Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Riemann-Liouville fractional Hermite-Hadamard inequalities. Part I: for once differentiable geometric-arithmetically *s*-convex functions

YuMei Liao[1,2], JianHua Deng[2] and JinRong Wang[1,2*]

*Correspondence:
sci.jrwang@gzu.edu.cn;
wangjinrong@gznc.edu.cn
[1]School of Mathematics and Computer Science, Guizhou Normal College, Guiyang, Guizhou 550018, P.R. China
[2]Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, P.R. China

Abstract

By using the definition of geometric-arithmetically *s*-convex functions in (Analysis 33:197-208, 2013) and first-order fractional integral identities in (Math. Comput. Model. 57:2403-2407, 2013; J. Appl. Math. Stat. Inform. 8:21-28, 2012; Comput. Math. Appl. 63:1147-1154, 2012), we present some interesting Riemann-Liouville fractional Hermite-Hadamard inequalities for differentiable geometric-arithmetically *s*-convex functions. Both beta function and incomplete beta function are used in the desired estimations.

MSC: 26A33; 26A51; 26D15

Keywords: fractional Hermite-Hadamard inequalities; Riemann-Liouville fractional integrals; geometric-arithmetically *s*-convex functions

1 Introduction

Fractional integrals and derivatives arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of a complex medium. In particular, the subject of fractional calculus has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics [1–8].

Fractional Hermite-Hadamard inequalities involving all kinds of fractional integrals have attracted by many researchers. Many authors paid attention to the study of fractional Hermite-Hadamard inequalities according to the first-order integral equalities and convex functions of different classes. For instance, the readers can refer to [9–11] for convex functions and [12] for nondecreasing functions, [13–15] for m-convex functions and [16] for (s,m)-convex functions, [17, 18] for functions satisfying s-e-condition, [19] for (α,m)-logarithmically convex functions and the references therein.

Very recently, the authors [20] introduced the new concept of geometric-arithmetically s-convex functions and established some interesting Hermite-Hadamard type inequalities for integer integrals of such functions. However, to our knowledge, fractional Hermite-Hadamard inequalities for geometric-arithmetically s-convex functions have not been reported. Motivated by [9, 10, 13, 20], we study Riemann-Liouville fractional Hermite-

Hadamard type inequalities for geometric-arithmetically s-convex functions by means of first-order fractional integral equalities.

2 Preliminaries

In this section, we introduce notations, definitions and preliminary facts.

Definition 2.1 (see [3]) Let $f \in L[a,b]$. The symbols $J_{a^+}^{\alpha}f$ and $J_{b^-}^{\alpha}f$ denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha \in R^+$ and are defined by

$$\big(J_{a^+}^{\alpha}f\big)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,dt \quad (0 \le a < x \le b)$$

and

$$\big(J_{b^-}^{\alpha}f\big)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\,dt \quad (0 \le a \le x < b),$$

respectively, here $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 (see [20]) Let $f : I \subseteq R^+ \to R^+$ and $s \in (0,1]$. A function $f(x)$ is said to be geometric-arithmetically s-convex on I if for every $x, y \in I$ and $t \in [0,1]$, we have

$$f\big(x^t y^{1-t}\big) \le t^s\big(f(x)\big) + (1-t)^s f(y).$$

Definition 2.3 (see [21]) The incomplete beta function is defined as follows:

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt,$$

where $x \in [0,1]$, $a, b > 0$.

The following inequality will be used in the sequel.

Lemma 2.4 (see [19]) *For* $t \in [0,1]$, *we have*

$$(1-t)^n \le 2^{1-n} - t^n \quad \textit{for } n \in [0,1],$$
$$(1-t)^n \ge 2^{1-n} - t^n \quad \textit{for } n \in [1,\infty).$$

The following elementary inequality was used in the proof directly in [20]. Here, we revisit this inequality from the point of our view and give a proof.

Lemma 2.5 *For* $t \in [0,1]$, $x, y > 0$, *we have*

$$tx + (1-t)y \ge y^{1-t}x^t.$$

Proof If $x \ge y$, then we consider the function $f(z) = tz + 1 - t - z^t$, $z \ge 1$. If $y \ge x$, then we consider the function $g(z) = t + (1-t)z - z^{1-t}$, $z \ge 1$. Clearly $f(\cdot)$ and $g(\cdot)$ are increasing functions for all $z \geq 1$, then $f(z) \geq f(1) = 0$ and $g(z) \geq g(1) = 0$. So, $f(\frac{x}{y}) \geq 0$ and $g(\frac{y}{x}) \geq 0$, *i.e.*, the statement holds. □

We collect the following first-order fractional integrals identities.

Lemma 2.6 ([9, Lemma 2]) *Let $f : [a,b] \to R$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L[a,b]$, then the following equality for fractional integrals holds*:

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\bigl[J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)\bigr] \\ &\quad = \frac{b-a}{2}\int_0^1 \bigl[(1-t)^\alpha - t^\alpha\bigr] f'\bigl(ta+(1-t)b\bigr)\,dt. \end{aligned}$$

Lemma 2.7 ([10, Lemma 2.1]) *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b). If $f' \in L[a,b]$, then the following equality for fractional integrals holds*:

$$\begin{aligned} &\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\bigl[J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)\bigr] - f\left(\frac{a+b}{2}\right) \\ &\quad = \frac{b-a}{2}\int_0^1 \bigl[h(t) - (1-t)^\alpha + t^\alpha\bigr] f'\bigl(ta+(1-t)b\bigr)\,dt, \end{aligned}$$

where

$$h(t) = \begin{cases} 1, & 0 < t < \frac{1}{2}, \\ -1, & \frac{1}{2} < t < 1. \end{cases}$$

