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Positive answer for a conjecture about stabilizable means

Mustapha Raïssouli*

*Correspondence: raissouli_10@hotmail.com
Department of Mathematics, Faculty of Science, Taibah University, P.O. Box 30097, Al Madinah Al Munawwarah, Zip Code 41477, Kingdom of Saudi Arabia
Department of Mathematics, Faculty of Science, Moulay Ismail University, Meknes, Morocco

Abstract

In an earlier paper (Raïssouli in Appl. Math. E-Notes 11:159-174, 2011), the author conjectured that for given stable means m_1 and m_2 such that $m_1 \leq m_2$, there exists a unique (m_1, m_2)-stabilizable mean satisfying that $m_1 \leq m \leq m_2$. In the present paper, a positive answer of this conjecture is given. Some examples, illustrating the theoretical study, are discussed.

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Keywords: means; stable means; stabilizable means

1 Introduction

In the recent past, the theory of means has been the subject of intensive research. Stability and stabilizability concepts for means have been recently introduced by the author in [1]. It has been proved to be a useful tool for theoretical viewpoint as well as for practical purposes [2–4]. In this section, we recall some basic notions about means in two variables that will be needed later. We understand by (bivariate) mean a binary map m between positive real numbers satisfying the following statements:

(i) $m(a,a) = a$ for all $a > 0$;
(ii) $m(a,b) = m(b,a)$ for all $a,b > 0$;
(iii) $m(ta,tb) = tm(a,b)$ for all $a,b,t > 0$;
(iv) $m(a,b)$ is an increasing function in a (and in b);
(v) $m(a,b)$ is a continuous function of a and b.

The maps $(a,b) \longmapsto \min(a,b)$ and $(a,b) \longmapsto \max(a,b)$ are (trivial) means which are denoted by min and max, respectively. The standard examples of means are given in the following, see [5–24] for instance.

$$A := A(a,b) = \frac{a+b}{2}; \qquad G := G(a,b) = \sqrt{ab}; \qquad H := H(a,b) = \frac{2ab}{a+b};$$
$$L := L(a,b) = \frac{b-a}{\ln b - \ln a}, \quad L(a,a) = a; \qquad I := I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}, \quad I(a,a) = a;$$

and are known as the arithmetic, geometric, harmonic, logarithmic and identric means, respectively.

For two means m_1 and m_2, we write $m_1 \leq m_2$ if and only if $m_1(a,b) \leq m_2(a,b)$ for every $a,b > 0$ and $m_1 < m_2$ if and only if $m_1(a,b) < m_2(a,b)$ for all $a,b > 0$ with $a \neq b$. The above means satisfy the known chain of inequalities

$$\min < H < G < L < I < A < \max.$$

We say that m is a strict mean if $m(a,b)$ is strictly increasing in a and in b. Also, every strict mean m satisfies that $m(a,b) = a \Longrightarrow a = b$. It is not hard to check that the trivial means min and max are not strict, while A, G, H, L, I, S, C are strict means.

For the sake of simplicity for the reader, we end this section by recalling some basic notions and results stated by the author in an earlier paper [1] and needed in the sequel.

Definition 1.1 Let m_1, m_2 and m_3 be three given means. For all $a, b > 0$, define

$$\mathcal{R}(m_1, m_2, m_3)(a,b) = m_1\big(m_2\big(a, m_3(a,b)\big), m_2\big(m_3(a,b), b\big)\big),$$

called the resultant mean-map of m_1, m_2 and m_3.

A study investigating the elementary properties of the resultant mean-map was stated in [1]. Here, we just recall the following result needed later.

Proposition 1.1 *Let m_1, m_2 and m_3 be three means. Then the map $(a,b) \longmapsto \mathcal{R}(m_1, m_2, m_3)(a,b)$ defines a mean. Further the mean-map $(m_1, m_2, m_3) \longmapsto \mathcal{R}(m_1, m_2, m_3)$ is pointwisely increasing with respect to each of its mean variables, that is,*

$$\big(m_1 \le m_1', m_2 \le m_2', m_3 \le m_3'\big) \quad \Longrightarrow \quad \mathcal{R}(m_1, m_2, m_3) \le \mathcal{R}\big(m_1', m_2', m_3'\big). \tag{1.1}$$

In [1, 4], the author gives a lot of examples about computations of $\mathcal{R}(m_1, m_2, m_3)$ when m_1, m_2 and m_3 are means belonging to the set of the above standard means.

