_{S^{n-1}} \big[f_{p,i}(K_1,u)\cdots f_{p,i}(K_{n-i},u)\big]^{\frac{1}{n+p-i}}\, dS(u). \tag{3.2}$$

Let $K_1 = \cdots = K_{n-i-j} = K$ and $K_{n-i-j+1} = \cdots = K_{n-i} = L$ ($j = 0, \ldots, n-i$), we define $\Omega_{p,j}^{(i)}(K,L) = \Omega_p^{(i)}(K, \ldots, K, L, \ldots, L)$ with $n-i-j$ copies of K and j copies of L. From this, if j is any real number, we can define the following.

Definition 3.3 For $K, L \in \mathcal{F}_{i,o}^n$, $i = 0, \ldots, n-1$, $p \geq 1$, $j \in \mathbb{R}$, the (i,j)-type L_p-mixed affine surface area, $\Omega_{p,j}^{(i)}(K,L)$, of K, L is defined by

$$\Omega_{p,j}^{(i)}(K,L) = \int_{S^{n-1}} f_{p,i}(K,u)^{\frac{n-i-j}{n+p-i}} f_{p,i}(L,u)^{\frac{j}{n+p-i}}\, dS(u). \tag{3.3}$$

Specially for the case $j = -p$, we have that

$$\Omega_{p,-p}^{(i)}(K,L) = \int_{S^{n-1}} f_{p,i}(K,u) f_{p,i}(L,u)^{\frac{-p}{n+p-i}}\, dS(u). \tag{3.4}$$

Take $L = B$ in (3.4) and write

$$\Omega_{p,j}^{(i)}(K) := \Omega_{p,j}^{(i)}(K,B). \tag{3.5}$$

Because for $u \in S^{n-1}$, $S_i(B,u) = S$, $h(B,u) = 1$, so by (2.8) and (1.13), we get $f_{p,i}(B,u) = 1$. This together with (3.3) and (3.5) yields

$$\Omega_{p,j}^{(i)}(K) = \int_{S^{n-1}} f_{p,i}(K,u)^{\frac{n-i-j}{n+p-i}}\, dS(u), \tag{3.6}$$

and $\Omega_{p,j}^{(i)}(K)$ is called the (i,j)-type L_p-mixed affine surface area of $K \in \mathcal{F}_{i,o}^n$. In particular, $\Omega_{p,j}^{(o)}(K) = \Omega_{p,j}(K)$ is called the jth L_p-mixed affine surface area of $K \in \mathcal{F}_o^n$ (see [26]).

Next, we give some propositions of L_p-mixed curvature image and (i,j)-type L_p-mixed affine surface area.

Proposition 3.1 *Let $K \in \mathcal{F}_{i,o}^n$, $i = 0,1,\ldots,n-1$, $j \in \mathbb{R}$. Then*

$$\Omega_{p,j}^{(i)}(K) = n\left(\frac{\omega_n}{\widetilde{W}_i(\Lambda_{p,i}K)}\right)^{\frac{n-i-j}{n+p-i}} \widetilde{W}_{i+j}(\Lambda_{p,i}K). \tag{3.7}$$

In particular, take $j = 0$ in (3.7), *then*

$$\Omega_p^{(i)}(K) = n\omega_n^{\frac{n-i}{n+p-i}}\, \widetilde{W}_i(\Lambda_{p,i}K)^{\frac{p}{n+p-i}}. \tag{3.8}$$

Proof From (3.6), (1.14) and (2.12), we have

$$\begin{aligned}
\Omega_{p,j}^{(i)}(K) &= \int_{S^{n-1}} f_{p,i}(K,u)^{\frac{n-i-j}{n+p-i}}\, dS(u) \\
&= \left(\frac{\omega_n}{\widetilde{W}_i(\Lambda_{p,i}K)}\right)^{\frac{n-i-j}{n+p-i}} \int_{S^{n-1}} \rho(\Lambda_{p,i}K,u)^{n-i-j}\, dS(u) \\
&= \left(\frac{\omega_n}{\widetilde{W}_i(\Lambda_{p,i}K)}\right)^{\frac{n-i-j}{n+p-i}} \widetilde{W}_{i+j}(\Lambda_{p,i}K).
\end{aligned}$$

