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# A note on the roots of some special type polynomials

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**Abstract**

In this study, we investigate the polynomials for $n \geq 2$ and positive integers $k$ and a positive real number $a$, with the initial values $G_0(x) = -a$, $G_1(x) = x - a$

$$G_{n+2}^{(k)}(x) = x^k G_{n+1}^{(k)}(x) + G_n^{(k)}(x).$$

We give some fundamental properties related to them. Also, we obtain asymptotic results for the roots of polynomials $G_n^{(k)}(x)$.

**MSC:** 11B39; 11B37

**Keywords:** Fibonacci polynomials; Binet formula; generating function

## 1 Introduction

The polynomials defined by Catalan, for $n \geq 0$, as follows

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x); \qquad F_1(x) = 1, \qquad F_2(x) = x \tag{1}$$

are called Fibonacci polynomials and denoted by $F_n(x)$, [1]. The Fibonacci-type polynomials $G_n(x)$, $n \geq 0$, are defined by

$$G_{n+2}(x) = xG_{n+1}(x) + G_n(x), \tag{2}$$

where $G_0(x)$ and $G_1(x)$ are seed polynomials. There are several studies about the properties of zeros of polynomials $G_n(x)$. However, there are no general formulas for zeros of Fibonacci-type polynomials. In [2, 3], the authors studied the limiting behavior of the maximal real roots of polynomials $G_n(x)$ with the initial values $G_0(x) = -1$, $G_1(x) = x - 1$. In [4], the authors generalized Moore's result for these polynomials. They considered the initial values $G_0(x) = a$, $G_1(x) = x + b$, where $a$ and $b$ are integer numbers. In [5], the author determined the absolute values of complex zeros of these polynomials. In [6], Ricci studied this problem in the case $a = 1$ and $b = 1$. In [7], Tewodros investigated the convergence of maximal real roots of different Fibonacci-type polynomials given by the following relation:

$$G_{n+2}^{(k)}(x) = x^k G_{n+1}^{(k)}(x) + G_n^{(k)}(x), \quad n \geq 0, \tag{3}$$

where $k$ is a positive integer number. The initial values of the recursive relation (3) are $G_0^{(k)}(x) = -1$ and $G_1^{(k)}(x) = x - 1$. In this study, firstly, we investigate some fundamental properties of Fibonacci-type polynomials. We give some combinatorial identities related to equation (3). Then, we investigate the limit of maximal real roots of these polynomials. We notice that Tewodros [7] studied a special case $a = 1$ of the polynomials we investigate.

## 2 Some fundamental properties of polynomials $G_n^{(k)}(x)$

In this section, we give some fundamental properties of polynomials $G_n^{(k)}(x)$, for $n \geq 0$, defined by the recursive formula as follows:

$$G_{n+2}^{(k)}(x) = x^k G_{n+1}^{(k)}(x) + G_n^{(k)}(x); \qquad G_0^{(k)}(x) = -a; \qquad G_1^{(k)}(x) = x - a. \tag{4}$$

The characteristic equation for (4) is $t^2 - x^k t - 1 = 0$ and its roots are

$$\alpha(x) = \frac{x^k + \sqrt{x^{2k} + 4}}{2}$$

and

$$\beta(x) = \frac{x^k - \sqrt{x^{2k} + 4}}{2}.$$

Note that $\alpha(x)\beta(x) = -1$, $\alpha(x) + \beta(x) = x^k$ and $\alpha(x) - \beta(x) = \sqrt{x^{2k} + 4}$. For relation (4), the Binet formula is

$$G_n^{(k)}(x) = A(x)\alpha^n(x) + B(x)\beta^n(x), \tag{5}$$

where

$$A(x) = \frac{2(x-a) + ax^k - a\sqrt{x^{2k} + 4}}{2\sqrt{x^{2k} + 4}}, \qquad B(x) = \frac{-2(x-a) - ax^k - a\sqrt{x^{2k} + 4}}{2\sqrt{x^{2k} + 4}}. \tag{6}$$

**Proposition 2.1** *For $n \geq 0$, the generating function for polynomials $G_n^{(k)}(x)$ is*

$$H_r^{(k)}(x,t) = \sum_{n\geq 0} G_{n+r}^{(k)}(x)t^n = \begin{cases} \frac{G_r^{(k)}(x) + G_{r-1}^{(k)}(x)t}{1 - x^k t - t^2}, & r = 1, 2, 3, \ldots, \\ \frac{t(ax^k + x - a)}{1 - x^k t - t^2}, & r = 0. \end{cases} \tag{7}$$

