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# Properties of multivalent functions associated with the integral operator defined by the hypergeometric function

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## Abstract

In this paper, we introduce a new class of multivalent functions by using a generalized integral operator defined by the hypergeometric function. Some properties such as inclusion, radius problem and integral preserving are considered.

**MSC:** 30C45; 30C50

**Keywords:** Bazilevic functions; multivalent functions; integral operator; hypergeometric functions

## 1 Introduction and preliminaries

Let  $A_p$  denote the class of functions  $f(z)$  of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}), \quad (1.1)$$

which are analytic in the open unit disc  $E$ . Also  $A_1 = A$ , the usual class of analytic functions defined in the open unit disc  $E = \{z : |z| < 1\}$ . A function  $f \in A_p$  is a  $p$ -valent starlike function of order  $\rho$  if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \rho, \quad 0 \leq \rho < p, z \in E.$$

This class of functions is denoted by  $S_p^*(\rho)$ . It is noted that  $S_p^*(0) = S_p^*$ . Let  $f(z)$  and  $g(z)$  be analytic in  $E$ , we say  $f(z)$  is subordinate to  $g(z)$ , written  $f < g$  or  $f(z) < g(z)$  if there exists a Schwarz function  $w(z)$ ,  $w(0) = 0$  and  $|w(z)| < 1$  in  $E$ , then  $f(z) = g(w(z))$ . In particular, if  $g$  is univalent in  $E$ , then we have the following equivalence

$$f(z) < g(z) \iff f(0) = g(0) \quad \text{and} \quad f(E) \subset g(E).$$

For any two analytic functions  $f(z)$  and  $g(z)$  with

$$f(z) = \sum_{n=0}^{\infty} b_n z^{n+1} \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} c_n z^{n+1}, \quad z \in E,$$

the convolution (the Hadamard product) is given by

$$(f * g)(z) = \sum_{n=0}^{\infty} b_n c_n z^{n+1}, \quad z \in E.$$

A function  $f \in A$  is said to be in the class, denoted by  $SD(k, \delta)$  ( $0 \leq \delta < 1$ ), if and only if

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > k \left| \frac{zf'(z)}{f(z)} - 1 \right| + \delta, \quad k \geq 0, z \in E. \tag{1.2}$$

Similarly, a function  $f \in A$  is said to be in the class, denoted by  $CD(k, \delta)$  of  $k$ -uniformly convex of order  $\delta$  ( $0 \leq \delta < 1$ ), if

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > k \left| \frac{zf''(z)}{f'(z)} \right| + \delta, \quad k \geq 0, z \in E. \tag{1.3}$$

**Geometric interpretation** The functions  $f \in SD(k, \delta)$  and  $f \in CD(k, \delta)$  if and only if  $\frac{zf'(z)}{f(z)}$  and  $\frac{zf''(z)}{f'(z)} + 1$ , respectively, take all the values in the conic domain  $\Omega_{k,\delta}$  defined by

$$\Omega_{k,\delta} = \{u + iv : u > k\sqrt{(u-1)^2 + v^2} + \delta\}$$

with  $p(z) = \frac{zf'(z)}{f(z)}$  or  $p(z) = \frac{zf''(z)}{f'(z)} + 1$  and considering the functions which map  $E$  onto the conic domain  $\Omega_{k,\delta}$  such that  $1 \in \Omega_{k,\delta}$ . One may rewrite the conditions (1.2) or (1.3) in the form

$$p(z) < q_{k,\delta}(z).$$

The function  $q_{k,\delta}(z)$  plays the role of extremal for these classes and is given by

$$q_{k,\delta}(z) = \begin{cases} \frac{1+(1-2\delta)z}{1-z}, & k = 0, \\ 1 + \frac{2\delta\gamma}{\pi^2} (\log \frac{1+\sqrt{z}}{1-\sqrt{z}})^2, & k = 1, \\ 1 + \frac{2\delta}{1-k^2} \sinh^2[(\frac{2}{\pi} \arccos k) \arctan h\sqrt{z}], & 0 < k < 1, \\ 1 + \frac{\delta}{k^2-1} \sin(\frac{\pi}{2R(t)} \int_0^{\frac{u(z)}{\sqrt{t}}} \frac{1}{\sqrt{1-x^2}\sqrt{1-tx^2}} dx) + \frac{\delta}{k^2-1}, & k > 1. \end{cases} \tag{1.4}$$

