Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Approximation by complex Durrmeyer-Stancu type operators in compact disks

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Abstract
In this paper we introduce a class of complex Stancu-type Durrmeyer operators and study the approximation properties of these operators. We obtain a Voronovskaja-type result with quantitative estimate for these operators attached to analytic functions on compact disks. We also study the exact order of approximation. More important, our results show the overconvergence phenomenon for these complex operators.
MSC: 30E10; 41A25; 41A36

Keywords: complex Durrmeyer-Stancu type operators; Voronovskaja-type result; exact order of approximation; simultaneous approximation; overconvergence

1 Introduction

In 1986, some approximation properties of complex Bernstein polynomials in compact disks were initially studied by Lorentz [1]. Very recently, the problem of the approximation of complex operators has been causing great concern, which has become a hot topic of research. A Voronovskaja-type result with quantitative estimate for complex Bernstein polynomials in compact disks was obtained by Gal [2] Also, in [3–18] similar results for complex Bernstein-Kantorovich polynomials, Bernstein-Stancu polynomials, Kantorovich-Schurer polynomials, Kantorovich-Stancu polynomials, complex Favard-Szász-Mirakjan operators, complex Beta operators of first kind, complex Baskajov-Stancu operators, complex Bernstein-Durrmeyer polynomials, complex genuine Durrmeyer-Stancu polynomials and complex Bernstein-Durrmeyer operators based on Jacobi weights were obtained.

The aim of the present article is to obtain approximation results for complex Durrmeyer-Stancu type operators which are defined for $f:[0,1]\to \mathbf{C}$ continuous on $[0,1]$ by

$$M_n^{(\alpha,\beta)}(f;z) := n\sum_{k=1}^{n} p_{n,k}(z)\int_0^1 p_{n-1,k-1}(t)f\left(\frac{nt+\alpha}{n+\beta}\right)dt + f\left(\frac{\alpha}{n+\beta}\right)p_{n,0}(z), \tag{1}$$

where α, β are two given real parameters satisfying the condition $0\le\alpha\le\beta$, $z\in\mathbf{C}$, $n=1,2,\ldots$, and $p_{n,k}(z)=\binom{n}{k}z^k(1-z)^{n-k}$.

Note that, for $\alpha = \beta = 0$, these operators become the complex Durrmeyer-type operators $M_n(f;z) = M_n^{(0,0)}(f;z)$, this case has been investigated in [11].

2 Auxiliary results

In the sequel, we shall need the following auxiliary results.

Lemma 1 *Let $e_m(z) = z^m$, $m \in \mathbf{N} \cup \{0\}$, $z \in \mathbf{C}$, $n \in \mathbf{N}$, $0 \le \alpha \le \beta$, then we have that $M_n^{(\alpha,\beta)}(e_m;z)$ is a polynomial of degree less than or equal to* $\min(m,n)$ *and*

$$M_n^{(\alpha,\beta)}(e_m;z) = \sum_{j=0}^{m} \binom{m}{j} \frac{n^j \alpha^{m-j}}{(n+\beta)^m} M_n(e_j;z).$$

Proof By the definition given by (1), the proof is easy, here the proof is omitted.

Let $m = 0, 1, 2$, according to [11, Lemma 1], by a simple computation, we have

$$\begin{aligned}
M_n^{(\alpha,\beta)}(e_0;z) &= 1; \\
M_n^{(\alpha,\beta)}(e_1;z) &= \frac{n^2 z}{(n+1)(n+\beta)} + \frac{\alpha}{n+\beta}; \\
M_n^{(\alpha,\beta)}(e_2;z) &= \frac{n^2}{(n+\beta)^2}\left[\frac{2nz + n(n-1)z^2}{(n+1)(n+2)}\right] \\
&\quad + \frac{2n^2\alpha z}{(n+1)(n+\beta)^2} + \frac{\alpha^2}{(n+\beta)^2}.
\end{aligned}$$

□

Lemma 2 *Let $e_m(z) = z^m$, $m \in \mathbf{N} \cup \{0\}$, $z \in \mathbf{C}$, $n \in \mathbf{N}$, $0 \le \alpha \le \beta$, for all $|z| \le r$, $r \ge 1$, we have $|M_n^{(\alpha,\beta)}(e_m;z)| \le r^m$.*