*Proof* Let $H_r^{(k)}(x,t)$ be the generating function for polynomials $G_{n+r}^{(k)}(x)$. So, we write

$$H_r^{(k)}(x,t) = \sum_{n\geq 0} G_{n+r}^{(k)}(x)t^n. \tag{8}$$

If we multiply both sides of equation (8) by $x^k t$ and $t^2$, respectively, then we can get

$$x^k t H_r^{(k)}(x,t) = x^k G_r^{(k)}(x)t^1 + x^k G_{r+1}^{(k)}(x)t^2 + x^k G_{r+2}^{(k)}(x)t^3 + \cdots + x^k G_{r+n-1}^{(k)}(x)t^n + \cdots$$

and

$$t^2 H_r^{(k)}(x,t) = G_r^{(k)}(x)t^2 + G_{r+1}^{(k)}(x)t^3 + G_{r+2}^{(k)}(x)t^4 + \cdots + G_{r+n-2}^{(k)}(x)t^n + \cdots.$$

The last two equations give us the following equation:

$$\begin{aligned} H_r^{(k)}(x,t) - x^k t H_r^{(k)}(x,t) - t^2 H_r^{(k)}(x,t) = {} & G_r^{(k)}(x)t^0 + \bigl(G_{r+1}^{(k)}(x) - x^k G_r^{(k)}(x)\bigr)t \\ & + \bigl(G_{r+2}^{(k)}(x) - x^k G_{r+1}^{(k)}(x) - G_r^{(k)}(x)\bigr)t^2 + \cdots \\ & + \bigl(G_{n+r}^{(k)}(x) - x^k G_{n+r-1}^{(k)}(x) - G_{n+r-2}^{(k)}(x)\bigr)t^n + \cdots. \end{aligned}$$

If we use the recurrence relation and simplify it, we write

$$H_r^{(k)}(x,t) - x^k t H_r^{(k)}(x,t) - t^2 H_r^{(k)}(x,t) = G_r^{(k)}(x)t^0 + \bigl(G_{r+1}^{(k)}(x) - x^k G_r^{(k)}(x)\bigr)t,$$

*i.e.*,

$$H_r^k(x,t) = \begin{cases} \frac{G_r^{(k)}(x) + G_{r-1}^{(k)}(x)t}{1 - x^k t - t^2}, & r = 1, 2, 3, \ldots, \\ \frac{t(ax^k + x - a)}{1 - x^k t - t^2}, & r = 0. \end{cases}$$

Thus, the proof is completed. □

Let us give the well-known formula, which is called the Cassini-like formula, without proof.

**Proposition 2.2** (Cassini-like) *For $n \geq 0$, we have*

$$G_{n-1}^{(k)}(x) G_{n+1}^{(k)}(x) - \bigl[G_n^{(k)}(x)\bigr]^2 = (-1)^{n-1}\bigl[A(x)B(x)\bigr], \tag{9}$$

*where*

$$A(x) = \frac{2(x-a) + ax^k - a\sqrt{x^{2k}+4}}{2\sqrt{x^{2k}+4}}$$

*and*

$$B(x) = \frac{-2(x-a) - ax^k - a\sqrt{x^{2k}+4}}{2\sqrt{x^{2k}+4}}.$$

In the following propositions, we give some sums formulas related to polynomials $G_n^{(k)}(x)$.

**Proposition 2.3** *For $N \geq 0$, we have*

$$H_0^{(k)}(x,1) - H_{N+1}^{(k)}(x,1) = \sum_{r=0}^{N} G_r^{(k)}(x) = \frac{2a - x - ax^k + G_{N+1}^{(k)}(x) + G_N^{(k)}(x)}{x^k}. \tag{10}$$

*Proof* Proof of formula (10) follows now immediately from (7). □

**Proposition 2.4** *For $N \geq 0$, we have the following sum formulas*:

$$\sum_{r=0}^{N} G_{2r}^{(k)}(x) = \frac{-x^{k+1} - ax^k(x^k - 1) - G_{2N}^{(k)}(x) + G_{2N+2}^{(k)}(x)}{x^{2k}} \tag{11}$$

*and*

$$\sum_{r=0}^{N} G_{2r+1}^{(k)}(x) = \frac{ax^k - G_{2N+1}^{(k)}(x) + G_{2N+3}^{(k)}(x)}{x^{2k}}. \tag{12}$$