For  $f(z)$  in  $A_p$ , the operator  $D^{\mu+p-1} : A_p \rightarrow A_p$  is defined by

$$D^{\mu+p-1}f(z) = \frac{z^p}{(1-z)^{\mu+p}} * f(z) \quad (\mu > -p),$$

or equivalently

$$D^{\mu+p-1}f(z) = \frac{z^p (z^{\mu-1}f(z))^{\mu+p-1}}{(\mu+p-1)!}, \tag{1.5}$$

where  $\mu$  is any integer greater than  $-p$ . If  $f(z)$  is given by (1.1), then it follows that

$$D^{\mu+p-1}f(z) = z^p + \sum_{n=p+1}^{\infty} \frac{(\mu+n-1)!}{(n-p)!(\mu+p-1)!} a_n z^n.$$

The symbol  $D^{\mu+p-1}$  when  $p = 1$ , was introduced by Ruscheweyh [1] and  $D^{\mu+p-1}$  is called the  $(\mu + p - 1)$ th order Ruscheweyh derivative. We now introduce a function  $(z^p {}_2F_1(a, b, c; z))^{-1}$  given by

$$(z^p {}_2F_1(a, b, c; z)) * (z^p {}_2F_1(a, b, c; z))^{-1} = \frac{z^p}{(1-z)^{\mu+p}} \quad (\mu > -p),$$

and the following linear operator

$$I_{\mu,p}(a, b, c)f(z) = (z^p {}_2F_1(a, b, c; z))^{-1} * f(z), \tag{1.6}$$

where  $a, b, c$  are real or complex numbers other than  $0, -1, -2, \dots, \mu > -p, z \in E$  and  $f(z) \in A_p$ . This operator was recently introduced in [2]. In particular, for  $p = 1$ , this operator is studied by Noor [3]. For  $b = 1$ , this operator reduces to the well-known Cho-Kwon-Srivastava operator  $I_{\mu,p}(a, c)$ , which was studied by Cho *et al.* [4], and for  $\mu = 1, b = c, a = n + p$ , see [5]. For  $a = n + p, b = c = 1$ , this operator was investigated by Liu [6] and Liu and Noor [7].

Simple computations yield

$$I_{\mu,p}(a, b, c)f(z) = z^p + \sum_{n=p+1}^{\infty} \frac{(c)_n(\mu+p)_n}{(a)_n(b)_n} a_n z^n.$$

From (1.6), we note that

$$I_{\mu,1}(a, b, c)f(z) = I_{\mu}(a, b, c)f(z) \quad (\text{see [3]}),$$

$$I_{0,p}(a, p, a)f(z) = f(z), I_{1,p}(a, p, a)f(z) = \frac{zf'(z)}{p}.$$

Also, it can be easily seen that

$$z(I_{\mu,p}(a, b, c)f(z))' = (\mu + p)I_{\mu+1,p}(a, b, c)f(z) - \mu I_{\mu,p}(a, b, c)f(z), \tag{1.7}$$

and

$$z(I_{\mu,p}(a + 1, b, c)f(z))' = aI_{\mu,p}(a, b, c)f(z) - (a - p)I_{\mu,p}(a, b, c)f(z).$$

We define the following class of multivalent analytic functions by using the operator  $I_{\mu,p}(a, b, c)f(z)$  above.

**Definition 1.1** Let  $f \in A_p$  for  $p \in \mathbb{N}$ . Then  $f \in UB_{\mu,p}^{\alpha}(a, b, c; \gamma, k, \delta)$  for  $a, b, c \in \mathbb{R} \setminus \mathbb{Z}_0^-, \mu > -p, \alpha > 0, k \geq 0, 0 \leq \delta < 1$  and  $\gamma > 0$  if and only if

$$(1 - \gamma) \left( \frac{I_{\mu,p}(a, b, c)f(z)}{I_{\mu,p}(a, b, c)g(z)} \right)^{\alpha} + \gamma \frac{I_{\mu+1,p}(a, b, c)f(z)}{I_{\mu+1,p}(a, b, c)g(z)} \left( \frac{I_{\mu,p}(a, b, c)f(z)}{I_{\mu,p}(a, b, c)g(z)} \right)^{\alpha-1} < q_{k,\delta}(z), \tag{1.8}$$

where  $g \in A_p$  is such that

$$q(z) = \frac{I_{\mu+1,p}(a, b, c)g(z)}{I_{\mu,p}(a, b, c)g(z)} \in P(\rho), \quad \rho = \frac{k + \delta}{k + 1}, z \in E. \tag{1.9}$$

Furthermore, for different choices of parameters being involved, we obtain many other well-known subclasses of the class  $A_p$  and  $A$  as special cases.