As proved in [1, 3, 4], and will be again shown throughout this paper, the resultant mean-map stems its importance in the fact that it is a good tool for introducing the stability and stabilizability concepts as recalled below.

Definition 1.2 A mean m is said to be:

(a) stable if $\mathcal{R}(m, m, m) = m$;

(b) stabilizable if there exist two nontrivial stable means m_1 and m_2 satisfying the relation $\mathcal{R}(m_1, m, m_2) = m$. We then say that m is (m_1, m_2)-stabilizable.

In [1, 3], the author stated a developed study about the stability and stabilizability of the above standard means. In particular, he proved the following.

Theorem 1.2 *With the above, the following assertions hold true*:

(1) *The arithmetic, geometric and harmonic means A, G and H are stable.*

(2) *The logarithmic mean L is (H, A)-stabilizable and (A, G)-stabilizable.*

(3) *The identric mean I is (G, A)-stabilizable.*

2 Two needed results

The next definition [2], recalling another concept for means, will be needed in the sequel.

Definition 2.1 Let m_1 and m_2 be two means. The tensor product of m_1 and m_2 is the map, denoted $m_1 \otimes m_2$, defined by

$$\forall a,b,c,d > 0 \quad m_1 \otimes m_2(a,b,c,d) = m_1\big(m_2(a,b), m_2(c,d)\big).$$

A mean m will be called cross mean if the map $m^{\otimes 2} := m \otimes m$ is symmetric with its four variables.

It is proved in [1] that every cross mean is stable. The reverse of this latter assertion is still an open problem.

Now, let m_1, m_2 and m_3 be three given means. For the sake of simplicity, we set

$$\mathcal{M}(m_1, m_2, m_3) := m_1(m_2, m_3),$$

in the sense that

$$\mathcal{M}(m_1, m_2, m_3)(a,b) := m_1\big(m_2(a,b), m_3(a,b)\big) \tag{2.1}$$

for all $a, b > 0$.

Clearly, $\mathcal{M}(m_1, m_2, m_2) = m_2$ for all means m_1 and m_2. If m_1 is a strict mean, then $\mathcal{M}(m_1, m_2, m_3) = m_2$ (resp. $\mathcal{M}(m_1, m_2, m_3) = m_3$) if and only if $m_2 = m_3$. Further, we have $\mathcal{M}(m_1, m_2, m_3) = \mathcal{M}(m_1, m_3, m_2)$ for all means m_2 and m_3.

The next result will be of interest later.

Proposition 2.1 *Let m_1 and m_2 be two given means. Assume that m_1 is a cross mean, then we have*

$$\mathcal{R}(m_1, m_1, m_2) = \mathcal{M}(m_1, m_1, m_2).$$

Proof By Definition 1.1 and Definition 1.2, one has, for all $a, b > 0$,

$$\begin{aligned}\mathcal{R}(m_1, m_1, m_2)(a,b) &= m_1\big(m_1\big(a, m_2(a,b)\big), m_1\big(m_2(a,b), b\big)\big)\\ &= m_1 \otimes m_1\big(a, m_2(a,b), m_2(a,b), b\big).\end{aligned}$$

Since m_1 is a cross mean, then

$$\begin{aligned}\mathcal{R}(m_1, m_1, m_2)(a,b) &= m_1 \otimes m_1\big(a, b, m_2(a,b), m_2(a,b)\big)\\ &= m_1\big(m_1(a,b), m_2(a,b)\big) = \mathcal{M}(m_1, m_1, m_2)(a,b),\end{aligned}$$

which concludes the proof. □

The above proposition implies again that every cross mean is stable. The next theorem is more interesting.