*Proof* From the Binet formula, we can write

$$\sum_{r=0}^{N} G_{2r}^{(k)}(x) = \left(\frac{x-a+a\beta}{\alpha-\beta}\right)\sum_{r=0}^{N}\alpha^{2r} - \left(\frac{x-a+a\alpha}{\alpha-\beta}\right)\sum_{r=0}^{N}\beta^{2r}, \tag{13}$$

where $\alpha = \alpha(x)$, $\beta = \beta(x)$. If we substitute the equations

$$\sum_{r=0}^{N}\alpha^{2r} = \frac{1-\alpha^{2N+2}}{1-\alpha^2}, \qquad \sum_{r=0}^{N}\beta^{2r} = \frac{1-\beta^{2N+2}}{1-\beta^2}$$

and

$$\left(1-\alpha^2\right)\left(1-\beta^2\right) = -x^{2k}$$

into equation (13), then we can get

$$\sum_{r=0}^{N} G_{2r}^{(k)}(x) = \left(\frac{x-a+a\beta}{\alpha-\beta}\right)\frac{1-\alpha^{2N+2}}{1-\alpha^2} - \left(\frac{x-a+a\alpha}{\alpha-\beta}\right)\frac{1-\beta^{2N+2}}{1-\beta^2}.$$

If we rearrange the last equation, then we have

$$\begin{aligned}\sum_{r=0}^{N} G_{2r}^{(k)}(x) &= \frac{-a(\alpha-\beta)+(x-a)(\alpha^2-\beta^2)+a(\alpha^3-\beta^3)}{(1-\alpha^2)(1-\beta^2)(\alpha-\beta)} \\ &\quad - \frac{[(x-a+a\beta)\alpha^{2N+2}-(x-a+a\alpha)\beta^{2N+2}]}{(1-\alpha^2)(1-\beta^2)(\alpha-\beta)} \\ &\quad + \frac{\alpha^2\beta^2[(x-a+a\beta)\alpha^{2N}-(x-a+a\alpha)\beta^{2N}]}{(1-\alpha^2)(1-\beta^2)(\alpha-\beta)}.\end{aligned}$$

By taking aid of the Binet formula, we can write

$$\begin{aligned}\sum_{r=0}^{N} G_{2r}^{(k)}(x) &= \frac{-a+(x-a)(\alpha+\beta)+a(\alpha^2+\alpha\beta+\beta^2)}{(1-\alpha^2)(1-\beta^2)} \\ &\quad + \frac{G_{2N}^{(k)}(x)}{(1-\alpha^2)(1-\beta^2)} - \frac{G_{2N+2}^{(k)}(x)}{(1-\alpha^2)(1-\beta^2)}.\end{aligned}$$

If we substitute $\alpha+\beta = x^k$, $\alpha\beta = -1$, $(1-\alpha^2)(1-\beta^2) = -x^{2k}$ into the last equation, we obtain the following equation:

$$\sum_{r=0}^{N} G_{2r}^{(k)}(x) = \frac{x^{k+1} - ax^k + ax^{2k} - G_{2N+2}^{(k)}(x) + G_{2N}^{(k)}(x)}{-x^{2k}}.$$

Thus, the proof is completed. Similarly, the second part of the proposition can be seen. □

## 3 Asymptotic behaviors of the maximal roots for polynomials $G_n^{(k)}(x)$

In this section, firstly for $k = 2$, we investigate the roots of polynomials $G_n^{(k)}(x)$. After that, we generalize the obtained results for all positive real numbers $k$. When $k = 2$, we write

$$G_{n+2}^{(2)}(x) = x^2 G_{n+1}^{(2)}(x) + G_n^{(2)}(x), \tag{14}$$

where

$$G_0^{(2)}(x) = -a, \qquad G_1^{(2)}(x) = x - a$$

and $a$ is a positive real number. Now, we can give the following lemma to be used the later.

**Lemma 3.1** *If $r$ is a maximal root of a function $f$ with positive leading coefficient, then $f(x) > 0$ for all $x > r$. Conversely, if $f(x) > 0$ for all $x \geq t$, then $r < t$. If $f(s) < 0$ , then $s < r$* [2].