- (i)  $a = c, b = 1, k = 0, \mu = m \in \mathbb{N}_0$ , we have  $B_{m,p}^\alpha(\gamma, \delta)$  studied in [8].
- (ii)  $a = c = b = p = \gamma = 1, k = \mu = 0, g(z) = z$ , the class  $UB_{\mu,p}^\alpha(a, b, c; \gamma, k, \delta)$  reduces to the class

$$B^\alpha(\delta) = \left\{ f \in A(1) : \frac{zf'(z)}{f(z)} \left( \frac{f(z)}{z} \right)^\alpha \in P(\delta) \right\}$$

studied in [9].

- (iii)  $a = c = b = p = \gamma = 1, k = \mu = 0, g(z) = z$ , the  $UB_{\mu,p}^\alpha(a, b, c; \gamma, k, \delta)$  reduces to  $B(\alpha)$  is the class of Bazilevich functions investigated by Singh [10].
- (iv)  $a = c = b = p = \alpha = 1, \gamma = 0, k = \mu = 0, g(z) = z$ , the class  $UB_{\mu,p}^\alpha(a, b, c; \gamma, k, \delta)$  reduces to the class

$$B_\delta = \left\{ f \in A(1) : \frac{f(z)}{z} \in P(\delta) \right\},$$

the class studied by Chen [11].

Let  $f \in A_p$  and  $F_{\eta,p} : A_p \rightarrow A_p$  be defined by

$$F_{\eta,p}(z) = \frac{(\eta + p)}{z^\eta} \int_0^z t^{\eta-1} f(t) dt, \quad \eta > -p. \tag{1.10}$$

We need the following lemmas which will be used in our main results.

**Lemma 1.2** [12] *Let  $u = u_1 + iu_2$  and  $v = v_1 + iv_2$ , and let  $\psi : D \subset \mathbb{C}^2 \rightarrow \mathbb{C}$  be a complex-valued function satisfying the conditions:*

- (i)  $\psi(u, v)$  is continuous in a domain  $D \subset \mathbb{C}^2$ ,
  - (ii)  $(1, 0) \in D$  and  $\psi(1, 0) > 0$ ,
  - (iii)  $\operatorname{Re} \psi(iu_2, v_1) \leq 0$ , whenever  $(iu_2, v_1) \in D$  and  $v_1 \leq -\frac{1}{2}(1 + u_2^2)$ .
- If  $h(z) = 1 + c_1z + c_2z^2 + \dots$  is analytic in  $E$  such that  $(h, zh') \in D$  and  $\operatorname{Re} \psi(h(z), zh'(z)) > 0$  for  $z \in E$ , then  $\operatorname{Re} h(z) > 0$ .*

**Lemma 1.3** [13] *Let  $h$  be convex in the unit disc  $E$ , and let  $A \geq 0$ . Suppose that  $B(z)$  is analytic in  $E$  with  $\operatorname{Re} B(z) \geq A$ . If  $g$  is analytic in  $E$  and  $g(0) = h(0)$ . Then*

$$Az^2g''(z) + B(z)zg'(z) + g(z) < h(z) \quad \text{implies that} \quad g(z) < h(z).$$

**Lemma 1.4** [14] *Let  $F$  be analytic and convex in  $E$ . If  $f, g \in A_p$  and  $f, g < F$ . Then*

$$\sigma f + (1 - \sigma)g < F, \quad 0 \leq \sigma \leq 1.$$

**Lemma 1.5** [15] *Let  $h$  be convex in  $E$  with  $h(0) = a$  and  $\beta \in \mathbb{C}$  such that  $\operatorname{Re} \beta \geq 0$ . If  $p \in H[a, n]$  and*

$$p(z) + \frac{zp'(z)}{\beta} \prec h(z),$$

*then  $p(z) \prec q(z) \prec h(z)$ , where*

$$q(z) = \frac{\beta}{nz^{\beta/n}} \int_0^z h(t)t^{\beta/n-1} dt$$

*and  $q(z)$  is the best dominant.*

## 2 Main results

**Theorem 2.1** *Let  $f \in UB_{\mu,p}^\alpha(a, b, c; \gamma, k, \delta)$  for  $a, b, c \in \mathbb{R} \setminus \mathbb{Z}_0^-, \mu > -p, p \in \mathbb{N}, \alpha > 0, k \geq 0, 0 \leq \delta < 1$  and  $\gamma > 0$ . Then  $f \in UB_{\mu,p}^\alpha(a, b, c; 0, k, \delta)$ .*