□

Proposition 3.2 *Let $p \geq 1$, $K \in \mathcal{F}_{i,o}^n$ and $i = 0,1,\ldots,n-1$. Then*

$$W_{p,i}(K,Q^*) = \frac{\omega_n}{\widetilde{W}_i(\Lambda_{p,i}K)}\widetilde{W}_{-p,i}(\Lambda_{p,i}K,Q) \tag{3.9}$$

for each $Q \in \mathcal{K}_o^n$.

Proof For each $Q \in \mathcal{K}_o^n$, from (2.10), (1.14), (2.2) and (2.16), we have

$$\begin{aligned}
W_{p,i}(K,Q^*) &= \frac{1}{n}\int_{S^{n-1}} \rho^{-p}(Q,u) f_{p,i}(K,u)\, dS(u) \\
&= \frac{\omega_n}{n\widetilde{W}_i(\Lambda_{p,i}K)}\int_{S^{n-1}} \rho(Q,u)^{-p}\rho(\Lambda_{p,i}K,u)^{n+p-i}\, dS(u) \\
&= \frac{\omega_n}{\widetilde{W}_i(\Lambda_{p,i}K)}\widetilde{W}_{-p,i}(\Lambda_{p,i}K,Q).
\end{aligned}$$

□

Proposition 3.3 *If* $p \geq 1$, $L \in \mathcal{F}^n_{i,o}$, *then*

$$\Omega_p^{(i)}(L)^{n+p-i} \leq n^{n+p-i} W_{p,i}\big(L, K^*\big)^{n-i} \widetilde{W}_i(K)^p \tag{3.10}$$

for all $K \in \mathcal{S}^n_o$ *with equality if and only if* K *and* $\Lambda_{p,i}L$ *are dilates.*

Proof Let $L \in \mathcal{F}^n_{i,o}$ and each $K \in \mathcal{S}^n_o$, then from (1.20), (2.2), (2.7), (2.12) and Hölder's inequality, we have

$$\begin{aligned}
\Omega_p^{(i)}(L)^{n+p-i} &= \left[\int_{S^{n-1}} f_{p,i}(L,u)^{\frac{n-i}{n+p-i}}\, dS(u)\right]^{n+p-i} \\
&= \left[\int_{S^{n-1}} \big(\rho(K,u)^{-p} f_{p,i}(L,u)\big)^{\frac{n-i}{n+p-i}} \big(\rho(K,u)^{n-i}\big)^{\frac{p}{n+p-i}}\, dS(u)\right]^{n+p-i} \\
&\leq n^{n+p-i}\left(\frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{-p} f_{p,i}(L,u)\, dS(u)\right)^{n-i}\left(\frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-i}\, dS(u)\right)^{p} \\
&= n^{n+p-i} W_{p,i}\big(L,K^*\big)^{n-i}\widetilde{W}_i(K)^p.
\end{aligned}$$

From this, we immediately get (3.10).

According to the condition of equality to hold in the Hölder inequality, we know that equality holds in (3.10) if and only if

$$\frac{\rho(K,u)^{-p} f_{p,i}(L,u)}{\rho(K,u)^{n-i}} = c$$

for any $u \in S^{n-1}$, where c is a constant. Combined with the definition of L_p-mixed curvature image, for any $u \in S^{n-1}$, we have

$$\frac{\rho(\Lambda_{p,i}L,u)^{n+p-i}}{\rho(K,u)^{n+p-i}} = \frac{c\widetilde{W}_i(\Lambda_{p,i}L)}{\omega_n},$$

this shows that K and $\Lambda_{p,i}L$ are dilates. Therefore, the equality holds in inequality (3.10) if and only if K and $\Lambda_{p,i}L$ are dilates. The proof is complete. □

Now, according to Proposition 3.3, we can give an expansion of the definition of the $(i,0)$-type L_p-mixed affine surface area of $K \in \mathcal{K}^n_o$ as follows.

