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Properties for certain subclasses of meromorphic functions defined by a multiplier transformation

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Abstract
Some inclusion and convolution properties of certain subclasses of meromorphic functions associated with a family of multiplier transformations, which are defined by means of the Hadamard product (or convolution), are investigated. We also obtain closure properties for certain integral operators.

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1 Introduction

Let $\mathcal{A}$ denote the class of analytic functions f in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ with the usual normalization $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ denote the subclasses of $\mathcal{A}$ consisting of starlike and convex functions of order α $(0 \le \alpha < 1)$ and let $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(0) = \mathcal{K}$. If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g in $\mathbb{U}$, written as $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function w such that $f(z) = g(w(z))$ $(z \in \mathbb{U})$.

A function $f \in \mathcal{A}$ is said to be prestarlike of order α in $\mathbb{U}$ if

$$\frac{z}{(1-z)^{2(1-\alpha)}} * f(z) \in \mathcal{S}^*(\alpha) \quad (0 \le \alpha < 1),$$

where $f * g$ denotes the familiar Hadamard product (or convolution) of two analytic functions f and g in $\mathbb{U}$. We denote this class by $\mathcal{R}(\alpha)$ (see, for details, [1]). We note that $\mathcal{R}(0) = \mathcal{K}$ and $\mathcal{R}(1/2) = \mathcal{S}^*(1/2)$.

Let $\mathcal{N}$ be the class of all functions h which are analytic and univalent in $\mathbb{U}$ and for which $h(\mathbb{U})$ is convex with $h(0) = 1$.

Let $\mathcal{M}$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k$$

which are analytic in the punctured open unit disk $\mathbb{D} = \mathbb{U} \setminus \{0\}$.

For any $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, we denote the multiplier transformations D_λ^n of functions $f \in \mathcal{M}$ by

$$D_\lambda^n f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \left(\frac{k+1+l}{\lambda}\right)^n a_k z^k \quad (\lambda > 0; z \in \mathbb{D}).$$

Obviously, we have

$$D_\lambda^s \bigl(D_\lambda^t f(z)\bigr) = D_\lambda^{s+t} f(z)$$

for all nonnegative integers s and t. The operators D_λ^n and D_1^n are the multiplier transformations introduced and studied by Sarangi and Uraligaddi [2] and Uralegaddi and Somanatha [3, 4], respectively. Analogous to D_λ^n, we here define a new multiplier transformation $I_{\lambda,\mu}^n$ as follows.

Let $f_n(z) = 1/z + \sum_{k=0}^{\infty}((k+1+\lambda)/\lambda)^n z^k$, $n \in \mathbb{N}_0$, and let $f_{n,\mu}^{\dagger}$ be such that

$$f_n(z) * f_{n,\mu}^{\dagger}(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(\mu)_{k+1}}{(1)_{k+1}} z^k \quad (\mu > 0; z \in \mathbb{D}),$$

where $(\nu)_k$ is the Pochhammer symbol (or the shifted factorial) defined (in terms of the gamma function) by

$$(\nu)_k := \frac{\Gamma(\nu+k)}{\Gamma(\nu)} = \begin{cases} 1 & \text{if } k = 0 \text{ and } \nu \in \mathbb{C} \setminus \{0\}, \\ \nu(\nu+1)\cdots(\nu+k-1) & \text{if } k \in \mathbb{N} := \{1, 2, \ldots\} \text{ and } \nu \in \mathbb{C}. \end{cases}$$

Then

$$I_{\lambda,\mu}^n f(z) = f_{n,\mu}^{\dagger}(z) * f(z). \tag{1.1}$$

We note that $I_{1,2}^0 f(z) = zf'(z) + 2f(z)$ and $I_{1,2}^1 f(z) = f(z)$. It is easily verified from (1.1) that

$$z\bigl(I_{\lambda,\mu}^{n+1} f(z)\bigr)' = \lambda I_{\lambda,\mu}^n f(z) - (\lambda+1) I_{\lambda,\mu}^{n+1} f(z) \tag{1.2}$$

and

$$z\bigl(I_{\lambda,\mu}^n f(z)\bigr)' = \mu I_{\lambda,\mu+1}^n f(z) - (\mu+1) I_{\lambda,\mu}^n f(z). \tag{1.3}$$

The definition (1.1) of the multiplier transformation $I_{\lambda,\mu}^n$ is motivated essentially by the Choi-Saigo-Srivastava operator [5] for analytic functions, which includes the Noor integral operator studied by Liu [6] (also, see [7–9]).