*Proof* Consider

$$h(z) = \left( \frac{I_{\mu,p}(a, b, c)f(z)}{I_{\mu,p}(a, b, c)g(z)} \right)^\alpha, \tag{2.1}$$

where  $h$  is analytic in  $E$  with  $h(0) = 1$ , and  $g \in A_p$  satisfies condition (1.9). Differentiating (2.1) logarithmically and using (1.7), we have

$$\frac{zh'(z)}{h(z)} = \alpha(\mu + p) \left\{ \left( \frac{I_{\mu+1,p}(a, b, c)f(z)}{I_{\mu,p}(a, b, c)f(z)} - \frac{I_{\mu+1,p}(a, b, c)g(z)}{I_{\mu,p}(a, b, c)g(z)} \right) \right\}.$$

Using (2.1) and simplifying, we obtain

$$h(z) + \frac{\gamma zh'(z)}{\alpha(\mu + p)q(z)} = (1 - \gamma) \left( \frac{I_{\mu,p}(a, b, c)f(z)}{I_{\mu,p}(a, b, c)g(z)} \right)^\alpha + \gamma \frac{I_{\mu+1,p}(a, b, c)f(z)}{I_{\mu+1,p}(a, b, c)g(z)} \left( \frac{I_{\mu,p}(a, b, c)f(z)}{I_{\mu,p}(a, b, c)g(z)} \right)^{\alpha-1}.$$

Since  $f \in UB_{\mu,p}^\alpha(a, b, c; \gamma, k, \delta)$ , therefore, we can write

$$h(z) + \frac{\gamma zh'(z)}{\alpha(\mu + p)q(z)} \prec q_{k,\delta}(z), \quad z \in E.$$

Now using Lemma 1.3 for  $A = 0$  and  $B(z) = \frac{\gamma}{\alpha(\mu+p)q(z)}$  with  $\operatorname{Re} q(z) > 0$ , we have  $\operatorname{Re} B(z) \geq 0$ , therefore,  $h(z) \prec q_{k,\delta}(z)$ . Hence  $f \in UB_{\mu,p}^\alpha(a, b, c; 0, k, \delta)$ .  $\square$

**Theorem 2.2** *Let  $f \in UB_{\mu,p}^\alpha(a, b, c; \gamma, 0, \delta)$  for  $a, b, c \in \mathbb{R} \setminus \mathbb{Z}_0^-, \mu > -p, p \in \mathbb{N}$ . Then  $f \in UB_{\mu,p}^\alpha(a, b, c; 0, 0, \delta_1)$ , where*

$$\delta_1 = \frac{2\alpha\delta(p + \mu)|q(z)|^2 + \gamma\rho}{2\alpha(p + \mu)|q(z)|^2 + \gamma\rho}.$$

*Proof* Consider

$$h(z) = \frac{1}{(1 - \delta_1)} \left\{ \left( \frac{I_{\mu,p}(a, b, c)f(z)}{I_{\mu,p}(a, b, c)g(z)} \right)^\alpha - \delta_1 \right\}, \tag{2.2}$$

where  $h$  is analytic in  $E$  with  $h(0) = 1$ , and  $g \in A_p$  satisfies condition (1.9). Differentiating (2.2), we have

$$\begin{aligned} \frac{(1 - \delta_1)}{\alpha} h'(z) &= \left( \frac{I_{\mu,p}(a, b, c)f(z)}{I_{\mu,p}(a, b, c)g(z)} \right)^{\alpha-1} \\ &\times \left\{ \frac{(I_{\mu,p}(a, b, c)f(z))'}{I_{\mu,p}(a, b, c)g(z)} - \frac{I_{\mu,p}(a, b, c)f(z)}{I_{\mu,p}(a, b, c)g(z)} \frac{(I_{\mu,p}(a, b, c)g(z))'}{I_{\mu,p}(a, b, c)g(z)} \right\}. \end{aligned}$$

Using (1.7) and simplifying, we obtain

$$\begin{aligned} (1 - \gamma) \left( \frac{I_{\mu,p}(a, b, c)f(z)}{I_{\mu,p}(a, b, c)g(z)} \right)^\alpha + \gamma \frac{I_{\mu+1,p}(a, b, c)f(z)}{I_{\mu+1,p}(a, b, c)g(z)} \left( \frac{I_{\mu,p}(a, b, c)f(z)}{I_{\mu,p}(a, b, c)g(z)} \right)^{\alpha-1} \\ = (1 - \delta_1)h(z) + \delta_1 + \frac{\gamma(1 - \delta_1)zh'(z)}{\alpha(p + \mu)q(z)}. \end{aligned}$$

Since  $f \in UB_{\mu,p}^\alpha(a, b, c; \gamma, 0, \delta)$ , therefore we have

$$(1 - \delta_1)h(z) + \delta_1 + \frac{\gamma(1 - \delta_1)zh'(z)}{\alpha(p + \mu)q(z)} < \frac{1 + (1 - 2\delta)z}{1 - z}, \quad 0 \leq \delta < 1, z \in E.$$