Theorem 2.2 *Let m_1, m_2, m_3 and m_4 be four given means. Assume that m_1 is a cross mean, then the following holds*:

$$\mathcal{R}\big(m_1, \mathcal{M}(m_1, m_3, m_4), m_2\big) = \mathcal{M}\big(m_1, \mathcal{R}(m_1, m_3, m_2), \mathcal{R}(m_1, m_4, m_2)\big). \tag{2.2}$$

Proof By Definition 1.1, we have, for all $a, b > 0$,

$$\begin{aligned}
&\mathcal{R}\big(m_1, \mathcal{M}(m_1, m_3, m_4), m_2\big)(a,b)\\
&\quad = m_1\big(\mathcal{M}(m_1, m_3, m_4)\big(a, m_2(a,b)\big), \mathcal{M}(m_1, m_3, m_4)\big(m_2(a,b), b\big)\big),
\end{aligned}$$

which with (2.1) gives

$$\begin{aligned}
&\mathcal{R}\big(m_1, \mathcal{M}(m_1, m_3, m_4), m_2\big)(a,b)\\
&\quad = m_1\big(m_1\big(m_3\big(a, m_2(a,b)\big), m_4\big(a, m_2(a,b)\big)\big), m_1\big(m_3\big(m_2(a,b), b\big), m_4\big(m_2(a,b), b\big)\big)\big).
\end{aligned}$$

Since m_1 is a cross mean, then we have

$$\begin{aligned}
&\mathcal{R}\big(m_1, \mathcal{M}(m_1, m_3, m_4), m_2\big)(a,b)\\
&\quad = m_1\big(m_1\big(m_3\big(a, m_2(a,b)\big), m_3\big(m_2(a,b), b\big)\big), m_1\big(m_4\big(a, m_2(a,b)\big), m_4\big(m_2(a,b), b\big)\big)\big).
\end{aligned}$$

Again, by Definition 1.1 and (2.1), respectively, we obtain

$$\begin{aligned}
\mathcal{R}\big(m_1, \mathcal{M}(m_1, m_3, m_4), m_2\big)(a,b) &= m_1\big(\mathcal{R}(m_1, m_3, m_2)(a,b), \mathcal{R}(m_1, m_4, m_2)(a,b)\big)\\
&= \mathcal{M}\big(m_1, \mathcal{R}(m_1, m_3, m_2), \mathcal{R}(m_1, m_4, m_2)\big)(a,b),
\end{aligned}$$

which completes the proof. □

In [1], for defining an (m_1, m_2)-stabilizable mean, the author imposed that the means m_1 and m_2 should be nontrivial and stable. The fact that m_1 and m_2 are nontrivial is clear since the relation $\mathcal{R}(\min, m, \max) = m$ is valid for every mean m. However, the fact that m_1 and m_2 are stable was imposed only in the aim to characterize a stabilizable mean m (as L and I) in terms of m_1 and m_2 having simple expressions (as A, G and H). As example, we know that L is (H, A)-stabilizable, where H and A are (stable) means having expressions more simple as that of L. Analogous way for the fact that L is (A, G)-stabilizable and I is (G, A)-stabilizable can be stated.

3 Existence and uniqueness of a stabilizable mean

In [1], the author stated the following conjecture.

Conjecture *Let m_1 and m_2 be two nontrivial stable means such that $m_1 \leq m_2$. Then there exists one and only one mean m, which is (m_1, m_2)-stabilizable, satisfying that $m_1 \leq m \leq m_2$.*

The aim of this section is to prove that the above conjecture is true when we add convenient hypotheses for the means m_1 and m_2. Of course, following Definition 1.2, m_1 and m_2 will be assumed to be stable means. We can ask why it is interesting to solve the above conjecture. In fact, as we have seen before, the means L and I, having complicated expressions, are stabilizable with respect to A, G, H whose expressions are more simple. It follows that if for given (simple) means m_1 and m_2 we show that there exists a unique

(m_1, m_2)-stabilizable mean, we can then characterize new means in terms of known (simple) means. This can be also useful when we speak for means involving several variables or those with operator arguments, of course if the above conjecture can be extended for these classes of generalized means.