Lemma 2.8 ([13, Lemma 2]) *Let $f : [a,b] \to R$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L[a,b]$, then the following equality for fractional integrals holds*:

$$\begin{aligned} &\frac{(x-a)^\alpha + (b-x)^\alpha}{b-a} f(x) - \frac{\Gamma(\alpha+1)}{b-a}\bigl[J^\alpha_{x-}f(a) + J^\alpha_{x+}f(b)\bigr] \\ &\quad = \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^\alpha f'\bigl(tx+(1-t)a\bigr)\,dt \\ &\qquad - \frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^\alpha f'\bigl(tx+(1-t)b\bigr)\,dt. \end{aligned}$$

3 Main results

In this section, we use Lemmas 2.6, 2.7 and 2.8 via geometric-arithmetically s-convex functions to derive the main results in this paper.

3.1 The first results

By using Lemma 2.6, we can obtain the main results in this section.

Theorem 3.1 *Let $f : [0,b] \to R$ be a differentiable mapping. If $|f'|$ is measurable and $|f'|$ is decreasing and geometric-arithmetically s-convex on $[0,b]$ for some fixed $\alpha \in (0,\infty)$,*

$s \in (0,1]$, $0 \le a < b$, then the following inequality for fractional integrals holds:

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right| \\
&\quad \le \frac{(b-a)(2^{\alpha+s}|f'(b)| - |f'(a)| - |f'(b)|)}{(\alpha+s+1)2^{\alpha+s+1}} \\
&\qquad + (b-a)\left|f'(a)\right|\left[0.5B(s+1,\alpha+1) - B_{0.5}(\alpha+1,s+1)\right] \\
&\qquad + (b-a)\left|f'(b)\right|\left[B_{0.5}(s+1,\alpha+1) - 0.5B(s+1,\alpha+1)\right],
\end{aligned}$$

where

$$B(s+1,\alpha+1) = \int_0^1 t^s(1-t)^\alpha\,dt,$$

and

$$B_{0.5}(s+1,\alpha+1) = \int_0^{0.5} t^s(1-t)^\alpha\,dt.$$

Proof By using Definition 2.2, Definition 2.3, Lemma 2.5 and Lemma 2.6, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right| \\
&\quad \le \frac{b-a}{2}\int_0^1 \left|(1-t)^\alpha - t^\alpha\right|\left|f'\left(ta+(1-t)b\right)\right|dt \\
&\quad \le \frac{b-a}{2}\int_{\frac{1}{2}}^1 \left(t^\alpha - (1-t)^\alpha\right)\left|f'\left(ta+(1-t)b\right)\right|dt \\
&\qquad + \frac{b-a}{2}\int_0^{\frac{1}{2}} \left((1-t)^\alpha - t^\alpha\right)\left|f'\left(ta+(1-t)b\right)\right|dt \\
&\quad \le \frac{b-a}{2}\int_{\frac{1}{2}}^1 \left(t^\alpha - (1-t)^\alpha\right)\left|f'\left(a^t b^{1-t}\right)\right|dt \\
&\qquad + \frac{b-a}{2}\int_0^{\frac{1}{2}} \left((1-t)^\alpha - t^\alpha\right)\left|f'\left(a^t b^{1-t}\right)\right|dt \\
&\quad \le \frac{b-a}{2}\int_{\frac{1}{2}}^1 \left(t^\alpha - (1-t)^\alpha\right)\left[t^s\left|f'(a)\right| + (1-t)^s\left|f'(b)\right|\right]dt \\
&\qquad + \frac{b-a}{2}\int_0^{\frac{1}{2}} \left((1-t)^\alpha - t^\alpha\right)\left[t^s\left|f'(a)\right| + (1-t)^s\left|f'(b)\right|\right]dt \\
&\quad \le \frac{(b-a)|f'(a)|}{2}\int_{\frac{1}{2}}^1 t^{\alpha+s}\,dt - \frac{(b-a)|f'(a)|}{2}\int_{\frac{1}{2}}^1 t^s(1-t)^\alpha\,dt \\
&\qquad - \frac{(b-a)|f'(b)|}{2}\int_{\frac{1}{2}}^1 (1-t)^{\alpha+s}\,dt + \frac{(b-a)|f'(b)|}{2}\int_{\frac{1}{2}}^1 t^\alpha(1-t)^s\,dt \\
&\qquad - \frac{(b-a)|f'(a)|}{2}\int_0^{\frac{1}{2}} t^{\alpha+s}\,dt + \frac{(b-a)|f'(a)|}{2}\int_0^{\frac{1}{2}} t^s(1-t)^\alpha\,dt
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(b-a)|f'(b)|}{2}\int_0^{\frac{1}{2}}(1-t)^{\alpha+s}\,dt-\frac{(b-a)|f'(b)|}{2}\int_0^{\frac{1}{2}}t^{\alpha}(1-t)^{s}\,dt\\
&\le (b-a)\big|f'(a)\big|\int_{\frac{1}{2}}^{1}t^{\alpha+s}\,dt-(b-a)\big|f'(a)\big|\int_{\frac{1}{2}}^{1}t^{s}(1-t)^{\alpha}\,dt\\
&\quad-(b-a)\big|f'(b)\big|\int_{\frac{1}{2}}^{1}(1-t)^{\alpha+s}\,dt+(b-a)\big|f'(b)\big|\int_{\frac{1}{2}}^{1}t^{\alpha}(1-t)^{s}\,dt\\
&\quad-(b-a)\big|f'(a)\big|\int_0^1 t^{\alpha+s}\,dt+\frac{(b-a)|f'(a)|}{2}\int_0^1 t^{s}(1-t)^{\alpha}\,dt\\
&\quad+\frac{(b-a)|f'(b)|}{2}\int_0^1(1-t)^{\alpha+s}\,dt-\frac{(b-a)|f'(b)|}{2}\int_0^1 t^{\alpha}(1-t)^{s}\,dt\\
&\le -(b-a)\big|f'(a)\big|\frac{1}{(\alpha+s+1)2^{\alpha+s+1}}\\
&\quad+(b-a)\big|f'(a)\big|\frac{1}{\alpha+s+1}-(b-a)\big|f'(a)\big|B_{0.5}(\alpha+1,s+1)\\
&\quad-(b-a)\big|f'(b)\big|\frac{1}{(\alpha+s+1)2^{\alpha+s+1}}+(b-a)\big|f'(b)\big|B_{0.5}(s+1,\alpha+1)\\
&\quad-(b-a)\big|f'(a)\big|\frac{1}{\alpha+s+1}+\frac{(b-a)|f'(a)|}{2}B(s+1,\alpha+1)\\
&\quad+\frac{(b-a)|f'(b)|}{2}\frac{1}{\alpha+s+1}-\frac{(b-a)|f'(b)|}{2}B(s+1,\alpha+1)\\
&\le\frac{(b-a)(2^{\alpha+s}|f'(b)|-|f'(a)|-|f'(b)|)}{(\alpha+s+1)2^{\alpha+s+1}}\\
&\quad+(b-a)\big|f'(a)\big|\big[0.5B(s+1,\alpha+1)-B_{0.5}(\alpha+1,s+1)\big]\\
&\quad+(b-a)\big|f'(b)\big|\big[B_{0.5}(s+1,\alpha+1)-0.5B(s+1,\alpha+1)\big].
\end{aligned}
$$