Proof The proof follows directly Lemma 1 and [11, Lemma 2]. □

Lemma 3 *Let $e_m(z) = z^m$, $m, n \in \mathbf{N}$, $z \in \mathbf{C}$ and $0 \le \alpha \le \beta$, then we have*

$$\begin{aligned}
M_n^{(\alpha,\beta)}(e_{m+1};z) &= \frac{z(1-z)n}{(n+\beta)(m+n+1)}\big(M_n^{(\alpha,\beta)}(e_m;z)\big)' \\
&\quad + \frac{(m+nz)n + \alpha(1+2m+n)}{(n+\beta)(m+n+1)} M_n^{(\alpha,\beta)}(e_m;z) \\
&\quad - \frac{\alpha m(n+\alpha)}{(n+\beta)^2(m+n+1)} M_n^{(\alpha,\beta)}(e_{m-1};z). \qquad (2)
\end{aligned}$$

Proof Let

$$\begin{aligned}
T_{n-1,k-1}^{(\alpha,\beta)}(f) &:= \int_0^1 p_{n-1,k-1}(t) f\left(\frac{nt+\alpha}{n+\beta}\right) dt, \\
\widetilde{T}_{n-1,k-1}^{(\alpha,\beta)}(f) &:= \int_0^1 p_{n-1,k-1}(t)\, t f\left(\frac{nt+\alpha}{n+\beta}\right) dt, \\
\widehat{T}_{n-1,k-1}^{(\alpha,\beta)}(f) &:= \int_0^1 p_{n-1,k-1}(t)\, t^2 f\left(\frac{nt+\alpha}{n+\beta}\right) dt, \\
E_n^{(\alpha,\beta)}(f;z) &:= n \sum_{k=1}^{n} p_{n,k}(z) T_{n-1,k-1}^{(\alpha,\beta)}(f),
\end{aligned}$$

then we have

$$
\begin{aligned}
M_n^{(\alpha,\beta)}(f;z) &= E_n^{(\alpha,\beta)}(f;z) + f\left(\frac{\alpha}{n+\beta}\right)p_{n,0}(z),\\
\widetilde{T}_{n-1,k-1}^{(\alpha,\beta)}(e_m) &= \int_0^1 p_{n-1,k-1}(t)\frac{n+\beta}{n}\left(\frac{nt+\alpha}{n+\beta}-\frac{\alpha}{n+\beta}\right)\left(\frac{nt+\alpha}{n+\beta}\right)^m dt\\
&= \frac{n+\beta}{n}T_{n-1,k-1}^{(\alpha,\beta)}(e_{m+1}) - \frac{\alpha}{n}T_{n-1,k-1}^{(\alpha,\beta)}(e_m),\\
\widehat{T}_{n-1,k-1}^{(\alpha,\beta)}(e_m) &= \int_0^1 p_{n-1,k-1}(t)\left(\frac{n+\beta}{n}\right)^2\left(\frac{nt+\alpha}{n+\beta}-\frac{\alpha}{n+\beta}\right)^2\left(\frac{nt+\alpha}{n+\beta}\right)^m dt\\
&= \left(\frac{n+\beta}{n}\right)^2 T_{n-1,k-1}^{(\alpha,\beta)}(e_{m+2}) - \frac{2\alpha(n+\beta)}{n^2}T_{n-1,k-1}^{(\alpha,\beta)}(e_{m+1})\\
&\quad + \left(\frac{\alpha}{n}\right)^2 T_{n-1,k-1}^{(\alpha,\beta)}(e_m).
\end{aligned}
$$

By a simple calculation, we obtain

$$
z(1-z)p'_{n,k}(z) = (k-nz)p_{n,k}(z), \qquad \bigl[(k-1)-(n-1)t\bigr]p_{n-1,k-1}(t) = t(1-t)p'_{n-1,k-1}(t).
$$

