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New Ostrowski-Grüss type inequalities with the derivatives bounded by functions

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Abstract

In this paper, we establish some new Ostrowski-Grüss type inequalities involving multiple interior points with the first-order derivative bounded by functions instead of constants, some of which provide sharp bounds. Then we establish a new 2D Ostrowski-Grüss type inequality involving multiple interior points with the second mixed partial derivative bounded by functions. For illustrating the applications of the Ostrowski-Grüss type inequalities established, we apply them to derive error bounds for some numerical integration formulae.

MSC: 26D10; 26D20

Keywords: Ostrowski-Grüss type inequality; sharp; numerical integration; error bound

1 Introduction

As is known, the Ostrowski-type inequality [1] can be used to estimate the absolute deviation of a function from its integral mean, while the Grüss inequality [2] can be used to estimate the absolute deviation of the integral of the product of two functions from the product of their respective integral. Recently, various generalizations of the Ostrowski inequality and the Grüss inequality have been established (for example, see [3–21] and the references therein). These inequalities can be used to provide explicit error bounds for numerical quadrature formulae such as the Simpson quadrature formula, trapezoid quadrature formula and so on. Among the generalizations, many Ostrowski-Grüss type inequalities have been established [13–21]. Now we list some important results in the literature.

In [13], Dragomir *et al.* presented an Ostrowski-Grüss type inequality for the first time as follows:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right|\le\frac{1}{4}(b-a)(\Gamma-\gamma)$$

for all $x\in[a,b]$, where γ, Γ are two constants such that $\gamma\le f'(t)\le\Gamma$, $t\in[a,b]$.

In [14], under the same conditions as above, Matić *et al.* improved the above result to the following form:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right|\le\frac{1}{4\sqrt{3}}(b-a)(\Gamma-\gamma).$$

Then in [15], Cheng presented a sharp inequality as follows:

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right| \le \frac{1}{8}(b-a)(\Gamma - \gamma). \tag{1}$$

This inequality is sharp in the sense that the constant $\frac{1}{8}$ cannot be replaced by a smaller one.

In [16], Feng *et al.* further generalized the inequalities above, and presented a sharp Ostrowski-Grüss type inequality involving multiple interior points as follows:

$$\begin{aligned}&\left|\frac{1}{b-a}\sum_{i=0}^{k}(m_{i+1}-m_i)f(x_i)\right.\\&\qquad\left. - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{(b-a)^2}\left[\frac{b^2-a^2}{2} - \sum_{i=0}^{k-1} m_{i+1}(x_{i+1}-x_i)\right]\right|\\&\quad\le \frac{1}{8}(b-a)(\Gamma-\gamma),\end{aligned} \tag{2}$$

where $x_i \in [a,b]$, $i = 1,\dots,k-1$, are interior points, $m_i \in [x_{i-1},x_i]$, $i = 1,2,\dots,k$, $x_0 = a$, $x_k = b$, $m_0 = a$, $m_{k+1} = b$, and γ, Γ are defined as before.

We notice that little attention is paid to the Ostrowski-Grüss type inequalities involving multiple interior points with $f'(t)$ bounded by functions instead of constants so far in the literature. So, in this paper, motivated by the above works, we extend the Ostrowski-Grüss type inequalities to the case involving multiple interior points with the bounds of $f'(t)$ shown as $\gamma(t) \le f'(t) \le \Gamma(t)$, $t \in [a,b]$. Some bounds will be derived based on the inequalities, and some of the bounds are sharp. A new 2D Ostrowski-Grüss type inequality will also be derived with the bounds of $\frac{\partial^2 f(x,y)}{\partial x \partial y}$ shown as $\gamma(x,y) \le \frac{\partial^2 f(x,y)}{\partial x \partial y} \le \Gamma(x,y)$, $x \in [a,b]$, $y \in [c,d]$. We also present some applications for the Ostrowski-Grüss type inequalities established, in which new error bounds for some numerical integration formulae are derived.

2 Main results

Lemma 2.1 [16, Lemma 2.1] *Let $I \subset R$ be an open interval, $a, b \in I$, $a < b$. $f : I \to R$ is a differential function. Furthermore, suppose that $t_i \in [a,b]$, $i = 0,1,\dots,k$, $I_k : a = t_0 < t_1 < \cdots < t_{k-1} < t_k = b$ is a division of the interval $[a,b]$, and $\delta_i \in [t_{i-1},t_i]$, $i = 1,2,\dots,k$, $\delta_0 = a$, $\delta_{k+1} = b$. Then we have*

$$\sum_{i=0}^{k}(\delta_{i+1}-\delta_i)f(t_i) = \int_a^b f(t)\,dt + \int_a^b \zeta(t,I_k)f'(t)\,dt, \tag{3}$$

where

$$\zeta(t,I_k) = \begin{cases} t-\delta_1, & t \in [t_0,t_1),\\ t-\delta_2, & t\in[t_1,t_2),\\ \cdots & \\ t-\delta_{k-1}, & t \in [t_{k-2},t_{k-1}),\\ t-\delta_k, & t\in[t_{k-1},t_k]. \end{cases} \tag{4}$$

Lemma 2.2 *Let*

$$\varphi(t,I_k)=\begin{cases} t-\delta_1-C, & t\in[t_0,t_1),\\ t-\delta_2-C, & t\in[t_1,t_2),\\ \cdots \\ t-\delta_{k-1}-C, & t\in[t_{k-2},t_{k-1}),\\ t-\delta_k-C, & t\in[t_{k-1},t_k],\end{cases} \tag{5}$$

where $C=\frac{1}{2}(b+a)-\frac{1}{b-a}\sum_{i=0}^{k-1}\delta_{i+1}(t_{i+1}-t_i)$. *Then* $\int_a^b\varphi(t,I_k)\,dt=0$.