We also define the function $\phi(a, c; z)$ by

$$\phi(a, c; z) := \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(a)_{k+1}}{(c)_{k+1}} z^k \quad \bigl(z \in \mathbb{U}; a \in \mathbb{R}; c \in \mathbb{R} \setminus \mathbb{Z}_0^-; \mathbb{Z}_0^- := \{-1, -2, \ldots\}\bigr). \tag{1.4}$$

By using the operator $I_{\lambda,\mu}^n$, we introduce the following class of analytic functions for $\gamma > 0$, $\lambda > 0$, $s \in \mathbb{R}$, $\mu > 0$ and $h \in \mathcal{N}$:

$$\mathcal{M}_{\lambda,\mu}^n(\gamma; h) := \bigl\{f \in \mathcal{M} : (1-\gamma) z I_{\lambda,\mu}^n f(z) + \gamma z^2 \bigl(I_{\lambda,\mu}^n f(z)\bigr)' \prec h(z)\bigr\}.$$

In the present paper, we derive some inclusion relations, convolution properties and integral preserving properties for the class $\mathcal{M}_{\lambda,\mu}^{n}(\gamma;h)$.

The following lemmas will be required in our investigation.

Lemma 1.1 [10, Lemma 2, p.192] *Let g be analytic in $\mathbb{U}$ and h be analytic and convex univalent in U with $h(0)=g(0)$. If*

$$g(z)+\frac{1}{\gamma}zg'(z)\prec h(z) \quad \bigl(\mathrm{Re}\{\gamma\}\geq 0;\gamma\neq 0\bigr), \tag{1.5}$$

then

$$g(z)\prec \widetilde{h}(z)=\gamma z^{-\gamma}\int_0^z t^{\gamma-1}h(t)\,dt\prec h(z)$$

and $\widetilde{h}$ is the best dominant of (1.5).

Lemma 1.2 [1, Theorem 2.4, p.54] *Let $f\in\mathcal{S}^{*}(\alpha)$ and $g\in\mathcal{R}(\alpha)$. Then for any analytic function F in $\mathbb{U}$,*

$$\frac{g*(fF)}{g*f}(\mathbb{U})\subset\overline{co}\bigl(F(\mathbb{U})\bigr),$$

where $\overline{co}(F(\mathbb{U}))$ denotes the convex hull of $F(\mathbb{U})$.

Lemma 1.3 [11, Lemma 5, p.656] *Let $0<a\leq c$. Then*

$$\mathrm{Re}\bigl\{z\phi(a,c;z)\bigr\}>\frac{1}{2} \quad (z\in\mathbb{U}),$$

where ϕ is given by (1.4).

2 Inclusion relations

Theorem 2.1 *If $0\leq\gamma_1<\gamma_2$, then*

$$\mathcal{M}_{\lambda,\mu}^{n}(\gamma_2;h)\subset\mathcal{M}_{\lambda,\mu}^{n}(\gamma_1;h).$$

Proof Let

$$g(z)=zI_{\lambda,\mu}^{n}f(z) \quad \bigl(f\in\mathcal{M}_{\lambda,\mu}^{n}(\gamma_2;h):z\in\mathbb{U}\bigr). \tag{2.1}$$

Then the function g is analytic in $\mathbb{U}$ with $g(0)=1$. Differentiating both sides of (2.1), we have

$$(1+\gamma_2)zI_{\lambda,\mu}^{n}f(z)+\gamma_2z^2\bigl(I_{\lambda,\mu}^{n}f(z)\bigr)'=g(z)+\gamma_2zg'(z)\prec h(z). \tag{2.2}$$