It follows that

$$
\begin{aligned}
&z(1-z)\bigl(E_n^{(\alpha,\beta)}(e_m;z)\bigr)'\\
&\quad = n\sum_{k=1}^{n}(k-nz)p_{n,k}(z)\int_0^1 p_{n-1,k-1}(t)\left(\frac{nt+\alpha}{n+\beta}\right)^m dt\\
&\quad = n\sum_{k=1}^{n}p_{n,k}(z)\int_0^1\bigl[(k-1)-(n-1)t+(n-1)t+1\bigr]p_{n-1,k-1}(t)\\
&\qquad \cdot\left(\frac{nt+\alpha}{n+\beta}\right)^m dt - nzE_n^{(\alpha,\beta)}(e_m;z),
\end{aligned}
$$

where

$$
\begin{aligned}
&n\sum_{k=1}^{n}p_{n,k}(z)\int_0^1\bigl[(k-1)-(n-1)t+(n-1)t+1\bigr]\\
&\qquad \cdot p_{n-1,k-1}(t)\left(\frac{nt+\alpha}{n+\beta}\right)^m dt\\
&\quad = n\sum_{k=1}^{n}p_{n,k}(z)\int_0^1 t(1-t)p'_{n-1,k-1}(t)\left(\frac{nt+\alpha}{n+\beta}\right)^m dt\\
&\qquad + n(n-1)\sum_{k=1}^{n}p_{n,k}(z)\widetilde{T}_{n-1,k-1}^{(\alpha,\beta)}(e_m) + n\sum_{k=1}^{n}p_{n,k}(z)T_{n-1,k-1}^{(\alpha,\beta)}(e_m)\\
&\quad = n\sum_{k=1}^{n}p_{n,k}(z)\int_0^1 t(1-t)p'_{n-1,k-1}(t)\left(\frac{nt+\alpha}{n+\beta}\right)^m dt\\
&\qquad + \frac{(n-1)(n+\beta)}{n}E_n^{(\alpha,\beta)}(e_{m+1};z) + \left[1-\frac{\alpha(n-1)}{n}\right]E_n^{(\alpha,\beta)}(e_m;z).
\end{aligned}
$$

Also, using integration by parts, we have

$$\begin{aligned}
&\int_0^1 t(1-t)p'_{n-1,k-1}(t)\left(\frac{nt+\alpha}{n+\beta}\right)^m dt \\
&\quad = -\int_0^1 p_{n-1,k-1}(t)(1-2t)\left(\frac{nt+\alpha}{n+\beta}\right)^m dt \\
&\qquad - \frac{mn}{n+\beta}\int_0^1 p_{n-1,k-1}(t)t(1-t)\left(\frac{nt+\alpha}{n+\beta}\right)^{m-1} dt \\
&\quad = -T_{n-1,k-1}^{(\alpha,\beta)}(e_m) + 2\widetilde{T}_{n-1,k-1}^{(\alpha,\beta)}(e_m) \\
&\qquad - \frac{mn}{n+\beta}\widetilde{T}_{n-1,k-1}^{(\alpha,\beta)}(e_{m-1}) + \frac{mn}{n+\beta}\widehat{T}_{n-1,k-1}^{(\alpha,\beta)}(e_{m-1}) \\
&\quad = \frac{n+\beta}{n}(m+2)T_{n-1,k-1}^{(\alpha,\beta)}(e_{m+1}) - \left(1+\frac{2\alpha}{n}+m+\frac{2\alpha m}{n}\right)T_{n-1,k-1}^{(\alpha,\beta)}(e_m) \\
&\qquad + \frac{\alpha m(\alpha+n)}{n(n+\beta)}T_{n-1,k-1}^{(\alpha,\beta)}(e_{m-1}).
\end{aligned}$$