Proof Based on the definition of $\varphi(t,I_k)$, we have

$$\begin{aligned}\int_a^b\varphi(t,I_k)\,dt&=\sum_{i=0}^{k-1}\left[\frac{(t_{i+1}-\delta_{i+1}-C)^2}{2}-\frac{(t_i-\delta_{i+1}-C)^2}{2}\right]\\ &=\frac{1}{2}\sum_{i=0}^{k-1}\left[(t_{i+1}-t_i)(t_{i+1}+t_i-2\delta_{i+1}-2C)\right]\\ &=\frac{1}{2}\left\{\sum_{i=0}^{k-1}\left(t_{i+1}^2-t_i^2\right)-2\sum_{i=0}^{k-1}(t_{i+1}-t_i)\delta_{i+1}-2C\sum_{i=0}^{k-1}(t_{i+1}-t_i)\right\}\\ &=\frac{1}{2}\left(b^2-a^2\right)-\sum_{i=0}^{k-1}(t_{i+1}-t_i)\delta_{i+1}-C(b-a)=0.\end{aligned} \tag{6}$$

□

Theorem 2.1 *Under the conditions of Lemma* 2.1, *if there exist two functions* $\gamma(t)$, $\Gamma(t)$ *with* $\gamma(t)\le f'(t)\le\Gamma(t)$, $t\in[a,b]$, *then the following inequality holds*:

$$\begin{aligned}&\left|\sum_{i=0}^{k}(\delta_{i+1}-\delta_i)f(t_i)-\int_a^b f(t)\,dt-\int_a^b\zeta(t,I_k)\left[\frac{\gamma(t)+\Gamma(t)}{2}\right]dt\right|\\ &\quad\le\frac{1}{2}\|\Gamma-\gamma\|_\infty\left\{\frac{1}{2}\sum_{i=0}^{k-1}\left[t_{i+1}^2+t_i^2\right]-\sum_{i=0}^{k-1}\delta_{i+1}(t_{i+1}+t_i)+\sum_{i=0}^{k-1}\delta_{i+1}^2\right\}.\end{aligned} \tag{7}$$

Proof Consider

$$\begin{aligned}&\int_a^b\zeta(t,I_k)\left[f'(t)-\frac{\gamma(t)+\Gamma(t)}{2}\right]dt\\ &\quad=\int_a^b\zeta(t,I_k)f'(t)\,dt-\int_a^b\zeta(t,I_k)\left[\frac{\gamma(t)+\Gamma(t)}{2}\right]dt,\end{aligned} \tag{8}$$

$$\begin{aligned}\int_a^b|\zeta(t,I_k)|\,dt&=\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}|t-\delta_{i+1}|\,dt\\ &=\sum_{i=0}^{k-1}\left[\frac{(t_{i+1}-\delta_{i+1})^2}{2}+\frac{(t_i-\delta_{i+1})^2}{2}\right]\\ &=\frac{1}{2}\sum_{i=0}^{k-1}\left[t_{i+1}^2+t_i^2\right]-\sum_{i=0}^{k-1}\delta_{i+1}(t_{i+1}+t_i)+\sum_{i=0}^{k-1}\delta_{i+1}^2\end{aligned} \tag{9}$$

and

$$\left|\int_a^b \zeta(t,I_k)\left[f'(t)-\frac{\gamma(t)+\Gamma(t)}{2}\right]dt\right| \le \int_a^b \left|\zeta(t,I_k)\right|\left|\left[f'(t)-\frac{\gamma(t)+\Gamma(t)}{2}\right]\right|dt$$
$$\le \frac{1}{2}\|\Gamma-\gamma\|_\infty \int_a^b \left|\zeta(t,I_k)\right| dt. \tag{10}$$

Combining (8)-(10) and Lemma 2.1, we obtain the desired result. □

Corollary 2.1 *Under the conditions of Lemma* 2.1, *if* $W_1 \le f(t) \le W_2$, $t \in [a,b]$, *where* W_1, W_2 *are two constants, then we have the following inequality*:

$$\left|\sum_{i=0}^{k}(\delta_{i+1}-\delta_i)f(t_i)-\int_a^b f(t)\,dt-\left(\frac{W_2+W_1}{2}\right)\left\{\frac{1}{2}\left(b^2-a^2\right)-\sum_{i=0}^{k-1}\delta_{i+1}(t_{i+1}-t_i)\right\}\right|$$
$$\le \left(\frac{W_2-W_1}{2}\right)\left\{\frac{1}{2}\sum_{i=0}^{k-1}\left[t_{i+1}^2+t_i^2\right]-\sum_{i=0}^{k-1}\delta_{i+1}(t_{i+1}+t_i)+\sum_{i=0}^{k-1}\delta_{i+1}^2\right\}. \tag{11}$$

Proof In fact, in (7) we have $\gamma(t) = W_1$, $\Gamma(t) = W_2$. On the other hand,

$$\int_a^b \zeta(t,I_k)\,dt = \sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}(t-\delta_{i+1})\,dt = \sum_{i=0}^{k-1}\left[\frac{(t_{i+1}-\delta_{i+1})^2}{2}-\frac{(t_i-\delta_{i+1})^2}{2}\right]$$
$$= \frac{1}{2}\sum_{i=0}^{k-1}\left[t_{i+1}^2-t_i^2\right]-\sum_{i=0}^{k-1}\delta_{i+1}(t_{i+1}-t_i)$$
$$= \frac{1}{2}\left(b^2-a^2\right)-\sum_{i=0}^{k-1}\delta_{i+1}(t_{i+1}-t_i). \tag{12}$$

Then, combining (7) and (12), we get the desired inequality (11). □

Next we present a sharp Ostrowski-Grüss type inequality containing multiple interior points with $f'(t)$ bounded by functions as follows.

Theorem 2.2 *Under the conditions of Lemma* 2.1, *if there exist two functions* $\gamma(t)$, $\Gamma(t)$ *with* $\gamma(t) \le f'(t) \le \Gamma(t)$, $t \in [a,b]$, *then we have the following inequality*:

$$\left|\sum_{i=0}^{k}(\delta_{i+1}-\delta_i)f(t_i)-\int_a^b f(t)\,dt-\frac{f(b)-f(a)}{b-a}\left[\frac{b^2-a^2}{2}-\sum_{i=0}^{k-1}\delta_{i+1}(t_{i+1}-t_i)\right]\right.$$
$$\left.-\int_a^b \varphi(t,I_k)\left[\frac{\Gamma(t)+\gamma(t)}{2}\right]dt\right|$$
$$\le \frac{1}{2}\|\Gamma-\gamma\|_\infty \int_a^b \left|\varphi(t,I_k)\right| dt, \tag{13}$$

where $\varphi(t,I_k)$ *is defined as in Lemma* 2.2.