Hence an application of Lemma 1.1 with $\mu=1/\gamma_2$ yields

$$g(z)\prec h(z). \tag{2.3}$$

Since $0 \le \gamma_1/\gamma_2 < 1$ and h is convex univalent in U, it follows from (2.1), (2.2) and (2.3) that

$$\begin{aligned}
&(1+\gamma_1)zI^n_{\lambda,\mu}f(z) + \gamma_1 z^2\bigl(I^n_{\lambda,\mu}f(z)\bigr)' \\
&\quad = \frac{\gamma_1}{\gamma_2}\bigl[(1-\gamma_2)zI^n_{\lambda,\mu}f(z) + \gamma_2 z^2\bigl(I^n_{\lambda,\mu}f(z)\bigr)'\bigr] + \left(1-\frac{\gamma_1}{\gamma_2}\right)g(z) \\
&\quad \prec h(z).
\end{aligned}$$

Therefore $f \in \mathcal{M}^n_{\lambda,\mu}(\gamma_1;h)$, and so we complete the proof of Theorem 2.1. □

Theorem 2.2 *If* $0 < \mu_1 \le \mu_2$, *then*

$$\mathcal{M}^n_{\lambda,\mu_2}(\gamma;h) \subset \mathcal{M}^n_{\lambda,\mu_1}(\gamma;h).$$

Proof Let $f \in \mathcal{M}^n_{\lambda,\mu_2}(\gamma;h)$. Then

$$\begin{aligned}
&(1+\gamma)zI^n_{\lambda,\mu_1}f(z) + \gamma z^2\bigl(I^n_{\lambda,\mu_1}f(z)\bigr)' \\
&\quad = z\phi(\mu_1,\mu_2;z) * \bigl[(1+\gamma)zI^n_{\lambda,\mu_2}f(z) + \gamma z^2\bigl(I^n_{\lambda,\mu_2}f(z)\bigr)'\bigr].
\end{aligned} \tag{2.4}$$

In view of Lemma 1.3, we see that the function $z\phi(\mu_1,\mu_2;z)$ has the Herglotz representation

$$z\phi(\mu_1,\mu_2;z) = \int_{|x|=1}\frac{d\mu(x)}{1-xz} \quad (z \in \mathbb{U}), \tag{2.5}$$

where $\mu(x)$ is a probability measure defined on the unit circle $|x| < 1$ and

$$\int_{|x|=1} d\mu(x) = 1.$$

Since h is convex univalent in $\mathbb{U}$, it follows from (2.4) and (2.5) that

$$(1+\gamma)zI^n_{\lambda,\mu_1}f(z) + \gamma z^2\bigl(I^n_{\lambda,\mu_1}f(z)\bigr)' = \int_{|x|=1} h(xz)\,d\mu(x) \prec h(z),$$

which completes the proof of Theorem 2.2. □

Theorem 2.3 *If* $\mu > 0$, *then*

$$\mathcal{M}^n_{\lambda,\mu+1}(\gamma;\widetilde{h}) \subset \mathcal{M}^n_{\lambda,\mu}(\gamma;h),$$

where

$$\widetilde{h}(z) = \mu z^{-\mu}\int_0^z t^{\mu-1}h(t)\,dt \prec h(z).$$

Proof Let

$$g(z) = (1+\gamma)zI^n_{\lambda,\mu}f(z) + \gamma z^2\bigl(I^n_{\lambda,\mu}f(z)\bigr)' \quad (f \in \mathcal{M}; z \in \mathbb{U}). \tag{2.6}$$

Then from (1.4) and (2.6), we have

$$z^{-1}g(z) = \gamma\mu I^n_{\lambda,\mu+1}f(z) + (1-\gamma\mu)I^n_{\lambda,\mu}f(z). \tag{2.7}$$