Before giving an affirmative response that we are waiting for, we state some needed notions. A sequence $(m_n)_n$ of means will be called point-wise convergent (in short, p-convergent) if, for all $a, b > 0$, the real sequence $(m_n(a,b))_n$ converges. Setting $m_\infty(a,b) = \lim_n m_n(a,b)$, it is easy to see that m_∞ is a mean. Similarly, we define the point-wise monotonicity of $(m_n)_n$. By virtue of the double inequality

$$\min(a,b) \le m_n(a,b) \le \max(a,b),$$

we deduce that every p-increasing (resp. p-decreasing) sequence $(m_n)_n$ is p-convergent.

Now, let m_1 and m_2 be two given means and define the following two mean-sequences:

$$\begin{cases} \Theta_{n+1} = \mathcal{R}(m_1, \Theta_n, m_2), \quad n \ge 0, \qquad \Theta_0 = m_1, \\ \Upsilon_{n+1} = \mathcal{R}(m_1, \Upsilon_n, m_2), \quad n \ge 0, \qquad \Upsilon_0 = m_2. \end{cases} \tag{3.1}$$

By mathematical induction, it is not hard to check that Θ_n and Υ_n are means for every $n \ge 0$. In the following, we study the p-convergence of the mean-sequences $(\Theta_n)_n$ and $(\Upsilon_n)_n$. We may state the next result.

Proposition 3.1 *Let m_1 and m_2 be two stable means with $m_1 \le m_2$. Then the following mean-inequalities*

$$m_1 \le \cdots \le \Theta_{n-1} \le \Theta_n \le \Upsilon_n \le \Upsilon_{n-1} \le \cdots \le m_2 \tag{3.2}$$

hold for all $n \ge 1$. Consequently, the mean-sequences $(\Theta_n)_n$ and $(\Upsilon_n)_n$ both p-converge.

Proof Since $\Theta_0 := m_1 \le m_2 := \Upsilon_0$, we deduce by simple mathematical induction, with the help of (1.1), that $\Theta_n \le \Upsilon_n$ for each $n \ge 0$. Now, using the fact that m_1 is stable, we can write, again with help of (1.1),

$$\Theta_0 = m_1 = \mathcal{R}(m_1, m_1, m_1) \le \mathcal{R}(m_1, m_1, m_2) = \mathcal{R}(m_1, \Theta_0, m_2) = \Theta_1.$$

This, with mathematical induction, shows that $\Theta_{n-1} \le \Theta_n$ for each $n \ge 1$. Analogously, we prove that $\Upsilon_n \le \Upsilon_{n-1}$ for every $n \ge 1$. Summarizing, we deduce that $(\Theta_n)_n$ is a p-increasing sequence p-upper bounded by m_2, while $(\Upsilon_n)_n$ is a p-decreasing sequence p-lower bounded by m_1. Then the desired result follows, and so this completes the proof. □

We explicitly notice that the above mean-sequences $(\Theta_n)_n$ and $(\Upsilon_n)_n$ p-converge for all comparable means m_1 and m_2, *i.e.*, $m_1 \le m_2$ (or $m_2 \le m_1$, see Remark 3.1 below). Now, a natural question arises from the above: under what conditions on m_1 and m_2 do the p-limits of $(\Theta_n)_n$ and $(\Upsilon_n)_n$ coincide? In what follows, we are interested in finding a positive answer to this question.