The inequality (13) *is sharp in the sense that the constant* $\frac{1}{2}$ *on the right-hand side cannot be replaced by a smaller one.*

Proof We have the following observations:

$$\begin{aligned}
&\int_a^b \varphi(t,I_k)\left[f'(t)-\frac{\Gamma(t)+\gamma(t)}{2}\right]dt \\
&\quad=\int_a^b \varphi(t,I_k)f'(t)\,dt-\int_a^b \varphi(t,I_k)\left[\frac{\Gamma(t)+\gamma(t)}{2}\right]dt \\
&\quad=\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}(t-\delta_{i+1}-C)f'(t)\,dt-\int_a^b \varphi(t,I_k)\left[\frac{\Gamma(t)+\gamma(t)}{2}\right]dt \\
&\quad=\sum_{i=0}^{k-1}\bigl[(t_{i+1}-\delta_{i+1}-C)f(t_{i+1})-(t_i-\delta_{i+1}-C)f(t_i)\bigr] \\
&\qquad-\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}f(t)\,dt-\int_a^b \varphi(t,I_k)\left[\frac{\Gamma(t)+\gamma(t)}{2}\right]dt \\
&\quad=\sum_{i=0}^{k}(\delta_{i+1}-\delta_i)f(t_i)-\int_a^b f(t)\,dt-\frac{f(b)-f(a)}{b-a}\left[\frac{b^2-a^2}{2}-\sum_{i=0}^{k-1}\delta_{i+1}(t_{i+1}-t_i)\right] \\
&\qquad-\int_a^b \varphi(t,I_k)\left[\frac{\Gamma(t)+\gamma(t)}{2}\right]dt,
\end{aligned} \tag{14}$$

and

$$\left|\int_a^b \varphi(t,I_k)\left[f'(t)-\frac{\Gamma(t)+\gamma(t)}{2}\right]dt\right|\le\frac{1}{2}\|\Gamma-\gamma\|_\infty\int_a^b\bigl|\varphi(t,I_k)\bigr|\,dt. \tag{15}$$

Combining (14) and (15), we obtain the desired inequality (13).

To prove the sharpness of (13), we take $k=2$, $\delta_0=a$, $\delta_1=\frac{b+5a}{6}$, $\delta_2=\frac{5b+a}{6}$, $\delta_3=b$, $t_1=\frac{b+a}{2}$, and

$$f(t)=\begin{cases}\sigma_1 t, & a\le t<\frac{b+5a}{6},\\ \sigma_2 t+\frac{b+5a}{6}\sigma_1-\frac{b+5a}{6}\sigma_2, & \frac{b+5a}{6}\le t<\frac{b+a}{2},\\ \sigma_1 t+\frac{b+a}{2}\sigma_2+\frac{b+5a}{6}\sigma_1-\frac{b+5a}{6}\sigma_2-\frac{b+a}{2}\sigma_1, & \frac{b+a}{2}\le t<\frac{5b+a}{6},\\ \sigma_2 t+\frac{5b+a}{6}\sigma_1+\frac{b+a}{2}\sigma_2+\frac{b+5a}{6}\sigma_1-\frac{b+5a}{6}\sigma_2 & \\ \quad-\frac{b+a}{2}\sigma_1-\frac{5b+a}{6}\sigma_2, & \frac{5b+a}{6}\le t\le b,\end{cases} \tag{16}$$

where σ_1, σ_2 are two constants with $\sigma_1<\sigma_2$. Then one can see

$$f'(t)=\begin{cases}\sigma_2, & t\in[\frac{b+5a}{6},\frac{b+a}{2})\cup[\frac{5b+a}{6},b],\\ \sigma_1, & t\in[a,\frac{b+5a}{6})\cup[\frac{b+a}{2},\frac{5b+a}{6}),\end{cases} \tag{17}$$

and

$$\varphi(t,I_k)=\begin{cases}t-\frac{b+5a}{6}, & t\in[a,\frac{b+a}{2}),\\ t-\frac{5b+a}{6}, & t\in[\frac{b+a}{2},b].\end{cases} \tag{18}$$

So, in fact, $\gamma(t) \equiv \sigma_1$, $\Gamma(t) \equiv \sigma_2$. According to the left-side hand of (15), we have

$$\left|\int_a^b \varphi(t,I_k)\left[f'(t) - \frac{\gamma(t)+\Gamma(t)}{2}\right]dt\right| = \left(\frac{\sigma_2-\sigma_1}{2}\right)\int_a^b \left|\varphi(t,I_k)\right|dt,$$

while according to the right-side hand of (15), we also have

$$\begin{aligned}
&\int_a^b \left|\varphi(t,I_k)\right|\left|\left[f'(t) - \frac{\gamma(t)+\Gamma(t)}{2}\right]\right|dt \\
&\quad= -\int_a^{\frac{b+5a}{6}} \varphi(t,I_k)\left|\frac{\sigma_1-\sigma_2}{2}\right|dt + \int_{\frac{b+5a}{6}}^{\frac{b+a}{2}} \varphi(t,I_k)\left|\frac{\sigma_2-\sigma_1}{2}\right|dt \\
&\qquad - \int_{\frac{b+a}{2}}^{\frac{5b+a}{6}} \varphi(t,I_k)\left|\frac{\sigma_1-\sigma_2}{2}\right|dt + \int_{\frac{5b+a}{6}}^{b} \varphi(t,I_k)\left|\frac{\sigma_1-\sigma_2}{2}\right|dt \\
&\quad= \left(\frac{\sigma_2-\sigma_1}{2}\right)\int_a^b \left|\varphi(t,I_k)\right|dt.
\end{aligned}$$