This implies that

$$\frac{1}{1 - \delta} \left\{ (1 - \delta_1)h(z) + \delta_1 - \delta + \frac{\gamma(1 - \delta_1)zh'(z)}{\alpha(p + \mu)q(z)} \right\} \in q_{0,0}(E) = P.$$

To obtain our desired result, we show that  $h \in P$ , for  $z \in E$ . Let  $u = u_1 + iu_2, v = v_1 + iv_2$ , and let  $\Psi : D \subset \mathbb{C}^2 \rightarrow \mathbb{C}$  be a complex-valued function such that  $u = h(z), v = zh'(z)$ . Then

$$\Psi(u, v) = (1 - \delta_1)u + \delta_1 - \delta + \frac{\gamma(1 - \delta_1)v}{\alpha(p + \mu)q(z)}.$$

The first two conditions of Lemma 1.2 are easily verified. To verify the third condition, we consider

$$\begin{aligned} \operatorname{Re} \Psi(iu_2, v_1) &= \operatorname{Re} \left\{ (1 - \delta_1)iu_2 + \delta_1 - \delta + \frac{\gamma(1 - \delta_1)v_1}{\alpha(p + \mu)q(z)} \right\} \\ &= \delta_1 - \delta + \operatorname{Re} \frac{\gamma(1 - \delta_1)v_1}{\alpha(p + \mu)q(z)} \\ &\leq \delta_1 - \delta - \operatorname{Re} \frac{\gamma(1 - \delta_1)(1 + u_2^2)\overline{q(z)}}{2\alpha(p + \mu)|q(z)|^2} \\ &\leq \delta_1 - \delta - \frac{\gamma(1 - \delta_1)(1 + u_2^2)\rho}{2\alpha(p + \mu)|q(z)|^2} = \frac{A + Bu^2}{C}, \end{aligned}$$

where  $A = 2\alpha(p + \mu)(\delta_1 - \delta)|q(z)|^2 - \gamma\rho(1 - \delta_1)$ ,  $B = -\gamma\rho(1 - \delta_1) \leq 0$  if  $0 \leq \delta_1 < 1$  and  $C = 2\alpha(p + \mu)|q(z)|^2 > 0$ . From the relation  $\delta_1 = \frac{2\alpha\delta(p+\mu)|q(z)|^2 + \gamma\rho}{2\alpha(p+\mu)|q(z)|^2 + \gamma\rho}$ , we have  $A \leq 0$ . This implies that  $\operatorname{Re} \Psi(iu_2, v_1) \leq 0$ . Using Lemma 1.2, we have  $h \in P$  for  $z \in E$ . This completes the proof.  $\square$

**Theorem 2.3** *Let  $a, b, c \in \mathbb{R} \setminus \mathbb{Z}_0^-$ ,  $\mu > -p$ ,  $p \in \mathbb{N}$ ,  $\alpha > 0$ ,  $k \geq 0$  and  $0 \leq \delta < 1$ . Then*

$$UB_{\mu,p}^\alpha(a, b, c; \gamma_2, k, \delta) \subset UB_{\mu,p}^\alpha(a, b, c; \gamma_1, k, \delta), \quad 0 \leq \gamma_1 < \gamma_2, z \in E.$$

*Proof* Since  $f \in UB_{\mu,p}^\alpha(a, b, c; \gamma_2, k, \delta)$ , therefore, we have

$$\begin{aligned} (1 - \gamma_2) \left( \frac{I_{\mu,p}(a, b, c)f(z)}{I_{\mu,p}(a, b, c)g(z)} \right)^\alpha + \gamma_2 \frac{I_{\mu+1,p}(a, b, c)f(z)}{I_{\mu+1,p}(a, b, c)g(z)} \left( \frac{I_{\mu,p}(a, b, c)f(z)}{I_{\mu,p}(a, b, c)g(z)} \right)^{\alpha-1} \\ = h_1(z) \prec q_{k,\delta}(z), \end{aligned} \tag{2.3}$$

where  $g \in A_p$  satisfies condition (1.9). From Theorem 2.1, we write

$$\left( \frac{I_{\mu,p}(a, b, c)f(z)}{I_{\mu,p}(a, b, c)g(z)} \right)^\alpha = h_2(z) \prec q_{k,\delta}(z), \quad z \in E. \tag{2.4}$$