The proof is done. □

Theorem 3.2 *Let $f:[0,b]\to R$ be a differentiable mapping and $1<q<\infty$. If $|f'|^q$ is measurable and $|f'|^q$ is decreasing and geometric-arithmetically s-convex on $[0,b]$ for some fixed $\alpha\in(0,\infty)$, $s\in(0,1]$, $0\le a<b$, then the following inequality for fractional integrals holds*:

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\big[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\big]\right|\\
&\quad\le\frac{b-a}{2}\left(\frac{|f'(a)|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\left(\frac{2-2^{1-p\alpha}}{p\alpha+1}\right)^{\frac{1}{p}},
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof By using Definition 2.2, Lemma 2.5, Hölder's inequality and Lemma 2.6, we have

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\big[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\big]\right|\\
&\quad\le\frac{b-a}{2}\int_0^1\big|(1-t)^{\alpha}-t^{\alpha}\big|\big|f'\big(ta+(1-t)b\big)\big|\,dt
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{b-a}{2}\left(\int_0^1 \left|(1-t)^\alpha - t^\alpha\right|^p dt\right)^{\frac{1}{p}} \left(\int_0^1 \left|f'\big(ta+(1-t)b\big)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{b-a}{2}\left(\int_0^1 \left|(1-t)^\alpha - t^\alpha\right|^p dt\right)^{\frac{1}{p}} \left(\int_0^1 \left|f'\big(a^t b^{1-t}\big)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{b-a}{2}\left(\int_0^1 \left|(1-t)^\alpha - t^\alpha\right|^p dt\right)^{\frac{1}{p}} \left(\int_0^1 \left[t^s\left|f'(a)\right|^q + (1-t)^s\left|f'(b)\right|^q\right] dt\right)^{\frac{1}{q}} \\
&\leq \frac{b-a}{2}\left(\frac{|f'(a)|^q + |f'(b)|^q}{s+1}\right)^{\frac{1}{q}} \left(\int_0^1 \left|(1-t)^\alpha - t^\alpha\right|^p dt\right)^{\frac{1}{p}} \\
&\leq \frac{b-a}{2}\left(\frac{|f'(a)|^q + |f'(b)|^q}{s+1}\right)^{\frac{1}{q}} \left(\int_{\frac{1}{2}}^1 \big(t^\alpha - (1-t)^\alpha\big)^p dt + \int_0^{\frac{1}{2}} \big((1-t)^\alpha - t^\alpha\big)^p dt\right)^{\frac{1}{p}} \\
&\leq \frac{b-a}{2}\left(\frac{|f'(a)|^q + |f'(b)|^q}{s+1}\right)^{\frac{1}{q}} \left(\int_{\frac{1}{2}}^1 \big(t^{p\alpha} - (1-t)^{p\alpha}\big) dt + \int_0^{\frac{1}{2}} \big((1-t)^{p\alpha} - t^{p\alpha}\big) dt\right)^{\frac{1}{p}} \\
&\leq \frac{b-a}{2}\left(\frac{|f'(a)|^q + |f'(b)|^q}{s+1}\right)^{\frac{1}{q}} \left(\frac{2-2^{1-p\alpha}}{p\alpha+1}\right)^{\frac{1}{p}}.
\end{aligned}
$$

The proof is done. □

3.2 The second results

By using Lemma 2.7, we can obtain the main results in this section.