So, in conclusion, we have

$$\begin{aligned}
z(1-z)\big(E_n^{(\alpha,\beta)}(e_m;z)\big)' = {} & \frac{n+\beta}{n}(m+n+1)E_n^{(\alpha,\beta)}(e_{m+1};z) \\
& - \left[\frac{\alpha(1+2m+n)}{n} + m + nz\right]E_n^{(\alpha,\beta)}(e_m;z) \\
& + \frac{\alpha m(n+\alpha)}{n(n+\beta)}E_n^{(\alpha,\beta)}(e_{m-1};z),
\end{aligned}$$

which implies the recurrence in the statement. □

Lemma 4 *Let* $n \in \mathbf{N}$, $m = 2,3,\ldots$, $e_m(z) = z^m$, $S_{n,m}^{(\alpha,\beta)}(z) := M_n^{(\alpha,\beta)}(e_m;z) - z^m$, $z \in \mathbf{C}$ *and* $0 \le \alpha \le \beta$, *we have*

$$\begin{aligned}
S_{n,m}^{(\alpha,\beta)}(z) = {} & \frac{z(1-z)n}{(n+\beta)(m+n)}\big(M_n^{(\alpha,\beta)}(e_{m-1};z)\big)' \\
& + \frac{(m-1+nz)n + \alpha(m-1+n)}{(n+\beta)(m+n)}S_{n,m-1}^{(\alpha,\beta)}(z) \\
& + \frac{\alpha m}{(n+\beta)(m+n)}M_n^{(\alpha,\beta)}(e_{m-1};z) \\
& - \frac{\alpha(m-1)(n+\alpha)}{(n+\beta)^2(m+n)}M_n^{(\alpha,\beta)}(e_{m-2};z) \\
& + \frac{(m-1+nz)n + \alpha(m-1+n)}{(n+\beta)(m+n)}z^{m-1} - z^m.
\end{aligned} \tag{3}$$

Proof Using the recurrence formula (2), by a simple calculation, we can easily get the recurrence (3), the proof is omitted. □

3 Main results

The first main result is expressed by the following upper estimates.

Theorem 1 *Let $0 \le \alpha \le \beta$, $1 \le r \le R$, $D_R = \{z \in \mathbf{C} : |z| < R\}$. Suppose that $f : D_R \to \mathbf{C}$ is analytic in D_R, i.e., $f(z) = \sum_{m=0}^{\infty} c_m z^m$ for all $z \in D_R$.*

(i) *For all $|z| \le r$ and $n \in \mathbf{N}$, we have*

$$\left|M_n^{(\alpha,\beta)}(f;z) - f(z)\right| \le \frac{K_r^{(\alpha,\beta)}(f)}{n},$$

where $K_r^{(\alpha,\beta)}(f) = (1+r)\sum_{m=1}^{\infty} |c_m| m(m+1+\alpha+\beta) r^{m-1} < \infty$.

(ii) (Simultaneous approximation) *If $1 \le r < r_1 < R$ are arbitrarily fixed, then for all $|z| \le r$ and $n, p \in \mathbf{N}$, we have*

$$\left|\left(M_n^{(\alpha,\beta)}(f;z)\right)^{(p)} - f^{(p)}(z)\right| \le \frac{K_{r_1}^{(\alpha,\beta)}(f) p! r_1}{(n+\beta)(r_1 - r)^{p+1}},$$

where $K_{r_1}^{(\alpha,\beta)}(f)$ is defined as in the above point (i).

Proof Taking $e_m(z) = z^m$, by the hypothesis that $f(z)$ is analytic in D_R, *i.e.*, $f(z) = \sum_{m=0}^{\infty} c_m z^m$ for all $z \in D_R$, it is easy for us to obtain

$$M_n^{(\alpha,\beta)}(f;z) = \sum_{m=0}^{\infty} c_m M_n^{(\alpha,\beta)}(e_m;z).$$

Therefore, we get

$$\begin{aligned}\left|M_n^{(\alpha,\beta)}(f;z) - f(z)\right| &\le \sum_{m=0}^{\infty} |c_m| \cdot \left|M_n^{(\alpha,\beta)}(e_m;z) - e_m(z)\right| \\ &= \sum_{m=1}^{\infty} |c_m| \cdot \left|M_n^{(\alpha,\beta)}(e_m;z) - e_m(z)\right|,\end{aligned}$$

as $M_n^{(\alpha,\beta)}(e_0;z) = e_0(z) = 1$.