Differentiating both sides of (2.6) and using (1.4), we obtain

$$\begin{aligned} &z^{-1}\bigl(zg'(z) + g(z)\bigr) \\ &\quad = \gamma\mu z\bigl(I^n_{\lambda,\mu+1}f(z)\bigr) + (1-\gamma\mu)\bigl(\mu I^n_{\lambda,\mu+1}f(z) - (\mu+1)I^n_{\lambda,\mu}f(z)\bigr). \end{aligned} \tag{2.8}$$

By a simple calculation with (2.7) and (2.8), we get

$$g(z) + \frac{zg'(z)}{\mu} = (1+\gamma)\frac{I^n_{\lambda,\mu+1}f(z)}{z} + \gamma\bigl(I^n_{\lambda,\mu+1}f(z)\bigr)'. \tag{2.9}$$

If $f \in \mathcal{M}^n_{\lambda,\mu+1}(\gamma;h)$, then it follows from (2.9) that

$$g(z) + \frac{zg'(z)}{\mu} \prec h(z) \quad (\mu > 0).$$

Hence an application of Lemma 1.1 yields

$$g(z) \prec \widetilde{h}(z) = \mu z^{-\mu}\int_0^z t^{\mu-1}h(t)\,dt \prec h(z),$$

which shows that

$$f \in \mathcal{M}^n_{\lambda,\mu+1}(\gamma;\widetilde{h}) \subset \mathcal{M}^n_{\lambda,\mu}(\gamma;h). \qquad \square$$

Theorem 2.4 *If $s \in \mathbb{R}$ and $\lambda > 0$, then*

$$\mathcal{M}^n_{\lambda,\mu}(\gamma;\widetilde{h}) \subset \mathcal{M}^{n+1}_{\lambda,\mu}(\gamma;h),$$

where

$$\widetilde{h}(z) = \lambda z^{-\lambda}\int_0^z t^{\lambda-1}h(t)\,dt \prec h(z).$$

Proof By using the same techniques as in the proof of Theorem 2.3 and (1.5), we have Theorem 2.4 and so we omit the detailed proof involved. □

Theorem 2.5 *Let $\gamma > 0$, $\beta > 0$ and $f \in \mathcal{M}^n_{\lambda,\mu}(\gamma;\beta h + 1 - \beta)$. If $\beta \le \beta_0$, where*

$$\beta_0 = \frac{1}{2}\left(1 - \frac{1}{\gamma}\int_0^1 \frac{u^{\frac{1}{\gamma}-1}}{1+u}\,du\right)^{-1}, \tag{2.10}$$

then $f \in \mathcal{M}^n_{\lambda,\mu}(0;h)$. The bound β_0 is sharp for the function

$$h(z) = \frac{1}{1-z} \quad (z \in \mathbb{U}).$$

Proof Let

$$g(z) = zI^n_{\lambda,\mu}f(z) \quad \bigl(f \in \mathcal{M}^n_{\lambda,\mu}(\gamma;\beta h + 1 - \beta); \gamma > 0; \beta > 0\bigr). \tag{2.11}$$

Then we have

$$\begin{aligned} g(z) + \gamma z g'(z) &= (1+\gamma) zI^n_{\lambda,\mu}f(z) + \gamma z^2 \bigl(I^n_{\lambda,\mu}f(z)\bigr)' \\ &\prec \beta h(z) + 1 - \beta. \end{aligned}$$

Hence an application of Lemma 1.1 yields

$$g(z) \prec \frac{\beta}{\gamma} z^{-\frac{1}{\gamma}} \int_0^z t^{\frac{1}{\gamma}-1} h(t)\,dt + 1 - \beta = (h * \psi)(z), \tag{2.12}$$

where

$$\psi(z) = \frac{\beta}{\gamma} z^{-\frac{1}{\gamma}} \int_0^z \frac{t^{\frac{1}{\gamma}-1}}{1-t}\,dt + 1 - \beta. \tag{2.13}$$