Theorem 3.3 *Let $f:[0,b]\to R$ be a differentiable mapping. If $|f'|$ is measurable and $|f'|$ is decreasing and geometric-arithmetically s-convex on $[0,b]$ for some fixed $\alpha\in(0,\infty)$, $s\in(0,1]$, $0\leq a<b$, then the following inequality for fractional integrals holds*:

$$
\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] - f\left(\frac{a+b}{2}\right) \\
&\quad\leq 0.5(b-a)\big(\left|f'(a)\right| + \left|f'(b)\right|\big) \\
&\qquad\times\left(B_{0.5}(\alpha+1,s+1) - B_{0.5}(s+1,\alpha+1) + \frac{2^{-\alpha-1}}{\alpha+s+1} + \frac{1}{s+1}\right).
\end{aligned}
$$

Proof By using Definition 2.2, Lemma 2.5 and Lemma 2.7, we have

$$
\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] - f\left(\frac{a+b}{2}\right)\right| \\
&\quad\leq \frac{b-a}{2}\int_0^1 \left|h(t) - (1-t)^\alpha + t^\alpha\right|\left|f'\big(ta+(1-t)b\big)\right| dt \\
&\quad\leq \frac{b-a}{2}\int_{\frac{1}{2}}^1 \left|-1 - (1-t)^\alpha + t^\alpha\right|\left|f'\big(ta+(1-t)b\big)\right| dt \\
&\qquad+ \frac{b-a}{2}\int_0^{\frac{1}{2}} \left|1 - (1-t)^\alpha + t^\alpha\right|\left|f'\big(ta+(1-t)b\big)\right| dt \\
&\quad\leq \frac{b-a}{2}\int_{\frac{1}{2}}^1 \big(1 + (1-t)^\alpha - t^\alpha\big)\left|f'\big(ta+(1-t)b\big)\right| dt
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{b-a}{2}\int_0^{\frac{1}{2}}\bigl(1-(1-t)^\alpha+t^\alpha\bigr)\bigl|f'\bigl(ta+(1-t)b\bigr)\bigr|\,dt\\
&\le\frac{b-a}{2}\int_{\frac{1}{2}}^1\bigl(1+(1-t)^\alpha-t^\alpha\bigr)\bigl|f'\bigl(a^tb^{1-t}\bigr)\bigr|\,dt\\
&+\frac{b-a}{2}\int_0^{\frac{1}{2}}\bigl(1-(1-t)^\alpha+t^\alpha\bigr)\bigl|f'\bigl(a^tb^{1-t}\bigr)\bigr|\,dt\\
&\le\frac{b-a}{2}\int_{\frac{1}{2}}^1\bigl(1+(1-t)^\alpha-t^\alpha\bigr)\bigl[t^s\bigl|f'(a)\bigr|+(1-t)^s\bigl|f'(b)\bigr|\bigr]\,dt\\
&+\frac{b-a}{2}\int_0^{\frac{1}{2}}\bigl(1-(1-t)^\alpha+t^\alpha\bigr)\bigl[t^s\bigl|f'(a)\bigr|+(1-t)^s\bigl|f'(b)\bigr|\bigr]\,dt\\
&\le-\frac{(b-a)|f'(a)|}{2}\int_{\frac{1}{2}}^1 t^{\alpha+s}\,dt+\frac{(b-a)|f'(a)|}{2}\int_{\frac{1}{2}}^1 t^s\,dt+\frac{(b-a)|f'(a)|}{2}\int_{\frac{1}{2}}^1 t^s(1-t)^\alpha\,dt\\
&+\frac{(b-a)|f'(b)|}{2}\int_{\frac{1}{2}}^1(1-t)^{\alpha+s}\,dt-\frac{(b-a)|f'(b)|}{2}\int_{\frac{1}{2}}^1 t^\alpha(1-t)^s\,dt\\
&+\frac{(b-a)|f'(b)|}{2}\int_{\frac{1}{2}}^1(1-t)^s\,dt+\frac{(b-a)|f'(a)|}{2}\int_0^{\frac{1}{2}} t^{\alpha+s}\,dt\\
&-\frac{(b-a)|f'(a)|}{2}\int_0^{\frac{1}{2}} t^s(1-t)^\alpha\,dt+\frac{(b-a)|f'(a)|}{2}\int_0^{\frac{1}{2}} t^s\,dt\\
&-\frac{(b-a)|f'(b)|}{2}\int_0^{\frac{1}{2}}(1-t)^{\alpha+s}\,dt+\frac{(b-a)|f'(b)|}{2}\int_0^{\frac{1}{2}} t^\alpha(1-t)^s\,dt\\
&+\frac{(b-a)|f'(b)|}{2}\int_0^{\frac{1}{2}}(1-t)^s\,dt\\
&\le-\frac{(b-a)|f'(a)|}{2}\frac{1-2^{-\alpha-s-1}}{\alpha+s+1}+\frac{(b-a)|f'(a)|}{2}\frac{1-2^{-s-1}}{s+1}\\
&+\frac{(b-a)|f'(a)|}{2}B_{0.5}(\alpha+1,s+1)+\frac{(b-a)|f'(b)|}{2}\frac{2^{-\alpha-s-1}}{\alpha+s+1}\\
&-\frac{(b-a)|f'(b)|}{2}B_{0.5}(s+1,\alpha+1)+\frac{(b-a)|f'(b)|}{2}\frac{2^{-s-1}}{s+1}\\
&+\frac{(b-a)|f'(a)|}{2}\frac{2^{-\alpha-s-1}}{\alpha+s+1}\\
&-\frac{(b-a)|f'(a)|}{2}B_{0.5}(s+1,\alpha+1)+\frac{(b-a)|f'(a)|}{2}\frac{2^{-s-1}}{s+1}\\
&-\frac{(b-a)|f'(b)|}{2}\frac{1-2^{-\alpha-s-1}}{\alpha+s+1}\\
&+\frac{(b-a)|f'(b)|}{2}\frac{1-2^{-s-1}}{s+1}+\frac{(b-a)|f'(b)|}{2}B_{0.5}(\alpha+1,s+1)\\
&\le\frac{(b-a)|f'(a)|}{2}\frac{2^{-\alpha-s}-1}{\alpha+s+1}+\frac{(b-a)|f'(a)|}{2}\bigl[B_{0.5}(\alpha+1,s+1)-B_{0.5}(s+1,\alpha+1)\bigr]\\
&+\frac{(b-a)|f'(a)|}{2(s+1)}+\frac{(b-a)|f'(b)|}{2}\frac{2^{-\alpha-s}-1}{\alpha+s+1}\\
&+\frac{(b-a)|f'(b)|}{2}\bigl[B_{0.5}(\alpha+1,s+1)-B_{0.5}(s+1,\alpha+1)\bigr]+\frac{(b-a)|f'(b)|}{2(s+1)}
\end{aligned}
$$