Definition 3.4 If $K \in \mathcal{K}^n_o$, $p \geq 1$, then the $(i,0)$-type L_p-mixed affine surface area, $\Omega_p^{(i)}(K)$, of K is defined by

$$n^{-\frac{p}{n-i}} \Omega_p^{(i)}(K)^{\frac{n+p-i}{n-i}} = \inf\big\{n W_{p,i}\big(K,Q^*\big)\widetilde{W}_i(Q)^{\frac{p}{n-i}} : Q \in \mathcal{S}^n_o\big\}. \tag{3.11}$$

For $i = 0$, the definition is just the definition of L_p-affine surface area by Lutwak proposed in [2].

4 Generalized L_p-Petty affine projection inequality

In this section, we complete the proof of Theorem 1.1 in the introduction. In fact, we prove the following more general conclusion.

Theorem 4.1 *Let $K \in \mathcal{K}_o^n$, $L \in \mathcal{F}_{i,o}^n$, $p \geq 1$, $0 \leq i < n-1$, then*

$$n\omega_n^{\frac{n-i}{n+p-i}} W_{p,i}(L, \Pi_{p,i}K)^{\frac{n-i}{n+p-i}} \geq \Omega_p^{(i)}(L) W_i(K)^{\frac{n-i-p}{n-i+p}}, \tag{4.1}$$

with equality in inequality (4.1) *for* $0 < i < n-1$ *if and only if K and L are balls of dilates which are centered at the origin*; *for* $i = 0$ *if and only if K and L are ellipsoids of dilates which are centered at the origin*.

In order to prove the theorems above, we first give the following three lemmas.

Lemma 4.1 (See [27]) *Suppose that $K \in \mathcal{K}_o^n$, $i \in \mathbb{R}$ and $0 \leq i < n$, then*

$$\widetilde{W}_i(K) \leq \omega_n^{\frac{i}{n}} V(K)^{\frac{n-i}{n}}, \tag{4.2}$$

with the equality for $0 < i < n$ *if and only if K is a ball which is centered at the origin*. *If* $i = 0$, *then* (4.2) *is identical*.

Lemma 4.2 (See [18]) *Suppose that $K \in \mathcal{K}_o^n$, $p > 1$ and $0 < i < n-1$, i is a positive integer, then*

$$\omega_n^{\frac{i}{n}} W_i(K)^{\frac{n-i-p}{p}} V\big(\Pi_{p,i}^* K\big)^{\frac{n-i}{n}} \leq \omega_n^{\frac{n-i}{p}}, \tag{4.3}$$

with equality if and only if K is a ball which is centered at the origin.

Remark 4.1 The conditions of inequality (4.3) can be relaxed to $p \geq 1$ and $0 \leq i < n-1$, while the conditions of the equality that holds can be given separately. For $0 < i < n-1$ and $p = 1$, the inequality (4.3) is proved by Lutwak with the equality holding if and only if K is a ball (see [7]). For $i = 0$ and $p > 1$, inequality (4.3) is proved by Lutwak *et al.* with the equality that holds if and only if K is an ellipsoid which is centered at the origin (see [6]). For $i = 0$ and $p = 1$, then (4.3) is the famous Petty projection inequality (see [28]), with the equality that holds if and only if K is an ellipsoid.

Lemma 4.3 *If $K, L \in \mathcal{K}_o^n$, $p \geq 1$, $i = 0, 1, \ldots, n-1$, then*

$$W_{p,i}(L, \Pi_{p,i}K) = W_{p,i}(K, \Pi_{p,i}L). \tag{4.4}$$

Proof From (1.16), (2.10) and the Fubini theorem, it is easy to prove Lemma 4.3. □

Proof of Theorem 4.1 For $L \in \mathcal{F}_{i,o}^n$ and any $Q \in \mathcal{K}_o^n$, by inequality (3.10) and Lemma 4.1, we have

$$\Omega_p^{(i)}(L)^{n+p-i} \leq n^{n+p-i} \omega_n^{\frac{pi}{n}} W_{p,i}(L, Q)^{n-i} V\big(Q^*\big)^{\frac{p(n-i)}{n}}, \tag{4.5}$$

with equality for $0 < i \leq n-1$ if and only if $\Lambda_{p,i}L$ and Q^* are centered balls of dilates; for $i = 0$ if and only if $\Lambda_{p,i}L$ and Q^* are dilates.