(i) For $m \in \mathbf{N}$, taking into account that $M_n^{(\alpha,\beta)}(e_{m-1};z)$ is a polynomial of degree $\le \min(m-1, n)$, by the well-known Bernstein inequality and Lemma 2, we get

$$\left|\left(M_n^{(\alpha,\beta)}(e_{m-1};z)\right)'\right| \le \frac{m-1}{r} \max\left\{\left|M_n^{(\alpha,\beta)}(e_{m-1};z)\right| : |z| \le r\right\} \le (m-1) r^{m-2}.$$

On the one hand, when $m = 1$, for $|z| \le r$, by Lemma 1, we have

$$\left|M_n^{(\alpha,\beta)}(e_1;z) - e_1(z)\right| = \left|\frac{n^2 z}{(n+1)(n+\beta)} + \frac{\alpha}{n+\beta} - z\right| \le \frac{1+r}{n}(2+\alpha+\beta).$$

When $m \ge 2$, for $n \in \mathbf{N}$, $|z| \le r$, $0 \le \alpha \le \beta$, in view of $|(m-1+nz)n + \alpha(m-1+n)| \le (n+\beta)(m+n)r$, using the recurrence formula (3) and the above inequality, we have

$$\begin{aligned}\left|M_n^{(\alpha,\beta)}(e_m;z) - e_m(z)\right| &= \left|S_{n,m}^{(\alpha,\beta)}(z)\right| \\ &\le \frac{r(1+r)}{n} \cdot (m-1) r^{m-2} + r\left|S_{n,m-1}^{(\alpha,\beta)}(z)\right|\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{\alpha}{n} r^{m-1} + \frac{\alpha}{n} r^{m-2} + \frac{m+1+\beta}{n}(1+r)r^{m-1} \\
&\leq \frac{m-1}{n}(1+r)r^{m-1} + r\left|S_{n,m-1}^{(\alpha,\beta)}(z)\right| \\
&\quad + \frac{\alpha}{n}(1+r)r^{m-1} + \frac{m+1+\beta}{n}(1+r)r^{m-1} \\
&= r\left|S_{n,m-1}^{(\alpha,\beta)}(z)\right| + \frac{2m+\alpha+\beta}{n}(1+r)r^{m-1}.
\end{aligned}
$$

By writing the last inequality, for $m = 2, \ldots,$ we easily obtain step by step the following:

$$
\begin{aligned}
&\left|M_n^{(\alpha,\beta)}(e_m;z) - e_m(z)\right| \\
&\quad \leq r\left(r\left|S_{n,m-2}^{(\alpha,\beta)}(z)\right| + \frac{2(m-1)+\alpha+\beta}{n}(1+r)r^{m-2}\right) \\
&\qquad + \frac{2m+\alpha+\beta}{n}(1+r)r^{m-1} \\
&\quad = r^2\left|S_{n,m-2}^{(\alpha,\beta)}(z)\right| + \frac{2(m-1+m)+2(\alpha+\beta)}{n}(1+r)r^{m-1} \\
&\quad \leq \cdots \leq \frac{1+r}{n} m(m+1+\alpha+\beta)r^{m-1}.
\end{aligned}
$$

In conclusion, for any $m, n \in \mathbf{N}$, $|z| \leq r$, $0 \leq \alpha \leq \beta$, we have

$$
\left|M_n^{(\alpha,\beta)}(e_m;z) - e_m(z)\right| \leq \frac{1+r}{n} m(m+1+\alpha+\beta)r^{m-1},
$$

from which it follows that

$$
\left|M_n^{(\alpha,\beta)}(f;z) - f(z)\right| \leq \frac{1+r}{n} \sum_{m=1}^{\infty} |c_m| m(m+1+\alpha+\beta)r^{m-1}.
$$

By assuming that $f(z)$ is analytic in D_R, we have $f^{(2)}(z) = \sum_{m=2}^{\infty} c_m m(m-1)z^{m-2}$ and the series is absolutely convergent in $|z| \leq r$, so we get $\sum_{m=2}^{\infty} |c_m| m(m-1)r^{m-2} < \infty$, which implies $K_r^{(\alpha,\beta)}(f) = (1+r)\sum_{m=1}^{\infty} |c_m| m(m+1+\alpha+\beta)r^{m-1} < \infty$.