$$\begin{aligned}
&\le 0.5(b-a)\big(\big|f'(a)\big| + \big|f'(b)\big|\big) \\
&\quad \times \left(B_{0.5}(\alpha+1, s+1) - B_{0.5}(s+1, \alpha+1) + \frac{2^{-\alpha-1}}{\alpha+s+1} + \frac{1}{s+1}\right).
\end{aligned}$$

The proof is done. □

Theorem 3.4 *Let $f : [0, b] \to R$ be a differentiable mapping and $1 < q < \infty$. If $|f'|^q$ is measurable and $|f'|^q$ is decreasing and geometric-arithmetically s-convex on $[0, b]$ for some fixed $\alpha \in (0, \infty)$, $s \in (0, 1]$, $0 \le a < b$, then the following inequality for fractional integrals holds*:

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\big[J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)\big] - f\left(\frac{a+b}{2}\right)\right| \\
&\quad \le \max\left\{\frac{b-a}{2}\left(\frac{|f'(a)|^q + |f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\left(\frac{(1+2^{1-\alpha})^p}{2} - \frac{2^p(1-2^{-p\alpha})}{(p\alpha+1)}\right)^{\frac{1}{p}},\right. \\
&\qquad \left.(b-a)\left(\frac{|f'(a)|^q + |f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\left(2^{p-1} - \frac{2^p(1-2^{-p\alpha-1})}{p\alpha+1}\right)^{\frac{1}{p}}\right\},
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof To achieve our aim, we divide our proof into two cases.

Case 1: $\alpha \in (0, 1)$. By using Definition 2.2, Lemma 2.4, Lemma 2.5, Hölder's inequality and Lemma 2.7, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\big[J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)\big] - f\left(\frac{a+b}{2}\right)\right| \\
&\quad \le \frac{b-a}{2}\int_0^1 \big|h(t) - (1-t)^\alpha + t^\alpha\big|\big|f'\big(ta + (1-t)b\big)\big|\,dt \\
&\quad \le \frac{b-a}{2}\left(\int_0^1 \big|h(t) - (1-t)^\alpha + t^\alpha\big|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \big|f'\big(ta + (1-t)b\big)\big|^q\,dt\right)^{\frac{1}{q}} \\
&\quad \le \frac{b-a}{2}\left(\int_0^1 \big|h(t) - (1-t)^\alpha + t^\alpha\big|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \big|f'\big(a^t b^{1-t}\big)\big|^q\,dt\right)^{\frac{1}{q}} \\
&\quad \le \frac{b-a}{2}\left(\int_0^1 \big|h(t) - (1-t)^\alpha + t^\alpha\big|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \big[t^s\big|f'(a)\big|^q + (1-t)^s\big|f'(b)\big|^q\big]\,dt\right)^{\frac{1}{q}} \\
&\quad \le \frac{b-a}{2}\left(\frac{|f'(a)|^q + |f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\left(\int_0^1 \big|h(t) - (1-t)^\alpha + t^\alpha\big|^p\,dt\right)^{\frac{1}{p}} \\
&\quad \le \frac{b-a}{2}\left(\frac{|f'(a)|^q + |f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\left(\int_{\frac{1}{2}}^1 \big(1 - t^\alpha + (1-t)^\alpha\big)^p\,dt\right. \\
&\qquad \left. + \int_0^{\frac{1}{2}} \big(1 - (1-t)^\alpha + t^\alpha\big)^p\,dt\right)^{\frac{1}{p}} \\
&\quad \le \frac{b-a}{2}\left(\frac{|f'(a)|^q + |f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\left(\int_{\frac{1}{2}}^1 \big[1 - t^\alpha + 2^{1-\alpha} - t^\alpha\big]^p\,dt\right.
\end{aligned}$$

$$+\int_0^{\frac{1}{2}}\big[\big(1+t^\alpha\big)-\big(1-t^\alpha\big)\big]^p\,dt\bigg)^{\frac{1}{p}}$$

$$\le\frac{b-a}{2}\bigg(\frac{|f'(a)|^q+|f'(b)|^q}{s+1}\bigg)^{\frac{1}{q}}\bigg(\int_{\frac{1}{2}}^1\big[\big(1+2^{1-\alpha}\big)^p-2^pt^{p\alpha}\big]\,dt+\int_0^{\frac{1}{2}}2^pt^{p\alpha}\,dt\bigg)^{\frac{1}{p}}$$

$$\le\frac{b-a}{2}\bigg(\frac{|f'(a)|^q+|f'(b)|^q}{s+1}\bigg)^{\frac{1}{q}}\bigg(\frac{(1+2^{1-\alpha})^p}{2}-\frac{2^p(1-2^{-p\alpha})}{(p\alpha+1)}\bigg)^{\frac{1}{p}}.$$