Take $Q = \Pi_{p,i}K$ with $K \in \mathcal{K}_o^n$ in (4.5), then

$$\Omega_p^{(i)}(L)^{n+p-i} \le n^{n+p-i}\omega_n^{\frac{pi}{n}} W_{p,i}(L, \Pi_{p,i}K)^{n-i} V\bigl(\Pi_{p,i}^* K\bigr)^{\frac{p(n-i)}{n}}, \tag{4.6}$$

with equality for $0 < i \le n-1$ if and only if $\Lambda_{p,i}L$ and $\Pi_{p,i}^*K$ are centered balls of dilates; for $i = 0$ if and only if $\Lambda_p L$ and $\Pi_p^* K$ are dilates.

Combining with inequalities (4.3) and (4.6), we give

$$\begin{aligned}
&\Omega_p^{(i)}(L)^{n+p-i} W_i(K)^{n-i-p} \\
&\quad\le n^{n+p-i}\omega_n^{\frac{pi}{n}} W_{p,i}(L, \Pi_{p,i}K)^{n-i} W_i(K)^{n-i-p} V\bigl(\Pi_{p,i}^* K\bigr)^{\frac{p(n-i)}{n}} \\
&\quad\le n^{n+p-i} W_{p,i}(L, \Pi_{p,i}K)^{n-i}\omega_n^{n-i},
\end{aligned}$$

which implies that inequality (4.1) holds.

Next, we discuss the conditions of equality that holds in inequality (4.1).

According to the condition of the equality that holds in inequality (4.3) and inequality (4.6) with Remark 4.1, the four steps will be given.

(1) For the case $p > 1$ and $0 < i < n-1$, the equality holds in (4.1) if and only if $\Pi_{p,i}^*K$ and $\Lambda_{p,i}L$ are balls of dilates which are centered at the origin, and K is a ball which is centered at the origin. Together with $\Lambda_{p,i}B = B$ and $\Pi_{p,i}B = B$, we know that K and L are balls of dilates which are centered at the origin.

(2) For the case $p = 1$ and $0 < i < n-1$, the equality holds in (4.1) if and only if $\Pi_{1,i}^*K$ and $\Lambda_{1,i}L$ are balls of dilates which are centered at the origin, and K is a ball. By using $\Pi_{p,i}B = B$ and (1.17), it is obtained that $\Pi_{1,i}^*(\lambda B) = \lambda^{1+i-n}\Pi_{1,p}^*B = \lambda^{1+i-n}B$ ($\lambda > 0$) is a ball which is centered at the origin. Because $\Pi_{1,i}^*K$ and $\Lambda_{1,i}L$ are balls of dilates which are centered at the origin, then $\Lambda_{1,i}L$ is a ball of dilates which are centered at the origin, and together with $\Lambda_{p,i}B = B$ and (1.15), L is a ball which is centered at the origin. However, K is a ball, so the equality holds in (4.1) if and only if K and L are balls of dilates which are centered at the origin.

(3) For the case $p > 1$ and $i = 0$, the equality holds in (4.1) if and only if Π_p^*K and $\Lambda_p L$ are dilates and K is an ellipsoid which is centered at the origin. Let $K = E$ be an ellipsoid which is centered at the origin, from (1.5), we know that $\Pi_p^*E = (\omega_n/V(E))^{\frac{1}{p}}E$ is an ellipsoid which is centered at the origin. Other, from the literature [2], we know that L is an ellipsoid E which is centered at the origin if and only if $\Lambda_p L$ are dilates of polar body E^* of this E. So we know that the equality holds in (4.1) if and only if L and K are ellipsoids which are centered at the origin and both are dilates.