If $0 < \beta \le \beta_0$, where β_0 is given by (2.10), then from (2.13), we have

$$\begin{aligned} \operatorname{Re}\bigl\{\psi(z)\bigr\} &= \frac{\beta}{\gamma} \int_0^1 u^{\frac{1}{\gamma}-1} \operatorname{Re}\left\{\frac{1}{1-uz}\,du\right\} + 1 - \beta \\ &> \frac{\beta}{\gamma} \int_0^1 \frac{u^{\frac{1}{\gamma}-1}}{1+u}\,du + 1 - \beta \\ &\ge \frac{1}{2}. \end{aligned}$$

By using the Herglotz representation for ψ, it follows from (2.11) and (2.12) that

$$zI^n_{\lambda,\mu}f(z) \prec (h * \psi)(z) \prec h(z)$$

since h is convex univalent in $\mathbb{U}$. This shows that $f \in \mathcal{M}^n_{\lambda,\mu}(0;h)$.

For $h(z) = 1/(1-z)$ and $f \in \mathcal{M}$ defined by

$$zI^n_{\lambda,\mu}f(z) = \frac{\beta}{\gamma} z^{-\frac{1}{\gamma}} \int_0^z \frac{t^{\frac{1}{\gamma}-1}}{1-t}\,dt + 1 - \beta,$$

it is easy to verify that

$$(1+\gamma) zI^n_{\lambda,\mu}f(z) + \gamma z^2 \bigl(I^n_{\lambda,\mu}f(z)\bigr)' = \beta h(z) + 1 - \beta.$$

Thus $f \in \mathcal{M}^n_{\lambda,\mu}(\gamma;\beta h + 1 - \beta)$. Furthermore, for $\beta > \beta_0$, we have

$$\operatorname{Re}\bigl\{zI^n_{\lambda,\mu}f(z)\bigr\} \to \frac{\beta}{\gamma} \int_0^1 \frac{u^{\frac{1}{\gamma}-1}}{1+u}\,du + 1 - \beta < \frac{1}{2} \quad (z \to -1),$$

which implies that $f \notin \mathcal{M}^n_{\lambda,\mu}(0;h)$. Hence the bound β_0 cannot be increased when $h(z) = 1/(1-z)$ $(z \in \mathbb{U})$. □

3 Convolution properties

Theorem 3.1 *If $f \in \mathcal{M}^n_{\lambda,\mu}(\gamma;h)$ and*

$$\operatorname{Re}\{zg(z)\} > \frac{1}{2} \quad (g \in \mathcal{M}; z \in \mathbb{U}),$$

then

$$f * g \in \mathcal{M}^n_{\lambda,\mu}(\gamma;h).$$

Proof Let $f \in \mathcal{M}^n_{\lambda,\mu}(\gamma;h)$ and $g \in \mathcal{M}$. Then we have

$$(1+\gamma)zI^n_{\lambda,\mu}(f*g)(z) + \gamma z^2\left(I^n_{\lambda,\mu}(f*g)(z)\right)' = zg(z) * \psi(z),$$

where

$$\psi(z) = (1+\gamma)z\frac{I^n_{\lambda,\mu}f(z)}{+}\gamma z^2\left(I^n_{\lambda,\mu}f(z)\right)' \prec h(z).$$

The remaining part of the proof of Theorem 3.1 is similar to that of Theorem 2.2, and so we omit the details involved. □

Corollary 3.1 *Let $f \in \mathcal{M}^n_{\lambda,\mu}(\gamma;h)$ be given by* (1.1). *Then the function*

$$\sigma_m(z) = \int_0^1 tS_m(tz)\,dt \quad (z \in \mathbb{U}),$$

where

$$S_m(z) = \frac{1}{z} + \sum_{n=1}^{m-1} a_n z^{n-1} \quad \left(m \in \mathbb{N}\setminus\{1\}; z \in \mathbb{U}\right),$$

is also in the class $\mathcal{M}^n_{\lambda,\mu}(\gamma;h)$.