Case 2: $\alpha\in[1,\infty)$. By using Definition 2.2, Lemma 2.4 and Lemma 2.7, we have

$$\bigg|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\big[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\big]-f\bigg(\frac{a+b}{2}\bigg)\bigg|$$

$$\le\frac{b-a}{2}\bigg(\frac{|f'(a)|^q+|f'(b)|^q}{s+1}\bigg)^{\frac{1}{q}}\bigg(\int_{\frac{1}{2}}^1\big(1-t^\alpha+(1-t)^\alpha\big)^p\,dt$$

$$+\int_0^{\frac{1}{2}}\big(1-(1-t)^\alpha+t^\alpha\big)^p\,dt\bigg)^{\frac{1}{p}}$$

$$\le(b-a)\bigg(\frac{|f'(a)|^q+|f'(b)|^q}{s+1}\bigg)^{\frac{1}{q}}\bigg(\int_{\frac{1}{2}}^1\big(1-t^\alpha+1-t^\alpha\big)^p\,dt\bigg)^{\frac{1}{p}}$$

$$\le(b-a)\bigg(\frac{|f'(a)|^q+|f'(b)|^q}{s+1}\bigg)^{\frac{1}{q}}\bigg(\int_{\frac{1}{2}}^1 2^p\big(1-t^{p\alpha}\big)\,dt\bigg)^{\frac{1}{p}}$$

$$\le(b-a)\bigg(\frac{|f'(a)|^q+|f'(b)|^q}{s+1}\bigg)^{\frac{1}{q}}\bigg(2^{p-1}-\frac{2^p(1-2^{-p\alpha-1})}{p\alpha+1}\bigg)^{\frac{1}{p}}.$$

The proof is done. □

3.3 The third results

By using Lemma 2.8, we can obtain the main results in this section.

Theorem 3.5 *Let $f:[0,b]\to R$ be a differentiable mapping. If $|f'|$ is measurable and $|f'|$ is decreasing and geometric-arithmetically s-convex on $[0,b]$ for some fixed $\alpha\in(0,\infty)$, $s\in(0,1]$, $0\le a<x<b$, then the following inequality for fractional integrals holds*:

$$\bigg|\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{b-a}\big[J_{x-}^\alpha f(a)+J_{x+}^\alpha f(b)\big]\bigg|$$

$$\le\frac{|(x-a)^{\alpha+1}-(b-x)^{\alpha+1}|}{b-a}\bigg[\frac{|f'(a)|}{\alpha+s+1}+\big|f'(b)\big|B(\alpha+1,s+1)\bigg],$$

where

$$B(s+1,\alpha+1)=\int_0^1 t^s(1-t)^\alpha\,dt.$$

Proof By using Definition 2.2, Lemma 2.5 and Lemma 2.8, we have

$$\begin{aligned}
&\left|\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}^\alpha f(a)+J_{x^+}^\alpha f(b)\right]\right| \\
&\quad\le \frac{|(x-a)^{\alpha+1}-(b-x)^{\alpha+1}|}{b-a}\int_0^1 t^\alpha\left|f'\left(tx+(1-t)a\right)\right|dt \\
&\quad\le \frac{|(x-a)^{\alpha+1}-(b-x)^{\alpha+1}|}{b-a}\int_0^1 t^\alpha\left|f'\left(x^t a^{1-t}\right)\right|dt \\
&\quad\le \frac{|(x-a)^{\alpha+1}-(b-x)^{\alpha+1}|}{b-a}\int_0^1 t^\alpha\left[t^s\left|f'(a)\right|+(1-t)^s\left|f'(b)\right|\right]dt \\
&\quad\le \frac{|(x-a)^{\alpha+1}-(b-x)^{\alpha+1}|}{b-a}\int_0^1 \left[t^{\alpha+s}\left|f'(a)\right|+t^\alpha(1-t)^s\left|f'(b)\right|\right]dt \\
&\quad\le \frac{|(x-a)^{\alpha+1}-(b-x)^{\alpha+1}|}{b-a}\left[\frac{|f'(a)|}{\alpha+s+1}+\left|f'(b)\right|B(\alpha+1,s+1)\right].
\end{aligned}$$

The proof is done. □

Theorem 3.6 *Let* $f:[0,b]\to R$ *be a differentiable mapping and* $1<q<\infty$. *If* $|f'|^q$ *is measurable and* $|f'|^q$ *is decreasing and geometric-arithmetically s-convex on* $[0,b]$ *for some fixed* $\alpha\in(0,\infty)$, $s\in(0,1]$, $0\le a<b$, *then the following inequality for fractional integrals holds*:

$$\begin{aligned}
&\left|\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}^\alpha f(a)+J_{x^+}^\alpha f(b)\right]\right| \\
&\quad\le \frac{b-a}{2(p\alpha+1)}\left(\frac{|f'(a)|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}.
\end{aligned}$$