So, (15) holds in the equality form, which confirms the proof. □

Corollary 2.2 *Under the conditions of Lemma* 2.1, *if* $W_1 \le f(t) \le W_2$, $t \in [a,b]$, *where* W_1, W_2 *are two constants, then we have the following inequality*:

$$\begin{aligned}
&\left|\sum_{i=0}^{k}(\delta_{i+1}-\delta_i)f(t_i) - \int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\left[\frac{b^2-a^2}{2} - \sum_{i=0}^{k-1}\delta_{i+1}(t_{i+1}-t_i)\right]\right| \\
&\quad\le \left(\frac{W_2-W_1}{2}\right)\int_a^b \left|\varphi(t,I_k)\right|dt.
\end{aligned}\tag{19}$$

Remark 2.1 In Corollary 2.2, if we take $k = 1$, $\delta_0 = a$, $\delta_1 = \frac{a+b}{2}$, $\delta_2 = b$, $t_0 = a$, $t_1 = b$, then Corollary 2.2 becomes the trapezoid-type inequality, which provides error bound for the trapezoid quadrature formula in calculating $\int_a^b f(t)\,dt$.

$$\left|\frac{f(b)+f(a)}{2}(b-a) - \int_a^b f(t)\,dt\right| \le \left(\frac{W_2-W_1}{2}\right)\int_a^b \left|\varphi(t,I_k)\right|dt,$$

where $\varphi(t,I_k) = t - \frac{a+b}{2}$, $t \in [a,b]$.

If we take $k = 2$, $\delta_0 = a$, $\delta_1 = a$, $\delta_2 = b$, $\delta_3 = b$, $t_0 = a$, $t_1 = x$, $t_2 = b$, then Corollary 2.2 becomes the following Ostroski-Grüss type inequality as denoted in (1) [15, Th. 1.5]:

$$\begin{aligned}
\left|(b-a)f(x) - \int_a^b f(t)\,dt - \big[f(b)-f(a)\big]\left(x - \frac{a+b}{2}\right)\right| &\le \left(\frac{W_2-W_1}{2}\right)\int_a^b \left|\varphi(t,I_k)\right|dt \\
&= \left(\frac{W_2-W_1}{8}\right)(b-a)^2,
\end{aligned}$$

where

$$\varphi(t,I_k) = \begin{cases} t - a - (x - \frac{a+b}{2}), & t \in [a,x), \\ t - b - (x - \frac{a+b}{2}), & t \in [x,b]. \end{cases}$$

If we take $k = 2$, $\delta_0 = a$, $\delta_1 = \frac{5a+b}{6}$, $\delta_2 = \frac{a+5b}{6}$, $\delta_3 = b$, $t_0 = a$, $t_1 = \frac{a+b}{2}$, $t_2 = b$, then Corollary 2.2 becomes the following Simpson-type inequality, which provides better bound than the corresponding result in [21] (the constant on the right-side hand of the inequality is $\frac{1}{12}$ therein).

$$\left|\frac{1}{3}\left[\frac{f(b)+f(a)}{2} + 2f\left(\frac{(b+a)}{2}\right)\right](b-a) - \int_a^b f(t)\,dt\right| \leq \left(\frac{W_2 - W_1}{2}\right)\int_a^b \left|\varphi(t,I_k)\right|dt$$
$$= \frac{5(W_2 - W_1)}{72}(b-a)^2,$$

where

$$\varphi(t,I_k) = \begin{cases} t - \frac{5a+b}{6}, & t \in [a, \frac{a+b}{2}), \\ t - \frac{a+5b}{6}, & t \in [\frac{a+b}{2}, b]. \end{cases}$$

Remark 2.2 Under the conditions of Corollary 2.2, furthermore, assume that the conditions of [16, Th. 2.4] hold. Then, proceeding in the same manner as the proof in [16, Eqs. (13)-(14)], we obtain $\int_a^b |\varphi(t,I_k)|\,dt \leq \frac{(b-a)^2}{4}$. So, Corollary 2.2 provides better bound than the inequality in (2) [16, Th. 2.4].

In the following, we extend the result in Theorem 2.1 to 2D case, in which $\frac{\partial^2 f(x,y)}{\partial x\,\partial y}$ is bounded by two functions.

Theorem 2.3 *Let $f : [a,b] \times [c,d] \to R$ be an absolutely continuous function such that the second-order mixed partial derivative exists and there exist two functions $\gamma(x,y)$, $\Gamma(x,y)$ with $\gamma(x,y) \leq \frac{\partial^2 f(x,y)}{\partial x\,\partial y} \leq \Gamma(x,y)$, $x \in [a,b]$, $y \in [c,d]$. Suppose that $x_i \in [a,b]$, $y_i \in [c,d]$, $i = 0,1,\ldots,k$. $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ is a division of the interval $[a,b]$, while $J_k : a = y_0 < y_1 < \cdots < y_{k-1} < y_k = d$ is a division of the interval $[c,d]$. $\alpha_i \in [x_{i-1}, x_i]$, $\beta_i \in [y_{i-1}, y_i]$, $i = 1,2,\ldots,k$, $\alpha_0 = a$, $\alpha_{k+1} = b$, $\beta_0 = c$, $\beta_{k+1} = d$. Then*