Proof We have

$$\sigma_m(z) = \frac{1}{z} + \sum_{n=1}^{m-1} \frac{a_n}{n-1} z^{n+1} = (f * g_m)(z) \quad \left(m \in \mathbb{N}\setminus\{1\}\right), \tag{3.1}$$

where

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^{n-1} \in \mathcal{M}^n_{\lambda,\mu}(\gamma;h)$$

and

$$g_m(z) = \frac{1}{z} + \sum_{n=1}^{m-1} \frac{z^n}{n-1} \in \mathcal{M},$$

while, it is known [4] that

$$\mathrm{Re}\{zg_m(z)\} = \mathrm{Re}\left\{1 + \sum_{n=1}^{m-1} \frac{z^n}{n+1}\right\} > \frac{1}{2} \quad \bigl(m \in \mathbb{N} \setminus \{1\}; z \in \mathbb{U}\bigr). \tag{3.2}$$

In view of (3.1) and (3.2), an application of Theorem 3.1 leads to $\sigma_m \in \mathcal{M}^n_{\lambda,\mu}(\gamma; h)$. □

Theorem 3.2 *If* $f \in \mathcal{M}^n_{\lambda,\mu}(\gamma; h)$ *and*

$$z^2 g(z) \in R(\alpha) \quad (g \in \mathcal{M}; z \in \mathbb{U}),$$

then

$$(f * g) \in \mathcal{M}^n_{\lambda,\mu}(\gamma; h).$$

Proof By using a similar method as in the proof of Theorem 3.1, we have

$$(1+\gamma)zI^n_{\lambda,\mu}(f*g)(z) + \gamma z^2\bigl(I^n_{\lambda,\mu}(f*g)(z)\bigr)' = \frac{z^2 g(z) * (z\psi(z))}{z^2 g(z) * z} \quad (z \in \mathbb{U}), \tag{3.3}$$

where

$$\psi(z) = (1+\gamma)zI^n_{\lambda,\mu}f(z) + \gamma z^2\bigl(I^n_{\lambda,\mu}f(z)\bigr)' \prec h(z).$$

Since h is convex univalent in $\mathbb{U}$, it follows from (3.3) and Lemma 1.2 that Theorem 3.2 holds true. □

If we take $\alpha = 0$ and $\alpha = 1/2$ in Theorem 3.2, we have the following corollary.

Corollary 3.2 *If* $f \in \mathcal{M}^n_{\lambda,\mu}(\gamma; h)$ *and* $g \in \mathcal{M}$ *satisfies one of the following conditions*:
(i) $z^2 g(z)$ *is convex univalent in* $\mathbb{U}$
or
(ii) $z^2 g(z) \in S^*(\frac{1}{2})$,
then $(f * g) \in \mathcal{M}^n_{\lambda,\mu}(\gamma; h)$.

4 Integral operators

Theorem 4.1 *If* $f \in \mathcal{M}^n_{\lambda,\mu}(\gamma; h)$, *then the function* F *defined by*

$$F(z) = \frac{c-1}{z^c}\int_0^z t^{c-1} f(t)\,dt \quad \bigl(\mathrm{Re}\{c\} > 1\bigr) \tag{4.1}$$

is in the class $\mathcal{M}^n_{\lambda,\mu}(\gamma; \widetilde{h})$, *where*

$$\widetilde{h}(z) = (c-1)z^{-(c-1)}\int_0^z t^c h(t)\,dt \prec h(z).$$

Proof Let $f \in \mathcal{M}^n_{\lambda,\mu}(\gamma; h)$. Then from (4.1), we obtain

$$(c-1)f(z) = zF'(z) + cF(z). \tag{4.2}$$

Define the function G by

$$z^{-1}G(z) = (1+\gamma)I_{\lambda,\mu}^{n}F(z) + \gamma z\big(I_{\lambda,\mu}^{n}F(z)\big)' \quad (z \in \mathbb{D}). \tag{4.3}$$

Differentiating both sides of (4.3) with respect to z, we get

$$zG'(z) - G(z) = (1+\gamma)zI_{\lambda,\mu}^{n}\big(zF'(z)\big) + \gamma z^2\big(I_{\lambda,\mu}^{n}\big(zF'(z)\big)\big)'. \tag{4.4}$$