(4) For the case $p = 1$ and $i = 0$, the equality holds in (4.1) if and only if $\Lambda_1 L$ and Π_1^*K are dilates, and K is an ellipsoid. Suppose that $K = \lambda E + x_0$ with $\lambda > 0$, $x_0 \in \mathbb{R}^n$, and E is an ellipsoid which is centered at the origin, noting that $S(\lambda E + x_0, \cdot) = S(\lambda E, \cdot) = \lambda^{n-1}S(E, \cdot)$ (see [29]), this together with (1.5) $\Pi_1^*K = \Pi_1^*(\lambda E + x_0) = \Pi_1^*\lambda E = \lambda^{1-n}\Pi_1^*E = \lambda^{1-n}(\omega_n/V(E))E$ is an ellipsoid which is centered at the origin. Because $\Lambda_1 L$ and Π_1^*K are dilates, then $\Lambda_1 L$ is an ellipsoid which is centered at the origin. However, from [2], we know that L is an ellipsoid E which is centered at the origin if and only if $\Lambda_1 L$ are dilates of polar body E^* of this E. Therefore, the equality holds in (4.1) if and only if K and L are ellipsoids of the dilates which are centered at the origin.

To sum up, the equality holds in (4.1) for $p \geq 1$ and $0 < i < n-1$ if and only if K and L are balls of the dilates which are centered at the origin; for $p \geq 1$ and $i = 0$ if and only if K and L are ellipsoids of the dilates which are centered at the origin. The proof is complete. □

Proof of Theorem 1.1 Exchange K and L in inequality (4.1), we have for $L \in \mathcal{K}_o^n$, $K \in \mathcal{F}_{i,o}^n$, $p \geq 1$, $0 \leq i < n-1$

$$n\omega_n^{\frac{n-i}{n+p-i}} W_{p,i}(K, \Pi_{p,i}L)^{\frac{n-i}{n+p-i}} \geq \Omega_p^{(i)}(K) W_i(L)^{\frac{n-i-p}{n-i+p}}. \tag{4.7}$$

By using Lemmas 4.3 and (4.7), we have

$$\begin{aligned} n\omega_n^{\frac{n-i}{n+p-i}} W_{p,i}(L, \Pi_{p,i}K)^{\frac{n-i}{n+p-i}} &= n\omega_n^{\frac{n-i}{n+p-i}} W_{p,i}(K, \Pi_{p,i}L)^{\frac{n-i}{n+p-i}} \\ &\geq \Omega_p^{(i)}(K) W_i(L)^{\frac{n-i-p}{n-i+p}}. \end{aligned}$$

Taking $L = \Pi_{p,i}K$ in the inequality above, we immediately obtain inequality (1.21). The proof is complete. □

Combining with Theorem 1.1 and (3.8), we immediately obtain the following.

Corollary 4.1 *If* $K \in \mathcal{F}_{i,o}^n$, $i = 0, 1, \ldots, n-1$ *and* $p \geq 1$, *then*

$$W_i(\Pi_{p,i}K) \geq \widetilde{W}_i(\Lambda_{p,i}K), \tag{4.8}$$

with the equality in inequality (4.8) *for* $0 < i < n-1$ *if and only if* K *is a ball which is centered at the origin*; *for* $i = 0$ *if and only if* K *is an ellipsoid which is centered at the origin*.

Further, we have established the following results.

Theorem 4.2 *Let* $K, L \in \mathcal{F}_{i,o}^n$, $0 \leq i < n-1$, $p \geq 1$, *then*

$$n\omega_n^{\frac{n-i}{n+p-i}} W_{p,i}(\Pi_{p,i}K, \Pi_{p,i}L)^{\frac{p}{n+p-i}} \geq \Omega_p^{(i)}(K)^{\frac{n-p-i}{n-i}} \Omega_p^{(i)}(L)^{\frac{p}{n-i}}, \tag{4.9}$$

with the equality in inequality (4.9) *for* $0 < i < n-1$ *if and only if* K *and* L *are balls of dilates which are centered at the origin*; *for* $i = 0$ *if and only if* K *and* L *are ellipsoids of dilates which are centered at the origin*.