(ii) For the simultaneous approximation, denoting by Γ the circle of radius $r_1 > r$ and center 0, since for any $|z| \leq r$ and $\upsilon \in \Gamma$, we have $|\upsilon - z| \geq r_1 - r$. By Cauchy's formula, it follows that for all $|z| \leq r$ and $n \in \mathbf{N}$, we have

$$
\begin{aligned}
\left|\left(M_n^{(\alpha,\beta)}(f;z)\right)^{(p)} - f^{(p)}(z)\right| &= \frac{p!}{2\pi}\left|\int_{\Gamma} \frac{M_n^{(\alpha,\beta)}(f;\upsilon) - f(\upsilon)}{(\upsilon - z)^{p+1}}\, d\upsilon\right| \\
&\leq \frac{K_{r_1}^{(\alpha,\beta)}(f)}{n} \frac{p!}{2\pi} \frac{2\pi r_1}{(r_1 - r)^{p+1}} \\
&= \frac{K_{r_1}^{(\alpha,\beta)}(f)}{n} \cdot \frac{p! r_1}{(r_1 - r)^{p+1}},
\end{aligned}
$$

which proves the theorem. □

Theorem 2 *Let* $0 \leq \alpha \leq \beta$, $R > 1$, $D_R = \{z \in \mathbf{C} : |z| < R\}$. *Suppose that* $f : D_R \to \mathbf{C}$ *is analytic in* D_R, *i.e.*, $f(z) = \sum_{k=0}^{\infty} c_k z^k$ *for all* $z \in D_R$. *For any fixed* $r \in [1, R]$ *and all* $n \in \mathbf{N}$, $|z| \leq r$, *we* *have*

$$\left|M_n^{(\alpha,\beta)}(f;z)-f(z)-\frac{\alpha-(1+\beta)z}{n}f'(z)-\frac{z(1-z)}{n}f''(z)\right|$$
$$\le \frac{M_r(f)}{n^2}+\frac{M_{r,1}^{(\alpha,\beta)}(f)}{n(n+\beta)}+\frac{M_{r,2}^{(\alpha,\beta)}(f)}{(n+\beta)^2}, \tag{4}$$

where $M_r(f)=\sum_{k=1}^{\infty}|c_k|kB_{k,r}r^k<\infty$ *with* $B_{k,r}=r^2(2k^3+3k^2+3k+1)+r(4k^3+12k^2+14k+6)+(2k^3+9k^2+13k+6)$, $M_{r,1}^{(\alpha,\beta)}(f)=\sum_{k=1}^{\infty}|c_k|[2k(k-1)^2\alpha+2k^3\beta r+k^2\alpha\beta+k^2\beta^2 r]r^{k-1}$, $M_{r,2}^{(\alpha,\beta)}(f)=\sum_{k=1}^{\infty}|c_k|\frac{k(k-1)(\alpha^2+\beta^2r^2)}{2}r^{k-2}<\infty$.

Proof For all $z\in D_R$, we have

$$\begin{aligned}
&M_n^{(\alpha,\beta)}(f;z)-f(z)-\frac{\alpha-(1+\beta)z}{n}f'(z)-\frac{z(1-z)}{n}f''(z)\\
&=M_n^{(\alpha,\beta)}(f;z)-f(z)+\frac{z}{n}f'(z)-\frac{z(1-z)}{n}f''(z)-\frac{\alpha-\beta z}{n}f'(z)\\
&=\left[M_n(f;z)-f(z)-\frac{z(1-z)f''(z)-zf'(z)}{n}\right]\\
&\quad+\left[M_n^{(\alpha,\beta)}(f;z)-M_n(f;z)-\frac{\alpha-\beta z}{n}f'(z)\right]\\
&:=I_1+I_2.
\end{aligned}$$

By [11, Theorem 1], we have $|I_1|\le\frac{M_r(f)}{n^2}$, where $M_r(f)=\sum_{k=1}^{\infty}|c_k|kB_{k,r}r^k<\infty$ with $B_{k,r}=r^2(2k^3+3k^2+3k+1)+r(4k^3+12k^2+14k+6)+(2k^3+9k^2+13k+6)$.