**Lemma 3.2** *For $n \geq 2$, $G_n^{(2)}(x)$ has at least one real root on the interval $(a, a + 1)$ and $g_n \in (a, a + 1)$, where $g_n$ is the maximal root of polynomial $G_n^{(2)}(x)$.*

*Proof* Some of polynomials $G_n^{(2)}(x)$ are as follows:

$$G_2^{(2)}(x) = x^3 - ax^2 - a,$$
$$G_3^{(2)}(x) = x^5 - ax^4 - ax^2 + x - a,$$
$$G_4^{(2)}(x) = x^7 - ax^6 - ax^4 + 2x^3 - 2ax^2 - a,$$
$$\vdots$$

Note that polynomials $G_n^{(2)}(x)$ are monic polynomials with degree $n$ and constant term $-a$. If we write for $x = a$, then we have

$$G_1^{(2)}(a) = 0,$$
$$G_2^{(2)}(a) = -a < 0,$$
$$G_3^{(2)}(a) = -a^3 = a^2 G_2^{(2)}(a) < 0,$$
$$G_4^{(2)}(a) = -a^5 - a \leq -a^5 = a^2 G_3^{(2)}(a) < 0,$$
$$\vdots$$

For $k \geq 2$, if we suppose $G_k^{(2)}(a) \leq a^2 G_{k-1}^{(2)}(a) < 0$, then by using the recursive relation (14), we get

$$G_{k+1}^{(2)}(a) = a^2 G_k^{(2)}(a) + G_{k-1}^{(2)}(a) < 0.$$

Thus, for $x = a$, we get $G_n^{(2)}(x) < 0$. Similarly, when $x = a + 1$, we have $G_n^{(2)}(x) > 0$. Therefore, $G_n^{(2)}(x)$ has at least one real root on the interval $(a, a + 1)$, and we write $g_n \in (a, a + 1)$ for the maximal root of $G_n^{(2)}(x)$, which results easily from Lemma 3.1 and the recursive relation for $G_n^{(2)}(x)$. □

Let $g_n$ denote the maximal root of polynomial $G_n^{(2)}(x)$ for every $n \in \mathbb{N}$. Then we can give the following proposition to illustrate the monotonicity of $\{g_{2n-1}\}$ and $\{g_{2n}\}$.

**Proposition 3.3** *The sequence* $\{g_{2n-1}\}$ *is a monotonically increasing sequence and the sequence* $\{g_{2n}\}$ *is a monotonically decreasing sequence.*

*Proof* Firstly, we consider polynomials $G_n^{(2)}(x)$ with odd indices. By a direct computation, we get $G_3^{(2)}(a) = -a^3 < 0$, $g_3 > a$, $a = g_1$. Assume that $g_1 < g_3 < g_5 < \cdots < g_{2k-3} < g_{2k-1}$. We can write $G_{2k-3}^{(2)}(g_{2k-1}) > 0$. Thus, it can be easily seen that

$$G_{n+k}^{(2)}(g_n) = (-1)^{k+1} G_{n-k}^{(2)}(g_n). \tag{15}$$

By using equation (15), we can write

$$G_{2k+1}^{(2)}(g_{2k-1}) = G_{(2k-1)+2}^{(2)}(g_{2k-1}) = -G_{(2k-1)-2}^{(2)}(g_{2k-1}) = -G_{2k-3}^{(2)}(g_{2k-1}). \tag{16}$$

So, from equation (16) we write

$$G_{2k+1}^{(2)}(g_{2k-1}) < 0. \tag{17}$$

Therefore, polynomials $G_{2k+1}^{(2)}(x)$ must have a root greater than $g_{2k-1}$. So, we get

$$g_{2k+1} > g_{2k-1}. \tag{18}$$

After that we consider polynomials $G_n^{(2)}(x)$ with even indices. From the recursive relation (14), we can obtain

$$G_{2k+1}^{(2)}(g_{2k-1}) = g_{2k-1}^2 G_{2k}^{(2)}(g_{2k-1}) + G_{2k-1}^{(2)}(g_{2k-1}). \tag{19}$$

Since $G_{2k-1}^{(2)}(g_{2k-1}) = 0$, by using Lemma 3.1, we can get $G_{2k+1}^{(2)}(g_{2k-1}) < 0$. Thus, we get

$$g_{2k-1} < g_{2k}. \tag{20}$$

Again, by using Lemma 3.1, we can write

$$G_{2k-1}^{(2)}(g_{2k}) > 0. \tag{21}$$

From the recursive relation (14), we can write

$$G_{2k}^{(2)}(g_{2k}) = g_{2k}^2 G_{2k-1}^{(2)}(g_{2k}) + G_{2k-2}^{(2)}(g_{2k}). \tag{22}$$