Now, for  $\gamma_1 \geq 0$ , we obtain

$$\begin{aligned} (1 - \gamma_1) \left( \frac{I_{\mu,p}(a, b, c)f(z)}{I_{\mu,p}(a, b, c)g(z)} \right)^\alpha + \gamma_1 \frac{I_{\mu+1,p}(a, b, c)f(z)}{I_{\mu+1,p}(a, b, c)g(z)} \left( \frac{I_{\mu,p}(a, b, c)f(z)}{I_{\mu,p}(a, b, c)g(z)} \right)^{\alpha-1} \\ = \left( 1 - \frac{\gamma_1}{\gamma_2} \right) \left( \frac{I_{\mu,p}(a, b, c)f(z)}{I_{\mu,p}(a, b, c)g(z)} \right)^\alpha \\ + \frac{\gamma_1}{\gamma_2} \left\{ (1 - \gamma_2) \left( \frac{I_{\mu,p}(a, b, c)f(z)}{I_{\mu,p}(a, b, c)g(z)} \right)^\alpha + \gamma_2 \frac{I_{\mu+1,p}(a, b, c)f(z)}{I_{\mu+1,p}(a, b, c)g(z)} \left( \frac{I_{\mu,p}(a, b, c)f(z)}{I_{\mu,p}(a, b, c)g(z)} \right)^{\alpha-1} \right\} \\ = \frac{\gamma_1}{\gamma_2} h_1(z) + \left( 1 - \frac{\gamma_1}{\gamma_2} \right) h_2(z). \end{aligned}$$

Using the convexity of the class of the function  $q_{k,\delta}(z)$  and Lemma 1.4, we write

$$\frac{\gamma_1}{\gamma_2} h_1(z) + \left( 1 - \frac{\gamma_1}{\gamma_2} \right) h_2(z) \prec q_{k,\delta}(z), \quad z \in E,$$

where  $h_1$  and  $h_2$  are given by (2.3) and (2.4), respectively. This implies that  $f \in UB_{\mu,p}^\alpha(a, b, c; \gamma_1, k, \delta)$ . Hence the proof of the theorem is completed.  $\square$

**Theorem 2.4** *Let  $f \in UB_{\mu,p}^1(a, b, c; \gamma, k, \delta)$ ,  $a, b, c \in \mathbb{R} \setminus \mathbb{Z}_0^-$ ,  $\mu > -p$ ,  $p \in \mathbb{N}$ ,  $\alpha > 0$ ,  $k \geq 0$ ,  $\gamma \geq 1$  and  $0 \leq \delta < 1$ . Then  $f \in UB_{\mu+1,p}^1(a, b, c; 1, k, \delta)$ .*

*Proof* Since  $f \in UB_{\mu,p}^1(a, b, c; \gamma, k, \delta)$ , therefore, we have

$$(1 - \gamma) \left( \frac{I_{\mu,p}(a, b, c)f(z)}{I_{\mu,p}(a, b, c)g(z)} \right) + \gamma \frac{I_{\mu+1,p}(a, b, c)f(z)}{I_{\mu+1,p}(a, b, c)g(z)} \prec q_{k,\delta}(z).$$

Now, consider

$$\begin{aligned} \gamma \frac{I_{\mu+1,p}(a,b,c)f(z)}{I_{\mu+1,p}(a,b,c)g(z)} &= (1-\gamma) \left( \frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)} \right) + \gamma \frac{I_{\mu+1,p}(a,b,c)f(z)}{I_{\mu+1,p}(a,b,c)g(z)} \\ &+ (\gamma-1) \left( \frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)} \right). \end{aligned}$$

This implies that

$$\begin{aligned} \frac{I_{\mu+1,p}(a,b,c)f(z)}{I_{\mu+1,p}(a,b,c)g(z)} &= \frac{1}{\gamma} \left\{ (1-\gamma) \left( \frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)} \right) + \gamma \frac{I_{\mu+1,p}(a,b,c)f(z)}{I_{\mu+1,p}(a,b,c)g(z)} \right\} \\ &+ \left( 1 - \frac{1}{\gamma} \right) \left( \frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)} \right). \end{aligned}$$

Using Theorem 2.1, Lemma 1.4 and the convexity of  $q_{k,\delta}(z)$ , we have the required result.  $\square$

Now, using the operators  $I_{\mu,p}(a,b,c)$  and  $F_{\eta,p}$  defined by (1.7) and (1.10), respectively, we have

$$z(I_{\mu,p}(a,b,c)F_{\eta,p}(f)(z))' = (p+\eta)I_{\mu,p}^{\mu}(a,b,c)f(z) - \eta I_{\mu,p}^{\mu}(a,b,c)F_{\eta,p}(f)(z), \quad \eta > -p. \quad (2.5)$$