Next, let us estimate $|I_2|$.

By f is analytic in D_R, *i.e.*, $f(z)=\sum_{k=0}^{\infty}c_kz^k$ for all $z\in D_R$, we have

$$\begin{aligned}
|I_2| &= \left|\sum_{k=1}^{\infty}c_k\left[M_n^{(\alpha,\beta)}(e_k;z)-M_n(e_k;z)-\frac{\alpha-\beta z}{n}kz^{k-1}\right]\right|\\
&\le \sum_{k=1}^{\infty}|c_k|\left|M_n^{(\alpha,\beta)}(e_k;z)-M_n(e_k;z)-\frac{\alpha-\beta z}{n}kz^{k-1}\right|.
\end{aligned}$$

On the one hand, when $k\ge2$, since $\frac{n^k}{(n+\beta)^k}-1=-\sum_{j=0}^{k-1}\binom{k}{j}\frac{n^j\beta^{k-j}}{(n+\beta)^k}$, by Lemma 1, we obtain

$$\begin{aligned}
&M_n^{(\alpha,\beta)}(e_k;z)-M_n(e_k;z)-\frac{\alpha-\beta z}{n}kz^{k-1}\\
&=\sum_{j=0}^{k-1}\binom{k}{j}\frac{n^j\alpha^{k-j}}{(n+\beta)^k}M_n(e_j;z)+\left[\frac{n^k}{(n+\beta)^k}-1\right]M_n(e_k;z)\\
&\quad-\frac{\alpha-\beta z}{n}kz^{k-1}\\
&=\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j\alpha^{k-j}}{(n+\beta)^k}M_n(e_j;z)+\frac{kn^{k-1}\alpha}{(n+\beta)^k}M_n(e_{k-1};z)
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{j=0}^{k-1}\binom{k}{j}\frac{n^j\beta^{k-j}}{(n+\beta)^k}M_n(e_k;z)-\frac{\alpha-\beta z}{n}kz^{k-1}\\
&=\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j\alpha^{k-j}}{(n+\beta)^k}M_n(e_j;z)+\frac{kn^{k-1}\alpha}{(n+\beta)^k}\bigl[M_n(e_{k-1};z)-e_{k-1}(z)\bigr]\\
&\quad+\frac{kn^{k-1}\alpha}{(n+\beta)^k}z^{k-1}-\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j\beta^{k-j}}{(n+\beta)^k}M_n(e_k;z)\\
&\quad-\frac{kn^{k-1}\beta}{(n+\beta)^k}\bigl[M_n(e_k;z)-e_k(z)\bigr]-\frac{kn^{k-1}\beta}{(n+\beta)^k}z^k-\frac{\alpha-\beta z}{n}kz^{k-1}\\
&=\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j\alpha^{k-j}}{(n+\beta)^k}M_n(e_j;z)+\frac{kn^{k-1}\alpha}{(n+\beta)^k}\bigl[M_n(e_{k-1};z)-e_{k-1}(z)\bigr]\\
&\quad-\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j\beta^{k-j}}{(n+\beta)^k}M_n(e_k;z)-\frac{kn^{k-1}\beta}{(n+\beta)^k}\bigl[M_n(e_k;z)-e_k(z)\bigr]\\
&\quad-\left[\frac{1}{n}-\frac{n^{k-1}}{(n+\beta)^k}\right]k\alpha z^{k-1}+\left[\frac{1}{n}-\frac{n^{k-1}}{(n+\beta)^k}\right]k\beta z^k.
\end{aligned}
$$

By the proof of [11, Corollary 3], for any $k\in\mathbf{N}$, $|z|\le r$, $r\ge 1$, we have

$$
\bigl|M_n(e_k;z)\bigr|\le r^k,\qquad \bigl|M_n(e_k;z)-e_k\bigr|\le\frac{2k^2}{n}r^k.
$$