$$\Bigg|\Bigg\{\sum_{i=1}^{k-1}\sum_{j=1}^{k-1}(\alpha_{i+1} - \alpha_i)(\beta_{j+1} - \beta_j)f(x_i,y_j) + \sum_{i=1}^{k-1}(\alpha_{i+1} - \alpha_i)(y_k - \beta_k)f(x_i,y_k)$$
$$- \sum_{i=1}^{k-1}(\alpha_{i+1} - \alpha_i)(y_0 - \beta_1)f(x_i,y_0) + \sum_{j=1}^{k-1}(x_k - \alpha_k)(\beta_{j+1} - \beta_j)f(x_k,y_j)$$
$$+ (x_k - \alpha_k)(y_k - \beta_k)f(x_k,y_k) - (x_k - \alpha_k)(y_0 - \beta_1)f(x_k,y_0)$$
$$- \sum_{j=1}^{k-1}(x_0 - \alpha_1)(\beta_{j+1} - \beta_j)f(x_0,y_j) - (x_0 - \alpha_1)(y_k - \beta_k)f(x_0,y_k)$$
$$+ (x_0 - \alpha_1)(y_0 - \beta_1)f(x_0,y_0) - \sum_{j=1}^{k-1}\int_a^b(\beta_{j+1} - \beta_j)f(x,y_j)\,dx$$
$$- \int_a^b\big[(y_k - \beta_k)f(x,y_k) - (y_0 - \beta_1)f(x,y_0)\big]\,dx - \sum_{i=1}^{k-1}\int_c^d(\alpha_{i+1} - \alpha_i)f(x_i,y)\,dy$$
$$- \int_c^d\big[(x_k - \alpha_k)f(x_k,y) - (x_0 - \alpha_1)f(x_0,y)\big]\,dy + \int_a^b\int_c^d f(x,y)\,dy\,dx\Bigg\}$$

$$
\begin{aligned}
&- \frac{[f(b,d)-f(b,c)-f(a,d)+f(a,c)]}{(b-a)(d-c)} \\
&\times \left[\frac{1}{2}\left(b^2-a^2\right)-\sum_{i=0}^{k-1}\alpha_{i+1}(x_{i+1}-x_i)\right]\left[\frac{1}{2}\left(d^2-c^2\right)-\sum_{j=0}^{k-1}\beta_{j+1}(y_{j+1}-y_j)\right] \\
&- \int_a^b\int_c^d \phi(x,y,I_k,J_k)\left[\frac{\gamma(x,y)+\Gamma(x,y)}{2}\right]dy\,dx\Bigg| \\
\le{}& \frac{1}{2}\|\Gamma-\gamma\|_\infty\left\{\frac{1}{2}\sum_{i=0}^{k-1}\left[x_{i+1}^2+x_i^2\right]-\sum_{i=0}^{k-1}\alpha_{i+1}(x_{i+1}+x_i)+\sum_{i=0}^{k-1}\alpha_{i+1}^2\right\} \\
&\times\left\{\frac{1}{2}\sum_{j=0}^{k-1}\left[y_{j+1}^2+y_j^2\right]-\sum_{j=0}^{k-1}\beta_{j+1}(y_{j+1}+y_j)+\sum_{j=0}^{k-1}\beta_{j+1}^2\right\},
\end{aligned}
\tag{20}
$$

where

$$
\phi(x,y,I_k,J_k)=(x-\alpha_{i+1})(y-\beta_{j+1}),\quad (x,y)\in[x_i,x_{i+1})\times[y_j,y_{j+1}), i,j=0,1,\ldots,k-1. \tag{21}
$$

Proof Similar to [16, Eq. (21)], we obtain that

$$
\begin{aligned}
&\int_a^b\int_c^d \phi(x,y,I_k,J_k)\frac{\partial^2 f(x,y)}{\partial x\,\partial y}\,dy\,dx \\
&= \sum_{i=1}^{k-1}\sum_{j=1}^{k-1}(\alpha_{i+1}-\alpha_i)(\beta_{j+1}-\beta_j)f(x_i,y_j)+\sum_{i=1}^{k-1}(\alpha_{i+1}-\alpha_i)(y_k-\beta_k)f(x_i,y_k) \\
&\quad - \sum_{i=1}^{k-1}(\alpha_{i+1}-\alpha_i)(y_0-\beta_1)f(x_i,y_0)+\sum_{j=1}^{k-1}(x_k-\alpha_k)(\beta_{j+1}-\beta_j)f(x_k,y_j) \\
&\quad + (x_k-\alpha_k)(y_k-\beta_k)f(x_k,y_k)-(x_k-\alpha_k)(y_0-\beta_1)f(x_k,y_0) \\
&\quad - \sum_{j=1}^{k-1}(x_0-\alpha_1)(\beta_{j+1}-\beta_j)f(x_0,y_j)-(x_0-\alpha_1)(y_k-\beta_k)f(x_0,y_k) \\
&\quad + (x_0-\alpha_1)(y_0-\beta_1)f(x_0,y_0)-\sum_{j=1}^{k-1}\int_a^b(\beta_{j+1}-\beta_j)f(x,y_j)\,dx \\
&\quad - \int_a^b\left[(y_k-\beta_k)f(x,y_k)-(y_0-\beta_1)f(x,y_0)\right]dx-\sum_{i=1}^{k-1}\int_c^d(\alpha_{i+1}-\alpha_i)f(x_i,y)\,dy \\
&\quad - \int_c^d\left[(x_k-\alpha_k)f(x_k,y)-(x_0-\alpha_1)f(x_0,y)\right]dy+\int_a^b\int_c^d f(x,y)\,dy\,dx.
\end{aligned}
\tag{22}
$$

On the other hand, we have

$$
\begin{aligned}
&\left|\int_a^b\int_c^d \phi(x,y,I_k,J_k)\left[\frac{\partial^2 f(x,y)}{\partial x\,\partial y}-\frac{\gamma(x,y)+\Gamma(x,y)}{2}\right]dy\,dx\right| \\
&\le \frac{1}{2}\int_a^b\int_c^d\left|\phi(x,y,I_k,J_k)\right|\left|\Gamma(x,y)-\gamma(x,y)\right|dy\,dx,
\end{aligned}
\tag{23}
$$

and

$$\begin{aligned}
&\int_a^b \int_c^d \left|\phi(x,y,I_k,J_k)\right| \,dy\,dx \\
&= \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} \left|(x-\alpha_{i+1})(t-\beta_{j+1})\right| \,dy\,dx \\
&= \left\{ \frac{1}{2}\sum_{i=0}^{k-1}\left[x_{i+1}^2 + x_i^2\right] - \sum_{i=0}^{k-1}\alpha_{i+1}(x_{i+1}+x_i) + \sum_{i=0}^{k-1}\alpha_{i+1}^2 \right\} \\
&\quad \times \left\{ \frac{1}{2}\sum_{j=0}^{k-1}\left[y_{j+1}^2 + y_j^2\right] - \sum_{j=0}^{k-1}\beta_{j+1}(y_{j+1}+y_j) + \sum_{j=0}^{k-1}\beta_{j+1}^2 \right\}. \qquad (24)
\end{aligned}$$