Proof By using Definition 2.2, Lemma 2.5, Hölder's inequality and Lemma 2.8, we have

$$\begin{aligned}
&\left|\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}^\alpha f(a)+J_{x^+}^\alpha f(b)\right]\right| \\
&\quad\le \frac{|(x-a)^{\alpha+1}-(b-x)^{\alpha+1}|}{b-a}\int_0^1 t^\alpha\left|f'\left(tx+(1-t)a\right)\right|dt \\
&\quad\le \frac{b-a}{2}\left(\int_0^1 t^{p\alpha}\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(ta+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad\le \frac{b-a}{2(p\alpha+1)}\left(\int_0^1\left|f'\left(ta+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad\le \frac{b-a}{2(p\alpha+1)}\left(\int_0^1\left|f'\left(a^t b^{1-t}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad\le \frac{b-a}{2(p\alpha+1)}\left(\int_0^1\left[t^s\left|f'(a)\right|^q+(1-t)^s\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&\quad\le \frac{b-a}{2(p\alpha+1)}\left(\frac{|f'(a)|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}},
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$. The proof is done. □

4 Applications to special means

Consider the following special means (see Pearce and Pečarić [22]) for arbitrary real numbers α, β, $\alpha \neq \beta$ as follows:

(i) $H(\alpha,\beta) = \frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}}$, $\alpha,\beta \in R \setminus \{0\}$;

(ii) $A(\alpha,\beta) = \frac{\alpha+\beta}{2}$, $\alpha,\beta \in R$;

(iii) $L(\alpha,\beta) = \frac{\beta-\alpha}{\ln|\beta|-\ln|\alpha|}$, $|\alpha| \neq |\beta|$, $\alpha\beta \neq 0$;

(iv) $L_n(\alpha,\beta) = [\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}]^{\frac{1}{n}}$, $n \in \mathbb{Z} \setminus \{-1,0\}$, $\alpha,\beta \in R$, $\alpha \neq \beta$.

Next, we apply the obtained results to give some applications to special means of real numbers.

Theorem 4.1 *For some* $s \in (0,1]$, $n \in \mathbb{Z} \setminus \{-1,0\}$, $0 \leq a < b$, *the following inequality for fractional integrals holds*:

$$\left|A\left(a^n,b^n\right) - L_n^n\left(a^n,b^n\right)\right| \leq \frac{n(b-a)}{2}\left(\frac{a^{(n-1)q}+b^{(n-1)q}}{s+1}\right)^{\frac{1}{q}}\left(\frac{2-2^{1-p}}{p+1}\right)^{\frac{1}{p}},$$

where

$$\frac{1}{p}+\frac{1}{q}=1, \quad 1 < q < \infty.$$

Proof Applying Theorem 3.2 for $f(x) = x^n$, $\alpha = 1$, one can obtain the result immediately. □

Theorem 4.2 *For some* $s \in (0,1]$, $n \in \mathbb{Z} \setminus \{-1,0\}$, $0 \leq a < b$, *the following inequality for fractional integrals holds*:

$$\begin{aligned}&\left|L_n^n\left(a^n,b^n\right) - A^n(a,b)\right| \\ &\quad \leq (b-a)\left(\frac{a^{(n-1)q}+b^{(n-1)q}}{s+1}\right)^{\frac{1}{q}}\max\left\{\left(2^{p-2}-\frac{2^{p-1}-0.5}{p+1}\right)^{\frac{1}{p}},\left(2^{p-1}-\frac{2^{p}-0.5}{p+1}\right)^{\frac{1}{p}}\right\},\end{aligned}$$

where

$$\frac{1}{p}+\frac{1}{q}=1, \quad 1 < q < \infty.$$

Proof Applying Theorem 3.4 for $f(x) = x^n$, $\alpha = 1$, one can obtain the result immediately. □

Theorem 4.3 *For some* $s \in (0,1]$, $n \in \mathbb{Z} \setminus \{-1,0\}$, $1 < q < \infty$, $0 \leq a < x < b$, *the following inequality for fractional integrals holds*:

$$\left|L_n^n\left(a^n,b^n\right) - x^n\right| \leq \frac{b-a}{2(p+1)}\left(\frac{|f'(a)|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}.$$

Proof Applying Theorem 3.6 for $f(x) = x^n$, $\alpha = 1$, one can obtain the result immediately. □

Theorem 4.4 *For some $s \in (0,1]$, $0 \le a < b$, the following inequality for fractional integrals holds*:

$$\left|\frac{1}{H(a,b)} - \frac{1}{L(a,b)}\right| \le \frac{b-a}{2}\left(\frac{a^{2q}+b^{2q}}{(s+1)a^{2q}b^{2q}}\right)^{\frac{1}{q}}\left(\frac{2-2^{1-p}}{p+1}\right)^{\frac{1}{p}},$$

where

$$\frac{1}{p} + \frac{1}{q} = 1, \quad 1 < q < \infty.$$

Proof Applying Theorem 3.2 for $f(x) = x^{-1}$, $\alpha = 1$, one can obtain the result immediately. □

Theorem 4.5 *For some $s \in (0,1]$, $0 \le a < b$, the following inequality for fractional integrals holds*:

$$\begin{aligned}&\left|\frac{1}{H(a,b)} - \frac{1}{A(a,b)}\right| \\ &\quad \le (b-a)\left(\frac{a^{2q}+b^{2q}}{(s+1)a^{2q}b^{2q}}\right)^{\frac{1}{q}} \max\left\{\left(\frac{(1+2^{1-\alpha})^p}{4} - \frac{2^{p-1}(1-2^{-p\alpha})}{(p\alpha+1)}\right)^{\frac{1}{p}},\right. \\ &\qquad \left.\left(2^{p-1} - \frac{2^p(1-2^{-p\alpha-1})}{p\alpha+1}\right)^{\frac{1}{p}}\right\},\end{aligned}$$

where

$$\frac{1}{p} + \frac{1}{q} = 1, \quad 1 < q < \infty.$$

Proof Applying Theorem 3.4 for $f(x) = x^{-1}$, $\alpha = 1$, one can obtain the result immediately. □