Proposition 3.2 *Let m_1 and m_2 be two means. Assume that m_1 is a cross mean, then the mean-sequences $(\Theta_n)_n$ and $(\Upsilon_n)_n$ satisfy the following relationship*:

$$\forall n \geq 0 \quad \Theta_{n+1} = \mathcal{M}(m_1, \Theta_n, \Upsilon_n) := m_1(\Theta_n, \Upsilon_n). \tag{3.3}$$

Proof We use mathematical induction. For $n = 0$, by (3.1) and Proposition 2.1, we have

$$\Theta_1 = \mathcal{R}(m_1, \Theta_0, m_2) = \mathcal{R}(m_1, m_1, m_2) = \mathcal{M}(m_1, m_1, m_2) = \mathcal{M}(m_1, \Theta_0, \Upsilon_0).$$

Assume that (3.3) is true for n, by (3.1) we obtain

$$\Theta_{n+2} = \mathcal{R}(m_1, \Theta_{n+1}, m_2) = \mathcal{R}\big(m_1, \mathcal{M}(m_1, \Theta_n, \Upsilon_n), m_2\big).$$

This, with Theorem 2.2, yields

$$\Theta_{n+2} = \mathcal{M}\big(m_1, \mathcal{R}(m_1, \Theta_n, m_2), \mathcal{R}(m_1, \Upsilon_n, m_2)\big) = \mathcal{M}(m_1, \Theta_{n+1}, \Upsilon_{n+1}),$$

so proving the desired result. □

Now, we are in a position to state the next result.

Theorem 3.3 *Let m_1 and m_2 be two stable means with $m_1 \leq m_2$. Assume further that m_1 is strict and a cross mean. Then, the mean-sequences $(\Theta_n)_n$ and $(\Upsilon_n)_n$ both p-converge to the same limit m which is (m_1, m_2)-stabilizable and satisfying $m_1 \leq m \leq m_2$.*

Proof According to Proposition 3.1, the sequences $(\Theta_n)_n$ and $(\Upsilon_n)_n$ both p-converge. Call their limits Θ and Υ, respectively. By the continuity of m_1, relationship (3.3) gives, when $n \to \infty$, $\Theta = m_1(\Theta, \Upsilon)$. This, with the fact that m_1 is strict, yields $\Theta = \Upsilon := m$. Letting $n \to \infty$ in the first (or second) relation of (3.1), we obtain, with the help of continuity of m_1, $m = \mathcal{R}(m_1, m, m_2)$, which means that m is (m_1, m_2)-stabilizable. Inequalities (3.2) imply that $m_1 \leq m \leq m_2$, which completes the proof. □

Corollary 3.4 *Let m_1 and m_2 be as in the above theorem. Let m be a (m_1, m_2)-stabilizable mean such that $m_1 \leq m \leq m_2$. Then m is the common p-limit of the above sequences (Θ_n) and (Υ_n).*

Proof We show, by mathematical induction, that $\Theta_n \leq m \leq \Upsilon_n$ for all $n \geq 0$. For $n = 0$, it is true by virtue of $\Theta_0 = m_1 \leq m \leq m_2 = \Upsilon$. Assume that $\Theta_n \leq m \leq \Upsilon_n$, the recursive relations (3.1), with the help of (1.1), give

$$\Theta_{n+1} = \mathcal{R}(m_1, \Theta_n, m_2) \leq \mathcal{R}(m_1, m, m_2) \leq \mathcal{R}(m_1, \Upsilon_n, m_2) = \Upsilon_{n+1}.$$

This, with the fact that m is (m_1, m_2)-stabilizable, *i.e.*, $m = \mathcal{R}(m_1, m, m_2)$, yields $\Theta_{n+1} \leq m \leq \Upsilon_{n+1}$. It follows that $\Theta_n \leq m \leq \Upsilon_n$ for all $n \geq 0$. Since the sequences (Θ_n) and (Υ_n) both p-converge to the same limit, we deduce the desired result. □

The above corollary tells us that every (m_1, m_2)-stabilizable mean is the common p-limit of the above sequences (Θ_n) and (Υ_n). This, with the uniqueness of the p-limit, implies immediately the next result, which gives an affirmative answer of the above conjecture.

Corollary 3.5 *Let m_1 and m_2 be as in the above theorem. Then there exists one and only one (m_1, m_2)-stabilizable mean m such that $m_1 \leq m \leq m_2$.*

Remark 3.1 (i) If the means m_1 and m_2 are such that $m_1 \geq m_2$, analogous results as those above can be stated in a similar way. We leave to the reader the task to formulate these results in a detailed manner. In particular, with convenient means m_1 and m_2, there exists one and only one (m_1, m_2)-stabilizable mean satisfying that $m_2 \leq m \leq m_1$.