Proof From inequality (4.1), we know that for $Q \in \mathcal{K}_o^n$, $L \in \mathcal{F}_{i,o}^n$, $p \geq 1$, $0 \leq i \leq n-1$,

$$n\omega_n^{\frac{n-i}{n+p-i}} W_{p,i}(L, \Pi_{p,i}Q)^{\frac{n-i}{n+p-i}} \geq \Omega_p^{(i)}(L) W_i(Q)^{\frac{n-i-p}{n-i+p}}. \tag{4.10}$$

Take $Q = \Pi_{p,i}K$ in (4.9), and using Corollary 4.1 and Lemma 4.3, we have

$$\begin{aligned} \Omega_p^{(i)}(L) \widetilde{W}_i(\Lambda_{p,i}K)^{\frac{n-i-p}{n-i+p}} &\leq \Omega_p^{(i)}(L) W_i(\Pi_{p,i}K)^{\frac{n-i-p}{n-i+p}} \\ &\leq n\omega_n^{\frac{n-i}{n+p-i}} W_{p,i}(L, \Pi_{p,i}\Pi_{p,i}K)^{\frac{n-i}{n+p-i}} \\ &= n\omega_n^{\frac{n-i}{n+p-i}} W_{p,i}(\Pi_{p,i}K, \Pi_{p,i}L)^{\frac{n-i}{n+p-i}}. \end{aligned} \tag{4.11}$$

Combining inequality (4.11) with (3.8), we immediately obtain inequality (4.8). According to the condition of the equality holding in inequalities (4.1) and (4.9), the condition of the equality that holds in inequality (4.8) is easily obtained. The proof is complete. □

5 Generalized L_p-Busemann-Petty centroid inequality and dual inequality

In this section, we give the extension of the well-known L_p-Busemann-Petty centroid inequality (1.8). Namely, we complete the proof of Theorem 1.2 and Theorem 1.3 (*i.e.*, dual inequality of Theorem 1.2) in the introduction.

Lemma 5.1 *If* $K \in \mathcal{F}_{i,o}^n$, $0 = 0, 1, \ldots, n-1$, $p \geq 1$, *then*

$$\Pi_{p,i} K = \Gamma_{p,i} \Lambda_{p,i} K. \tag{5.1}$$

Proof Using definition (1.16) of L_p-mixed projection body and definition (1.14) of L_p-mixed curvature image, it is easy to prove (5.1). □

Lemma 5.2 *If* $K \in \mathcal{K}_o^n$, $L \in \mathcal{S}_o^n$, $p \geq 1$, $i = 0, 1, \ldots, n-1$, *then*

$$\frac{W_{p,i}(K, \Gamma_{p,i} L)}{W_i(K)} = \frac{\widetilde{W}_{-p,i}(L, \Gamma_{-p,i} K)}{\widetilde{W}_i(L)}. \tag{5.2}$$

Proof By (1.18), (1.19), (2.3), (2.7), (2.12) and (2.16), it is easy to prove (5.2). □

Proof of Theorem 1.2 For $L \in \mathcal{F}_{i,o}^n$, using (5.1) and Corollary 4.1, we have

$$W_i(\Gamma_{p,i} \Lambda_{p,i} L) = W_i(\Pi_{p,i} L) \geq \widetilde{W}_i(\Lambda_{p,i} L),$$

taking $K = \Lambda_{p,i} L$ in the inequality above, we immediately get inequality (1.22).

According to the condition of the equality that holds in inequalities (4.8) and (1.8), and noting that $\Lambda_{p,i} B = B$ and (1.15), we know with the equality in inequality (1.22) for $p \geq 1$ and $0 < i < n-1$ if and only if K is a ball which is centered at the origin; for $p \geq 1$ and $i = 0$ if and only if K is an ellipsoid which is centered at the origin. □

Proof of Theorem 1.3 Take $L = \Gamma_{-p,i} K$ in (5.2), we have

$$W_i(K) = W_{p,i}(K, \Gamma_{p,i} \Gamma_{-p,i} K).$$

Using the Minkowski inequality (2.11) of the L_p-mixed quermassintegrals, we have

$$W_i(K) \geq W_i(K)^{\frac{n-i-p}{n-i}} W_i(\Gamma_{p,i} \Gamma_{-p,i} K)^{\frac{p}{n-i}}. \tag{5.3}$$

Together with inequality (1.22), we can get

$$W_i(K) \geq W_i(\Gamma_{p,i} \Gamma_{-p,i} K) \geq \widetilde{W}_i(\Gamma_{-p,i} K),$$

from this, we can get inequality (1.23).