**Theorem 2.5** *Let  $f \in A_p$  and  $F_{\eta,p}$  be given by (1.10). If*

$$(1-\gamma) \frac{I_{\mu,p}(a,b,c)F_{\eta,p}(f)(z)}{z^p} + \gamma \frac{I_{\mu,p}(a,b,c)f(z)}{z^p} < q_{k,\delta}(z), \quad z \in E, \quad (2.6)$$

with  $a, b, c \in \mathbb{R} \setminus \mathbb{Z}_0^-, \mu, \eta > -p, p \in \mathbb{N}, \gamma > 0$ , then

$$\frac{I_{\mu,p}(a,b,c)F_{\eta,p}(f)(z)}{z^p} < h(z) < q_{k,\delta}(z), \quad z \in E,$$

where

$$h(z) = \frac{p+\eta}{\gamma z^{(p+\eta)/\gamma}} \int_0^z q_{k,\delta}(t) t^{(p+\eta)/\gamma - 1} dt.$$

*Proof* Let

$$\frac{I_{\mu,p}(a,b,c)F_{\eta,p}(f)(z)}{z^p} = h_1(z), \quad z \in E,$$

where  $h_1$  is analytic in  $E$  with  $h_1(0) = 1$ . Then

$$z(I_{\mu,p}(a,b,c)F_{\eta,p}(f)(z))' = pz^p h_1(z) + z^{p+1} h_1'(z).$$

Using (2.5), we have

$$\gamma(p+\eta) \frac{I_{\mu,p}(a,b,c)f(z)}{z^p} - \gamma \eta \frac{I_{\mu,p}(a,b,c)F_{\eta,p}(f)(z)}{z^p} = p\gamma h_1(z) + \gamma z h_1'(z).$$



Thus,

$$(1 - \gamma) \frac{I_{\mu,p}(a, b, c)F_{\eta,p}(f(z))}{z^p} + \gamma \frac{I_{\mu,p}(a, b, c)f(z)}{z^p} = h_1(z) + \gamma \frac{zh'_1(z)}{p + \eta}. \tag{2.7}$$

From (2.7), it follows that

$$h_1(z) + \gamma \frac{zh'_1(z)}{p + \eta} < q_{k,\delta}(z), \quad z \in E.$$

Using Lemma 1.5, for  $\beta_1 = \frac{p+\eta}{\gamma}$ ,  $n = 1$  and  $a = 1$ , we obtain  $h_1(z) < h(z) < q_{k,\delta}(z)$ . That is,  $\frac{I_{\mu,p}(a,b,c)F_{\eta,p}(f(z))}{z^p} < q_{k,\delta}(z)$ .  $\square$

**Theorem 2.6** Let  $f \in UB_{\mu,p}^\alpha(a, b, c; 0, 0, \delta)$  for  $a, b, c \in \mathbb{R} \setminus \mathbb{Z}_0^-, \mu > -p, p \in \mathbb{N}, \alpha, \gamma > 0, 0 \leq \delta < 1$ . Then  $f \in UB_{\mu,p}^\alpha(a, b, c; \gamma, 0, \delta)$ , for  $|z| < r_0$ , where

$$r_0 = \frac{\alpha(p}{+} \mu) + \gamma - \sqrt{\gamma^2 + 2\alpha\gamma(p + \mu)\alpha(p + \mu)}. \tag{2.8}$$

*Proof* Let  $f \in UB_{\mu,p}^\alpha(a, b, c; 0, 0, \delta)$ . Then we have

$$\left( \frac{I_{\mu,p}(a, b, c)f(z)}{I_{\mu,p}(a, b, c)g(z)} \right)^\alpha = (1 - \delta)h(z) + \delta, \tag{2.9}$$

where  $g \in A_p$  satisfies the condition

$$q(z) = \frac{I_{\mu+1,p}(a, b, c)g(z)}{I_{\mu,p}(a, b, c)g(z)} \in P, \quad z \in E$$

and  $h \in P$ . Differentiating (2.9) and then using (1.7), we obtain

$$\gamma \frac{I_{\mu+1,p}(a, b, c)f(z)}{I_{\mu+1,p}(a, b, c)g(z)} \left( \frac{I_{\mu,p}(a, b, c)f(z)}{I_{\mu,p}(a, b, c)g(z)} \right)^{\alpha-1} - \gamma \left( \frac{I_{\mu,p}(a, b, c)f(z)}{I_{\mu,p}(a, b, c)g(z)} \right)^\alpha = \frac{\gamma(p - \delta)zh'(z)}{\alpha(p + \mu)q(z)}.$$