Combining (22)-(24), we get the desired inequality (20). □

Corollary 2.3 *Under the conditions of Theorem* 2.3, *if there exist two constants* W_1, W_2 *such that* $W_1 \le f(x,y) \le W_2$, *then we have the following inequality*:

$$\begin{aligned}
&\Bigg|\Bigg\{ \sum_{i=1}^{k-1}\sum_{j=1}^{k-1}(\alpha_{i+1}-\alpha_i)(\beta_{j+1}-\beta_j)f(x_i,y_j) + \sum_{i=1}^{k-1}(\alpha_{i+1}-\alpha_i)(y_k-\beta_k)f(x_i,y_k) \\
&\quad - \sum_{i=1}^{k-1}(\alpha_{i+1}-\alpha_i)(y_0-\beta_1)f(x_i,y_0) + \sum_{j=1}^{k-1}(x_k-\alpha_k)(\beta_{j+1}-\beta_j)f(x_k,y_j) \\
&\quad + (x_k-\alpha_k)(y_k-\beta_k)f(x_k,y_k) - (x_k-\alpha_k)(y_0-\beta_1)f(x_k,y_0) \\
&\quad - \sum_{j=1}^{k-1}(x_0-\alpha_1)(\beta_{j+1}-\beta_j)f(x_0,y_j) - (x_0-\alpha_1)(y_k-\beta_k)f(x_0,y_k) \\
&\quad + (x_0-\alpha_1)(y_0-\beta_1)f(x_0,y_0) - \sum_{j=1}^{k-1}\int_a^b (\beta_{j+1}-\beta_j)f(x,y_j)\,dx \\
&\quad - \int_a^b \left[(y_k-\beta_k)f(x,y_k) - (y_0-\beta_1)f(x,y_0)\right]dx - \sum_{i=1}^{k-1}\int_c^d (\alpha_{i+1}-\alpha_i)f(x_i,y)\,dy \\
&\quad - \int_c^d \left[(x_k-\alpha_k)f(x_k,y) - (x_0-\alpha_1)f(x_0,y)\right]dy + \int_a^b\int_c^d f(x,y)\,dy\,dx \Bigg\} \\
&\quad - \frac{[f(b,d)-f(b,c)-f(a,d)+f(a,c)]}{(b-a)(d-c)} \\
&\quad \times \left[\frac{1}{2}(b^2-a^2) - \sum_{i=0}^{k-1}\alpha_{i+1}(x_{i+1}-x_i)\right]\left[\frac{1}{2}(d^2-c^2) - \sum_{j=0}^{k-1}\beta_{j+1}(y_{j+1}-y_j)\right] \\
&\quad - \left(\frac{W_2+W_1}{2}\right)\left[\frac{1}{2}(b^2-a^2) - \sum_{i=0}^{k-1}\alpha_{i+1}(x_{i+1}-x_i)\right] \\
&\quad \times \left[\frac{1}{2}(d^2-c^2) - \sum_{j=0}^{k-1}\beta_{j+1}(y_{j+1}-y_j)\right]\Bigg|
\end{aligned}$$

$$\le \left(\frac{W_2 - W_1}{2}\right)\left\{\frac{1}{2}\sum_{i=0}^{k-1}\left[x_{i+1}^2 + x_i^2\right] - \sum_{i=0}^{k-1}\alpha_{i+1}(x_{i+1} + x_i) + \sum_{i=0}^{k-1}\alpha_{i+1}^2\right\}$$
$$\times \left\{\frac{1}{2}\sum_{j=0}^{k-1}\left[y_{j+1}^2 + y_j^2\right] - \sum_{j=0}^{k-1}\beta_{j+1}(y_{j+1} + y_j) + \sum_{j=0}^{k-1}\beta_{j+1}^2\right\}. \tag{25}$$

Proof In fact, in (20) we have $\gamma(x,y) = W_1$, $\Gamma(x,y) = W_2$. Furthermore, we have the following observation:

$$\int_a^b \int_c^d \phi(x,y,I_k,J_k)\,dy\,dx$$
$$= \sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}(x-\alpha_{i+1})(y-\beta_{j+1})\,dy\,dx$$
$$= \left[\frac{1}{2}\left(b^2 - a^2\right) - \sum_{i=0}^{k-1}\alpha_{i+1}(x_{i+1} - x_i)\right]\left[\frac{1}{2}\left(d^2 - c^2\right) - \sum_{j=0}^{k-1}\beta_{j+1}(y_{j+1} - y_j)\right]. \tag{26}$$

Combining (20) and (26), we obtain the desired result. □

Remark 2.3 In Corollary 2.3, if we take $k = 2$, $\alpha_0 = a$, $\alpha_1 = \frac{5a+b}{6}$, $\alpha_2 = \frac{a+5b}{6}$, $\alpha_3 = b$, $x_0 = a$, $x_1 = \frac{a+b}{2}$, $x_2 = b$, $\beta_0 = c$, $\beta_1 = \frac{5c+d}{6}$, $\beta_2 = \frac{c+5d}{6}$, $\beta_3 = d$, $y_0 = c$, $y_1 = \frac{c+d}{2}$, $y_2 = d$, then Corollary 2.3 becomes the 2D Simpson-type inequality.

3 Applications to numerical quadrature formulae

In this section, we present some applications of the results established above, and derive error bounds for some numerical quadrature formulae.