Theorem 4.6 *For some $s \in (0,1]$, $0 \le a < x < b$, the following inequality for fractional integrals holds*:

$$\left|\frac{1}{H(a,b)} - \frac{1}{x}\right| \le \frac{b-a}{2(p+1)}\left(\frac{a^{2q}+b^{2q}}{(s+1)a^{2q}b^{2q}}\right)^{\frac{1}{q}},$$

where

$$\frac{1}{p} + \frac{1}{q} = 1, \quad 1 < q < \infty.$$

Proof Applying Theorem 3.6 for $f(x) = x^{-1}$, $\alpha = 1$, one can obtain the result immediately. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions

This work was carried out in collaboration between all authors. JRW raised these interesting problems in this research. YML, JHD and JRW proved the theorems, interpreted the results and wrote the article. All authors defined the research theme, read and approved the manuscript.

Acknowledgements

This work is supported by the National Natural Science Foundation of China (11201091), Key Projects of Science and Technology Research in the Chinese Ministry of Education (211169), Key Support Subject (Applied Mathematics) and Key project on the reforms of teaching contents and course system of Guizhou Normal College.

Received: 5 August 2013 Accepted: 23 August 2013 Published: 23 September 2013

References

1. Baleanu, D, Machado, JAT, Luo, ACJ: Fractional Dynamics and Control. Springer, Berlin (2012)
2. Diethelm, K: The Analysis of Fractional Differential Equations. Lecture Notes in Mathematics (2010)
3. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
4. Lakshmikantham, V, Leela, S, Devi, JV: Theory of Fractional Dynamic Systems. Cambridge Scientific Publishers, Cambridge (2009)
5. Miller, KS, Ross, B: An Introduction to the Fractional Calculus and Differential Equations. Wiley, New York (1993)
6. Michalski, MW: Derivatives of Noninteger Order and Their Applications. Dissertationes Mathematicae, vol. CCCXXVIII. Inst. Math., Polish Acad. Sci., Warsaw (1993)
7. Podlubny, I: Fractional Differential Equations. Academic Press, San Diego (1999)
8. Tarasov, VE: Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media. Springer, Berlin (2011)
9. Sarikaya, MZ, Set, E, Yaldiz, H, Başak, N: Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities. Math. Comput. Model. **57**, 2403-2407 (2013)
10. Zhu, C, Fečkan, M, Wang, J: Fractional integral inequalities for differentiable convex mappings and applications to special means and a midpoint formula. J. Appl. Math. Stat. Inform. **8**, 21-28 (2012)
11. Latif, MA, Dragomir, SS, Matouk, AE: New inequalities of Ostrowski type for co-ordinated convex functions via fractional integrals. J. Fract. Calc. Appl. **2**, 1-15 (2012)
12. Wang, J, Li, X, Zhu, C: Refinements of Hermite-Hadamard type inequalities involving fractional integrals. Bull. Belg. Math. Soc. Simon Stevin (2013, in press)
13. Set, E: New inequalities of Ostrowski type for mappings whose derivatives are *s*-convex in the second sense via fractional integrals. Comput. Math. Appl. **63**, 1147-1154 (2012)
14. Wang, J, Deng, J, Fečkan, M: Hermite-Hadamard type inequalities for *r*-convex functions via Riemann-Liouville fractional integrals. Ukr. Math. J. **65**, 193-211 (2013)
15. Zhang, Y, Wang, J: On some new Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals. J. Inequal. Appl. **2013**, Article ID 220 (2013)
16. Wang, J, Li, X, Fečkan, M, Zhou, Y: Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity. Appl. Anal. (2012). doi:10.1080/00036811.2012.727986
17. Wang, J, Deng, J, Fečkan, M: Exploring *s*-*e*-condition and applications to some Ostrowski type inequalities via Hadamard fractional integrals. Math. Slovaca (2013, in press)
18. Wang, J, Zhu, C, Zhou, Y: New generalized Hermite-Hadamard type inequalities and applications to special means. J. Inequal. Appl. **2013**, Article ID 325 (2013)
19. Deng, J, Wang, J: Fractional Hermite-Hadamard inequalities for (α, m)-logarithmically convex functions. J. Inequal. Appl. **2013**, Article ID 364 (2013)
20. Shuang, Y, Yin, HP, Qi, F: Hermite-Hadamard type integral inequalities for geometric-arithmetically *s*-convex functions. Analysis **33**, 197-208 (2013)
21. DiDonato, AR, Jarnagin, MP: The efficient calculation of the incomplete beta-function ratio for half-integer values of the parameters. Math. Comput. **21**, 652-662 (1967)
22. Pearce, CEM, Pečarić, JE: Inequalities for differentiable mappings with application to special means and quadrature formula. Appl. Math. Lett. **13**, 51-55 (2000)

doi:10.1186/1029-242X-2013-443
Cite this article as: Liao et al.: **Riemann-Liouville fractional Hermite-Hadamard inequalities. Part I: for once differentiable geometric-arithmetically *s*-convex functions.** *Journal of Inequalities and Applications* 2013 **2013**:443.