Furthermore, it follows from (4.2), (4.3) and (4.4) that

$$\begin{aligned}
&(1+\gamma)zI_{\lambda,\mu}^{n}f(z) + \gamma z^2\big(I_{\lambda,\mu}^{n}f(z)\big)' \\
&\quad = (1+\gamma)zI_{\lambda,\mu}^{n}\left(\frac{zF'(z)+cF(z)}{c-1}\right) \\
&\qquad + \gamma z^2\left(I_{\lambda,\mu}^{n}\left(\frac{zF'(z)+cF(z)}{c-1}\right)\right)' \\
&\quad = \frac{c}{c-1}G(z) + \frac{1}{c-1}\big(zG'(z) - G(z)\big) \\
&\quad = G(z) + \frac{1}{c-1}zG'(z).
\end{aligned} \tag{4.5}$$

Since $f \in \mathcal{M}_{\lambda,\mu}^{n}(\gamma;h)$, from (4.5), we have

$$G(z) + \frac{1}{c-1}zG'(z) \prec h(z) \quad \big(\mathrm{Re}\{c\} > 1\big),$$

and so an application of Lemma 1.1 yields

$$G(z) \prec \widetilde{h}(z) = \frac{c-1}{z^{c-1}}\int_0^z t^c h(t)\,dt \prec h(z).$$

Therefore we conclude that

$$F \in \mathcal{M}_{\lambda,\mu}^{n}(\gamma;\widetilde{h}) \subset \mathcal{M}_{\lambda,\mu}^{n}(\gamma;h). \qquad \square$$

Theorem 4.2 *If $f \in \mathcal{M}$ and F are defined as in Theorem* 4.1, *if*

$$(1-\alpha)zI_{\lambda,\mu}^{n}F(z) + \alpha zI_{\lambda,\mu}^{n}f(z) \prec h(z) \quad (\alpha > 0), \tag{4.6}$$

then $F \in \mathcal{M}_{\lambda,\mu}^{n}(0;\widetilde{h})$, where

$$\widetilde{h}(z) = \frac{c-1}{\alpha}z^{-\frac{\alpha}{c-1}}\int_0^z t^{\frac{c-1}{\alpha}-1}h(t) \prec h(z) \quad \big(\mathrm{Re}\{c\} > 1\big).$$

Proof Let

$$G(z) = zI_{\lambda,\mu}^{n}F(z) \quad (z \in \mathbb{D}). \tag{4.7}$$

Then G is analytic in $\mathbb{U}$ with $G(0) = 1$ and

$$zG'(z) = z^2\big(I_{\lambda,\mu}^{n}F(z)\big)' + G(z). \tag{4.8}$$

It follows from (4.2), (4.6), (4.7) and (4.8) that

$$\begin{aligned}
&(1-\alpha)zI_{\lambda,\mu}^{n}F(z)+\alpha zI_{\lambda,\mu}^{n}f(z)\\
&\quad=(1-\alpha)z\frac{I_{\lambda,\mu}^{n}F(z)}{+}\frac{\alpha}{c-1}\bigl[czI_{\lambda,\mu}^{n}F(z)+z^{2}\bigl(I_{\lambda,\mu}^{n}F(z)\bigr)'\bigr]\\
&\quad=G(z)+\frac{\alpha}{c-1}zG'(z)\prec h(z)\quad\bigl(\mathrm{Re}\{c\}>1;\alpha>0\bigr).
\end{aligned}$$

Therefore, by Lemma 1.1, we conclude that Theorem 4.2 holds true as stated. □

Theorem 4.3 *Let $F\in\mathcal{M}_{\lambda,\mu}^{n}(\gamma;h)$. If the function f is defined by*

$$F(z)=\frac{c-1}{z^{c}}\int_{0}^{z}t^{c-1}f(t)\,dt\quad(c>1),\tag{4.9}$$

then

$$\sigma f(\sigma z)\in\mathcal{M}_{\lambda,\mu}^{n}(\gamma;h),$$

where

$$\sigma=\sigma(c)=\frac{\sqrt{1+(c-1)^{2}}-1}{c-1}.\tag{4.10}$$

The bound σ is sharp for the function

$$h(z)=\beta+(1-\beta)\frac{1+z}{1-z}\quad(\beta\neq1;z\in\mathbb{U}).\tag{4.11}$$