Example 1 Let $G(f,I_h)$ be a numerical quadrature formula for $\int_a^b f(t)\,dt$, and denote

$$E(t) = \left|\int_a^b f(t)\,dt - G(f,I_h)\right|. \tag{27}$$

Theorem 3.1 *For* (27), *if there exist two functions* $\gamma(t)$, $\Gamma(t)$ *with* $\gamma(t) \le f'(t) \le \Gamma(t)$, $t \in [a,b]$, *and*

$$G(f,I_h) = \sum_{i=0}^{n-1}\left\{\frac{1}{3}\left[\frac{f(T_{i+1}) + f(T_i)}{2} + 2f\left(\frac{(T_{i+1} + T_i)}{2}\right)\right]h_i - \int_{T_i}^{T_{i+1}}\varphi(t,I_{ik})\left[\frac{\Gamma(t) + \gamma(t)}{2}\right]dt\right\},$$

where $I_h : a = T_0 < T_1 < \cdots < T_{n-1} < T_n = b$ *is a division of* $[a,b]$, $h_i = T_{i+1} - T_i$, $i = 0,1,\ldots,n-1$, *and*

$$\varphi(t,I_{i,k}) = \begin{cases} t - \frac{5T_i + T_{i+1}}{6}, & t \in [T_i, \frac{T_i + T_{i+1}}{2}), \\ t - \frac{T_i + 5T_{i+1}}{6}, & t \in [\frac{T_i + T_{i+1}}{2}, T_{i+1}], \end{cases} \tag{28}$$

then we have the following estimate:

$$\left|E(t)\right| \le \frac{5}{72}\|\Gamma - \gamma\|_\infty \sum_{i=0}^{n-1} h_i^2. \tag{29}$$

Proof Applying Theorem 2.2 (with $[a,b]$ replaced by $[T_i, T_{i+1}]$, $k = 2$, $\delta_0 = T_i$, $\delta_1 = \frac{5T_i+T_{i+1}}{6}$, $\delta_2 = \frac{T_i+5T_{i+1}}{6}$, $\delta_3 = T_{i+1}$, $t_0 = T_i$, $t_1 = \frac{T_i+T_{i+1}}{2}$, $t_2 = T_{i+1}$, we obtain that

$$\begin{aligned} &\left|\frac{1}{3}\left[\frac{f(T_{i+1}) + f(T_i)}{2} + 2f\left(\frac{(T_{i+1} + T_i)}{2}\right)\right]h_i \right.\\ &\qquad \left. - \int_{T_i}^{T_{i+1}} f(t)\,dt - \int_{T_i}^{T_{i+1}} \varphi(t, I_{ik})\left[\frac{\Gamma(t) + \gamma(t)}{2}\right]dt\right| \\ &\quad \le \frac{1}{2}\|\Gamma - \gamma\|_\infty \int_{T_i}^{T_{i+1}} \left|\varphi(t, I_{i,k})\right| dt, \end{aligned} \tag{30}$$

where

$$\varphi(t, I_{i,k}) = \begin{cases} t - \frac{5T_i+T_{i+1}}{6} - C_i, & t \in [T_i, \frac{T_i+T_{i+1}}{2}), \\ t - \frac{T_i+5T_{i+1}}{6} - C_i, & t \in [\frac{T_i+T_{i+1}}{2}, T_{i+1}]. \end{cases} \tag{31}$$

Furthermore,

$$C_i = \frac{1}{2}(T_i + T_{i+1}) - \frac{1}{T_{i+1} - T_i}\sum_{i=0}^{1} \delta_{i+1}(t_{i+1} - t_i) = 0, \tag{32}$$

and

$$\int_{T_i}^{T_{i+1}} \left|\varphi(t, I_{i,k})\right| dt = \frac{5}{36}h_i^2. \tag{33}$$

Combining (30), (32) and (33) and a summation with respect to i from 0 to $n-1$ yield (29). □

Corollary 3.1 *Under the conditions of Theorem* 3.1, *if there exist two constants* W_1, W_2 *with* $W_1 \le f'(t) \le W_2$, $t \in [a,b]$, $h_i \equiv h$, $i = 0, 1, \ldots, n-1$, *and* $G(f, I_h)$ *denote the Simpson complex formula*, *that is*,

$$G(f, I_h) = \sum_{i=0}^{n-1}\left\{\frac{1}{3}\left[\frac{f(T_{i+1}) + f(T_i)}{2} + 2f\left(\frac{(T_{i+1} + T_i)}{2}\right)\right]h_i\right\},$$

then we have

$$\left|E(t)\right| \le \frac{5}{72}(W_2 - W_1)(b-a)h. \tag{34}$$

Proof Considering $\int_{T_i}^{T_{i+1}} \varphi(t, I_{i,k})\,dt = 0$ and $b - a = nh$, from (29) we can obtain (33). □

Example 2 Let $G(f,I_h,J_h)$ be a numerical quadrature formula for $\int_a^b\int_c^d f(x,y)\,dx\,dy$, and denote

$$E(x,y)=\left|\int_a^b\int_c^d f(x,y)\,dx\,dy-G(f,I_h,J_h)\right|. \tag{35}$$

Theorem 3.2 *For* (35), *if there exist two functions* $\gamma(x,y)$, $\Gamma(x,y)$ *with* $\gamma(x,y)\le\frac{\partial^2 f(x,y)}{\partial x\,\partial y}\le\Gamma(x,y)$, $x\in[a,b]$, $y\in[c,d]$, *and*

$$\begin{aligned}
G(f,I_h,J_h)={}&\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\biggl\{-\frac{(x_{i+1}-x_i)(y_{j+1}-y_j)}{9}\biggl[f\left(x_i,\frac{y_j+y_{j+1}}{2}\right)+f\left(x_{i+1},\frac{y_j+y_{j+1}}{2}\right)\\
&+f\left(\frac{x_i+x_{i+1}}{2},y_j\right)+f\left(\frac{x_i+x_{i+1}}{2},y_{j+1}\right)+4f\left(\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2}\right)\biggr]\\
&-\frac{(x_{i+1}-x_i)(y_{j+1}-y_j)}{36}\bigl[f(x_i,y_j)+f(x_i,y_{j+1})+f(x_{i+1},y_j)+f(x_{i+1},y_{j+1})\bigr]\\
&+\frac{(y_{j+1}-y_j)}{6}\int_{x_i}^{x_{i+1}}\left[f(x,y_j)+f\left(x,\frac{y_j+y_{j+1}}{2}\right)+f(x,y_{j+1})\right]dx\\
&+\frac{(x_{i+1}-x_i)}{6}\int_{y_j}^{y_{j+1}}\left[f(x_i,y)+f\left(\frac{x_i+x_{i+1}}{2},y\right)+f(x_{i+1},y)\right]dy\\
&+\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}\phi(s,t,I_{i,h},J_{j,h})\left[\frac{\gamma(x,y)+\Gamma(x,y)}{2}\right]dy\,dx\biggr\},
\end{aligned}$$