(ii) For m_1 and m_2 as in the above theorem, the last corollary tells us that the map $m \longmapsto \mathcal{R}(m_1, m, m_2)$ has one and only one mean-fixed point.

Example 3.1 As already pointed, the mean L is (H, A)-stabilizable. Following the above study, L is the unique (H, A)-stabilizable mean satisfying $H \leq L \leq A$, and so L can be characterized as the p-limit of an iterative algorithm involving the simple means H and A. The same can be said for the other stabilizable means mentioned in Theorem 2.2. We leave it for the reader to give more details about this latter point in a similar manner as previously explained.

It is worth mentioning that the reader will do well in distinguishing between the following two situations:

(a) There exists one and only one (m_1, m_2)-stabilizable mean for suitable means m_1 and m_2 as previously showed.

(b) A given mean m can be (m_1, m_2)-stabilizable and (m_1', m_2')-stabilizable for two distinct couples (m_1, m_2) and (m_1', m_2'). Indeed, as already pointed before, the logarithmic mean L is simultaneously (A, G)-stabilizable and (H, A)-stabilizable.

Finally, the following is of interest: Let m_1 and m_2 be as in the above theorem. For every mean m, we set

$$\Lambda_{m_1}^{m_2}(m) = \mathcal{R}(m_1, m, m_2).$$

Then, for fixed means m_1 and m_2, $\Lambda_{m_1}^{m_2}$ defines a map from the set of means into itself. If $(\Lambda_{m_1}^{m_2})^{(n)}$ denotes the n-iterate of $\Lambda_{m_1}^{m_2}$, *i.e.*, $(\Lambda_{m_1}^{m_2})^{(0)}(m) = m$ and, for $n \geq 1$,

$$\left(\Lambda_{m_1}^{m_2}\right)^{(n)} = \Lambda_{m_1}^{m_2} \circ \Lambda_{m_1}^{m_2} \circ \cdots \circ \Lambda_{m_1}^{m_2} \quad \left(n \text{ times of } \Lambda_{m_1}^{m_2}\right),$$

then the above study tells us that every (m_1, m_2)-stabilizable mean m can be written as

$$m = \lim_n \left(\Lambda_{m_1}^{m_2}\right)^{(n)}(m_1) = \lim_n \left(\Lambda_{m_1}^{m_2}\right)^{(n)}(m_2)$$

for the point-wise limit. Further, the following iterative inequalities hold true (if $m_1 \leq m_2$):

$$\forall n \geq 1 \quad \left(\Lambda_{m_1}^{m_2}\right)^{(n-1)}(m_1) \leq \left(\Lambda_{m_1}^{m_2}\right)^{(n)}(m_1) \leq m \leq \left(\Lambda_{m_1}^{m_2}\right)^{(n)}(m_2) \leq \left(\Lambda_{m_1}^{m_2}\right)^{(n-1)}(m_2).$$

Example 3.2 L is (H, A)-stabilizable. Then we have

$$\forall n \geq 1 \quad \left(\Lambda_H^A\right)^{(n-1)}(H) \leq \left(\Lambda_H^A\right)^{(n)}(H) \leq L \leq \left(\Lambda_H^A\right)^{(n)}(A) \leq \left(\Lambda_H^A\right)^{(n-1)}(A).$$

Simple computations lead to

$$\Lambda_H^A(H) = \frac{2G^2}{A+H}, \qquad \Lambda_H^A(A) = \frac{3}{4}A + \frac{1}{4}H.$$

We then obtain

$$H \le \frac{2G^2}{A+H} \le L \le \frac{3}{4}A + \frac{1}{4}H \le A,$$

which refines $H \le L \le A$. The procedure can be continued for obtaining more iterative refinements for this latter double inequality.

Competing interests
The author declares that he has no competing interests.

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