Proof We note that for $F\in\mathcal{M}$,

$$F(z)=F(z)*\frac{1}{z(1-z)}\quad\text{and}\quad zF'(z)=F(z)*\left(\frac{1}{z(1-z)^{2}}-\frac{2}{z^{2}(1-z)}\right).$$

Then from (4.9), we have

$$f(z)=\frac{cF(z)+zF'(z)}{c-1}=(F*g)(z)\quad(c>1;z\in\mathbb{D}),\tag{4.12}$$

where

$$g(z)=\frac{1}{c-1}\left((c-2)\frac{1}{z(1-z)}+\frac{1}{z(1-z)^{2}}\right)\in\mathcal{M}.\tag{4.13}$$

Next, we show that

$$\mathrm{Re}\bigl\{zg(z)\bigr\}>\frac{1}{2}\quad\bigl(|z|<\sigma\bigr),\tag{4.14}$$

where $\sigma=\sigma(c)$ is given by (4.10). Letting

$$\frac{1}{1-z}=Re^{i\theta}\quad\bigl(|z|=r<1;R>0\bigr),$$

we see that

$$\cos\theta = \frac{1 + R^2(1 - r^2)}{2R} \quad \text{and} \quad R \geq \frac{1}{1 + r}. \tag{4.15}$$

Then for (4.13) and (4.15), we have

$$\begin{aligned} 2\operatorname{Re}\{zg(z)\} &= \frac{2}{c-1}\bigl[(c-2)R\cos\theta + R^2\bigl(2\cos^2\theta - 1\bigr)\bigr] \\ &= \frac{R^2}{c-1}\bigl[c\bigl(1-r^2\bigr) + R^2\bigl(1-r^2\bigr)^2 - 2\bigr] + 1 \\ &\geq \frac{R^2}{c-1}\bigl[c - 1 - 2r - (c-1)r^2\bigr] + 1. \end{aligned}$$

This evidently gives (4.14), which is equivalent to

$$\operatorname{Re}\{\sigma z g(\sigma z)\} > \frac{1}{2} \quad (z \in \mathbb{U}). \tag{4.16}$$

Let $F \in \mathcal{M}^n_{\lambda,\mu}(\gamma; h)$. Then, by using (4.12) and (4.16), an application of Theorem 3.1 yields

$$\sigma f(\sigma z) = F(z) * \sigma g(\sigma z) \in \mathcal{M}^n_{\lambda,\mu}(\gamma; h).$$

For h given by (4.11), we consider the function $F \in \mathcal{M}$ defined by

$$(1+\gamma)zI^n_{\lambda,\mu}F(z) + \gamma z^2\bigl(I^n_{\lambda,\mu}F(z)\bigr)' = \beta + (1-\beta)\frac{1+z}{1-z} \quad (\beta \neq 1; z \in \mathbb{U}). \tag{4.17}$$

Then from (4.3), (4.5) and (4.17), we find that

$$\begin{aligned} &(1+\gamma)zI^n_{\lambda,\mu}f(z) + \gamma z^2\bigl(I^n_{\lambda,\mu}f(z)\bigr)' \\ &\quad = \beta + (1-\beta)\frac{1+z}{1-z} + \frac{z}{c-1}\left(\beta + (1-\beta)\frac{1+z}{1-z}\right)' \\ &\quad = \beta + \frac{(1-\beta)(c-1+2z-(c-1)z^2)}{(c-1)(1-z)^2} \\ &\quad = \beta \quad (z = -\sigma). \end{aligned}$$

Therefore we conclude that the bound $\sigma = \sigma(c)$ cannot be increased for each c $(c > 1)$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on the results and they read and approved the final manuscript.

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