This implies that

$$\begin{aligned} & \frac{1}{1 - \delta} \left\{ (1 - \gamma) \left( \frac{I_{\mu,p}(a, b, c)f(z)}{I_{\mu,p}(a, b, c)g(z)} \right)^\alpha + \gamma \frac{I_{\mu+1,p}(a, b, c)f(z)}{I_{\mu+1,p}(a, b, c)g(z)} \left( \frac{I_{\mu,p}(a, b, c)f(z)}{I_{\mu,p}(a, b, c)g(z)} \right)^{\alpha-1} - \delta \right\} \\ & = h(z) + \frac{\gamma zh'(z)}{\alpha(p + \mu)q(z)}, \quad z \in E. \end{aligned} \tag{2.10}$$

Now, using the well-known distortion result for class  $P$ , we have

$$|zh'(z)| \leq \frac{2r \operatorname{Re} h(z)}{1 - r^2} \quad \text{and} \quad \operatorname{Re} h(z) \geq \frac{1 - r}{1 + r}, \quad |z| < r < 1, z \in E.$$

Thus, due to the applications of these inequalities, we have

$$\begin{aligned}\operatorname{Re}\left(h(z) + \frac{\gamma zh'(z)}{\alpha(p + \mu)q(z)}\right) &\geq \operatorname{Re} h(z) - \frac{\gamma |zh'(z)|}{\alpha(p + \mu)|q(z)|} \\ &\geq \operatorname{Re} h(z) \left(1 - \frac{2\gamma r}{\alpha(p + \mu)(1 - r)^2}\right) \\ &= \operatorname{Re} h(z) \left(\frac{\alpha(p + \mu)(1 - r)^2 - 2\gamma r}{\alpha(p + \mu)(1 - r)^2}\right).\end{aligned}$$

For  $|z| < r_0$ , where  $r_0$  is given in (2.8), the inequality above is positive. Sharpness of the result follows by taking  $h(z) = \frac{1+z}{1-z}$ . Hence from (2.10), we have the required result.  $\square$

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

MR and SNM jointly discussed and presented the ideas of this article. MR made the text file and corresponded it to the journal. Both authors read and approved the final manuscript.

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#### References

1. Ruscheweyh, S: New criteria for univalent functions. *Proc. Am. Math. Soc.* **49**, 109-115 (1975)
2. Ul-Haq, W: Some Classes of Generalized Convex and Related Functions. Ph.D. thesis, COMSATS Institute of Information Technology, Islamabad, Pakistan (2011)
3. Noor, KI: Integral operators defined by convolution with hypergeometric functions. *Appl. Math. Comput.* **182**, 1872-1881 (2006)
4. Cho, NE, Kwon, OS, Srivastava, HM: Inclusion relationships and argument properties for certain subclasses of multivalent functions associated with a family of linear operators. *J. Math. Anal. Appl.* **292**, 470-483 (2004)
5. Patel, J, Cho, NE: Some classes of analytic functions involving Noor integral operator. *J. Math. Anal. Appl.* **312**, 564-575 (2005)
6. Liu, J-L: The Noor integral and strongly starlike functions. *J. Math. Anal. Appl.* **261**, 441-447 (2001)
7. Liu, J-L, Noor, KI: Some properties of Noor integral operator. *J. Nat. Geom.* **21**, 81-90 (2002)
8. Patel, J, Rout, S: Properties of certain analytic functions involving Ruscheweyh's derivatives. *Math. Jpn.* **39**, 509-518 (1994)
9. Ponnusamy, S, Karanukaran, V: Differential subordination and conformal mappings. *Complex Var. Theory Appl.* **11**, 70-86 (1988)
10. Singh, R: On Bazilevič functions. *Proc. Am. Math. Soc.* **38**, 261-271 (1973)
11. Chen, MP: On the regular functions satisfying  $\operatorname{Re}\{f(z)/z\}$ . *Bull. Inst. Math. Acad. Sin.* **3**, 65-70 (1975)
12. Miller, SS, Mocanu, PT: Differential subordination and inequalities in the complex plane. *J. Differ. Equ.* **67**, 199-211 (1987)
13. Miller, SS, Mocanu, PT: Second order differential inequalities in the complex plane. *J. Math. Anal. Appl.* **65**, 289-305 (1978)
14. Liu, M-S: On certain subclass of analytic functions. *J. South China Normal Univ. Natur. Sci. Ed.* **4**, 15-20 (2002) (in Chinese)
15. Hallenbeck, DJ, Ruscheweyh, S: Subordination by convex functions. *Proc. Am. Math. Soc.* **52**, 191-195 (1975)

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