According to the condition of the equality that holds in inequalities (2.11) and (1.22), we discuss the conditions of the equality that holds in (1.23) in the following four cases:

(1) For the case $p > 1$ and $0 < i < n-1$, the equality holds in (1.23) if and only if K and $\Gamma_{p,i}\Gamma_{-p,i}K$ are dilates, and $\Gamma_{-p,i}K$ is a ball which is centered at the origin. Because $\Gamma_{-p,i}B = B$, then $\Gamma_{-p,i}K$ is a ball which is centered at the origin if and only if K is a ball which is centered at the origin. Therefore, K is a ball which is centered at the origin. While the equality $\Gamma_{p,i}B = B$ shows that $\Gamma_{-p,i}K$ is a ball which is centered at the origin if and only if $\Gamma_{p,i}\Gamma_{-p,i}K$ is a ball which is centered at the origin. From this, for $p > 1$ and $0 < i < n-1$ the equality holds in inequality (1.23) if and only if K is a ball which is centered at the origin.

(2) For the case $p > 1$ and $i = 0$, the equality holds in (1.23) if and only if K and $\Gamma_p\Gamma_{-p}K$ are dilates and $\Gamma_{-p}K$ is an ellipsoid which is centered at the origin. Because $\Gamma_{-p}K$ is an ellipsoid which is centered at the origin, and together with (1.10), K is an origin-symmetric ellipsoid E if and only if $\Gamma_{-p}K$ is an origin-symmetric ellipsoid. On the other hand, the literature [11] tells us that if E is an ellipsoid which is centered at the origin, then $\Gamma_p E = E$. From this, $\Gamma_p\Gamma_{-p}K$ is an ellipsoid which is centered at the origin. Therefore, for $p > 1$ and $i = 0$, the equality holds in inequality (1.23) if and only if K is an ellipsoid which is centered at the origin.

(3) For the case $p = 1$ and $0 < i < n-1$, the equality holds in (1.23) if and only if K and $\Gamma_{1,i}\Gamma_{-1,i}K$ are homothetic, and $\Gamma_{-1,i}K$ is a ball which is centered at the origin. From $\Gamma_{-p,i}B = B$, we know that $\Gamma_{-1,i}B = B$, then K is a ball which is centered at the origin. This $\Gamma_{-1,i}B = B$ together with $\Gamma_{1,i}B = B$, then $\Gamma_{1,i}\Gamma_{-1,i}K$ is a ball which is centered at the origin. Therefore, for $p = 1$ and $0 < i < n-1$, the equality holds in inequality (1.23) if and only if K is a ball which is centered at the origin.

(4) For the case $p = 1$ and $i = 0$, the equality holds in (1.23) if K and $\Gamma\Gamma_{-1}K$ are homothetic, and $\Gamma_{-1}K$ is an ellipsoid which is centered at the origin. Because $\Gamma_{-1}K$ is an origin-symmetric ellipsoid E if and only if K is an origin-symmetric ellipsoid. On the other hand, from $\Gamma_{-p}E = E$, we know that $\Gamma_{-1}K$ is an ellipsoid which is centered at the origin if and only if $\Gamma_1\Gamma_{-1}K$ is an ellipsoid which is centered at the origin. Therefore, for $p = 1$ and $i = 0$, the equality holds in inequality (1.23) if and only if K is an ellipsoid which is centered at the origin.

To sum up, the equality holds in (1.23) for $p \geq 1$ and $0 < i < n-1$ if and only if K is a ball which is centered at the origin; for $p \geq 1$ and $i = 0$ if and only if K is an ellipsoid which is centered at the origin. The proof is complete. □

Competing interests

The author declares that they have no competing interests.

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