From equation (22), we can get

$$-g_{2k}^2 G_{2k-1}^{(2)}(g_{2k}) = G_{2k-2}^{(2)}(g_{2k}) < 0.$$

So, we have $g_{2k} < g_{2k-2}$. Thus, $\{g_{2n-1}\}$ is a monotonically increasing sequence and bounded above by the number $a + 1$. Similarly, $\{g_{2n}\}$ is a monotonically decreasing sequence and bounded below by the number $a$. If we denote the $\lim_{x\to\infty} g_{2n-1}$ by $g_{\text{odd}}$ and $\lim_{x\to\infty} g_{2n}$ by $g_{\text{even}}$, then we can write $g_{\text{odd}} = g_{\text{even}}$. □

**Proposition 3.4** *For polynomials $G^{(2)}_{2n-1}(x)$ and $G^{(2)}_{2n}(x)$, the sequences $\{g_{2n-1}\}$ and $\{g_{2n}\}$ converge to the following number $\zeta$:*

$$\zeta = \frac{\sqrt{(1-a^2)^2 + 8a^2} - (1-a^2)}{2a}. \tag{23}$$

*Proof* Using the Binet formula of relation (14), for all $[a, a+1]$, we can see that $\alpha(x) \geq \alpha(a) > 1$ and $|\beta(x)| = \frac{1}{\alpha(x)} \leq \frac{1}{\alpha(a)}$. Thus, we get

$$\lim_{n\to\infty} \alpha^n(x) = +\infty; \qquad \lim_{n\to\infty} \beta^n(x) = 0. \tag{24}$$

If we write $n = 2k-1$ and $x = g_{2k-1}$ in equation (5), then we have

$$A(g_{2k-1})\alpha^{2k-1}(g_{2k-1}) + B(g_{2k-1})\beta^{2k-1}(g_{2k-1}) = 0. \tag{25}$$

And from equation (25) we write

$$A(g_{2k-1}) = -B(g_{2k-1})\left(\frac{\beta^{2k-1}(g_{2k-1})}{\alpha^{2k-1}(g_{2k-1})}\right). \tag{26}$$

$A(x)$ and $B(x)$ are continuous on the interval $[a, a+1]$, this implies that $|A(x)|$ and $|B(x)|$ are bounded below and above on $[a, a+1]$. So, since $a \geq 1$, we get

$$\lim_{k\to\infty} A(g_{2k-1}) = A(g_{\text{odd}}) = 0. \tag{27}$$

From Binet formula (5), we have

$$\lim_{k\to\infty} g_{2k-1} = \frac{\sqrt{(1-a^2)^2 + 8a^2} - (1-a^2)}{2a}. \tag{28}$$

Also, by the aid of similar discussion, if we take $n = 2k$ and $x = g_{2k}$, then we find that

$$\lim_{k\to\infty} g_{2k} = \frac{\sqrt{(1-a^2)^2 + 8a^2} - (1-a^2)}{2a}.$$

That is,

$$\zeta = \frac{\sqrt{(1-a^2)^2 + 8a^2} - (1-a^2)}{2a}. \tag{29}$$

Notice that if we take $a = 1$ in equation (29), then our result coincides with the result of Tewodros [7]. □

For $\zeta$ numbers in equation (23), from Proposition 3.4 we can deduce the following result.

**Corollary 3.5** *For every positive integer $a$, we have*

$$a < \zeta < a + 1. \tag{30}$$

Now, we give a proposition for the maximal real roots of $G_n^{(k)}(x)$ without proof.

**Proposition 3.6** *The maximal real roots of $G_n^{(k)}(x)$ provide the following equation*:

$$g - 2a + ag^k - a^2 g^{k-1} = 0, \tag{31}$$

*where the numbers $g = g_n = g_n(k)$ are the maximal real roots of $G_n^{(k)}(x)$, that is,*

$$ag_n = a^2 - g_n^{2-k} + 2ag_n^{1-k}, \tag{32}$$

*which implies*

$$\frac{a}{1 + a(a+1)^{k-1}} < g_n(k) - a < \frac{a}{1 + a^k},$$

*whenever $k > 2$.*

$$\frac{a-1}{a(a+1)} < \frac{2a - g_n(2)}{ag_n(2)} = g_n(2) - a < \frac{1}{a+1}$$

*and*

$$\lim_{k \to \infty} g_n(k) = a,$$

*whenever $a > 1$ for every $n \in \mathbb{N}$.*

*Proof* The proof can be easily seen as being similar to the proof of Proposition 3.4 □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors completed the paper together. All authors read and approved the final manuscript.

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