where $I_h: a=X_0<X_1<\cdots<X_{n-1}<X_n=b$ *is a division of the interval* $[a,b]$, *while* $J_h: a=Y_0<Y_1<\cdots<Y_{n-1}<Y_n=d$ *is a division of the interval* $[c,d]$, $h_{1,i}=x_{i+1}-x_i$, $h_{2,j}=y_{j+1}-y_j$, $i,j=0,1,\ldots,n-1$, *and*

$$\phi(x,y,I_{i,h},J_{j,h})=\begin{cases}
(x-\frac{5X_i+X_{i+1}}{6})(y-\frac{5Y_j+Y_{j+1}}{6}),\\
\quad x\in[X_i,\frac{X_i+X_{i+1}}{2}),y\in[Y_j,\frac{Y_j+Y_{j+1}}{2}),\\
(x-\frac{5X_i+X_{i+1}}{6})(y-\frac{Y_j+5Y_{j+1}}{6}),\\
\quad x\in[X_i,\frac{X_i+X_{i+1}}{2}),y\in[\frac{Y_j+Y_{j+1}}{2},Y_{j+1}],\\
(x-\frac{X_i+5X_{i+1}}{6})(y-\frac{5Y_j+Y_{j+1}}{6}),\\
\quad x\in[\frac{X_i+X_{i+1}}{2},X_{i+1}],y\in[Y_j,\frac{Y_j+Y_{j+1}}{2}),\\
(x-\frac{X_i+5X_{i+1}}{6})(y-\frac{Y_j+5Y_{j+1}}{6}),\\
\quad x\in[\frac{X_i+X_{i+1}}{2},X_{i+1}],y\in[\frac{Y_j+Y_{j+1}}{2},Y_{j+1}],
\end{cases} \tag{36}$$

then we have the following estimate:

$$\bigl|E(x,y)\bigr|\le\frac{25}{2592}\|\Gamma-\gamma\|_\infty\left(\sum_{i=0}^{n-1}h_{1,i}^2\right)\left(\sum_{i=0}^{n-1}h_{2,i}^2\right). \tag{37}$$

The desired result can be obtained by a suitable application of Theorem 2.3 (with $[a,b]$ replaced by $[x_i,x_{i+1}]$, $[c,d]$ replaced by $[y_j,y_{j+1}]$, $k=2$, $\alpha_0=x_i$, $\alpha_1=\frac{5x_i+x_{i+1}}{6}$, $\alpha_2=\frac{x_i+5x_{i+1}}{6}$, $\alpha_3=x_{i+1}$, $x_0=x_i$, $x_1=\frac{x_i+x_{i+1}}{2}$, $x_2=x_{i+1}$, $\beta_0=y_j$, $\beta_1=\frac{5y_j+y_{j+1}}{6}$, $\beta_2=\frac{y_j+5y_{j+1}}{6}$, $\beta_3=y_{j+1}$, $y_0=y_j$, $y_1=\frac{y_j+y_{j+1}}{2}$, $y_2=y_{j+1}$), and a summation with respect to i,j from 0 to $n-1$.

In Theorem 3.2, after a simple computation, one can see $\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}\phi(x,y,I_{i,h},J_{j,h})\,dy\,dx = 0$. So, we obtain the following corollary.

Corollary 3.2 *Under the conditions of Theorem* 3.2, *if there exist two constants* W_1, W_2 *with* $W_1 \le \frac{\partial^2 f(x,y)}{\partial x\,\partial y} \le W_2$, $x \in [a,b]$, $y \in [c,d]$, $h_{1,i} = h_{2,j} \equiv h$, $i,j = 0,1,\ldots,n-1$, *and* $G(f,I_h,J_h)$ *denote the* 2*D Simpson complex formula*, *that is*,

$$\begin{aligned} G(f,I_h,J_h) = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\Biggl\{ &-\frac{(x_{i+1}-x_i)(y_{j+1}-y_j)}{9}\biggl[f\left(x_i,\frac{y_j+y_{j+1}}{2}\right)+f\left(x_{i+1},\frac{y_j+y_{j+1}}{2}\right) \\ &+f\left(\frac{x_i+x_{i+1}}{2},y_j\right)+f\left(\frac{x_i+x_{i+1}}{2},y_{j+1}\right)+4f\left(\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2}\right)\biggr] \\ &-\frac{(x_{i+1}-x_i)(y_{j+1}-y_j)}{36}\bigl[f(x_i,y_j)+f(x_i,y_{j+1})+f(x_{i+1},y_j)+f(x_{i+1},y_{j+1})\bigr] \\ &+\frac{(y_{j+1}-y_j)}{6}\int_{x_i}^{x_{i+1}}\left[f(x,y_j)+f\left(x,\frac{y_j+y_{j+1}}{2}\right)+f(x,y_{j+1})\right]dx \\ &+\frac{(x_{i+1}-x_i)}{6}\int_{y_j}^{y_{j+1}}\left[f(x_i,y)+f\left(\frac{x_i+x_{i+1}}{2},y\right)+f(x_{i+1},y)\right]dy\Biggr\}, \end{aligned}$$

then we have the following estimate:

$$\bigl|E(x,y)\bigr| \le \frac{25}{2592}(W_2-W_1)(b-a)(d-c)h^2. \tag{38}$$

4 Conclusions

We have established some new Ostrowski-Grüss type inequalities involving multiple interior points with the derivatives bounded by functions in both 1D and 2D cases and derived some sharp bounds related to them. As one can see, the inequalities are of new forms and provide better bounds than some existing results in the literature. As for applications, new error bounds for some numerical quadrature formulae are derived based on the inequalities established.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

QF carried out the main part of this article. All authors read and approved the final manuscript.

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