Hence, for any $k\ge 2$, $|z|\le r$, $r\ge 1$, we can get

$$
\begin{aligned}
&\left|\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j\alpha^{k-j}}{(n+\beta)^k}M_n(e_j;z)\right|\\
&\quad\le\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j\alpha^{k-j}}{(n+\beta)^k}r^{k-2}\\
&\quad=\sum_{j=0}^{k-2}\frac{k(k-1)}{(k-j)(k-j-1)}\binom{k-2}{j}\frac{n^j\alpha^{k-2-j}}{(n+\beta)^{k-2}}\cdot\frac{\alpha^2}{(n+\beta)^2}r^{k-2}\\
&\quad\le\frac{k(k-1)}{2}\cdot\frac{\alpha^2}{(n+\beta)^2}\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{n^j\alpha^{k-2-j}}{(n+\beta)^{k-2}}r^{k-2}\\
&\quad\le\frac{k(k-1)}{2}\cdot\frac{\alpha^2}{(n+\beta)^2}r^{k-2}
\end{aligned}
$$

and

$$
\left|\frac{kn^{k-1}\alpha}{(n+\beta)^k}\bigl[M_n(e_{k-1};z)-e_{k-1}(z)\bigr]\right|\le\frac{2k(k-1)^2\alpha}{n(n+\beta)}r^{k-1}.
$$

Also, using

$$
\frac{1}{n}-\frac{n^{k-1}}{(n+\beta)^k}=\frac{\sum_{j=0}^{k-1}\binom{k}{j}n^j\beta^{k-j}}{n(n+\beta)^k}\le\frac{k\beta}{n(n+\beta)}
$$

for any $k \geq 2$, $|z| \leq r$, $r \geq 1$, we get

$$\begin{aligned}
&\left| M_n^{(\alpha,\beta)}(e_k;z) - M_n(e_k;z) - \frac{\alpha - \beta z}{n} k z^{k-1} \right| \\
&\quad \leq \frac{k(k-1)}{2} \cdot \frac{\alpha^2}{(n+\beta)^2} r^{k-2} + \frac{2k(k-1)^2\alpha}{n(n+\beta)} r^{k-1} + \frac{k(k-1)}{2} \cdot \frac{\beta^2}{(n+\beta)^2} r^k \\
&\qquad + \frac{2k^3\beta}{n(n+\beta)} r^k + \frac{k^2\alpha\beta}{n(n+\beta)} r^{k-1} + \frac{k^2\beta^2}{n(n+\beta)} r^k \\
&\quad = \frac{r^{k-1}}{n(n+\beta)} \left[2k(k-1)^2\alpha + 2k^3\beta r + k^2\alpha\beta + k^2\beta^2 r \right] \\
&\qquad + \frac{r^{k-2}}{(n+\beta)^2} \cdot \frac{k(k-1)(\alpha^2 + \beta^2 r^2)}{2}.
\end{aligned}$$

On the other hand, when $k = 1$, using Lemma 1 and $M_n(e_1;z) = \frac{nz}{n+1}$ (see [19]), by a simple calculation, we can get $|M_n^{(\alpha,\beta)}(e_1;z) - M_n(e_1;z) - \frac{\alpha-\beta z}{n}| \leq \frac{1}{n(n+\beta)}(2\beta r + \alpha\beta + \beta^2 r)$.

So, for any $k \in \mathbf{N}$, $|z| \leq r$, $r \geq 1$, we have

$$\begin{aligned}
&\left| M_n^{(\alpha,\beta)}(e_k;z) - M_n(e_k;z) - \frac{\alpha - \beta z}{n} k z^{k-1} \right| \\
&\quad \leq \frac{r^{k-1}}{n(n+\beta)} \left[2k(k-1)^2\alpha + 2k^3\beta r + k^2\alpha\beta + k^2\beta^2 r \right] \\
&\qquad + \frac{r^{k-2}}{(n+\beta)^2} \cdot \frac{k(k-1)(\alpha^2 + \beta^2 r^2)}{2}.
\end{aligned}$$

Hence, we have

$$|I_2| \leq \frac{M_{r,1}^{(\alpha,\beta)}(f)}{n(n+\beta)} + \frac{M_{r,2}^{(\alpha,\beta)}(f)}{(n+\beta)^2},$$

where $M_{r,1}^{(\alpha,\beta)}(f) = \sum_{k=1}^{\infty} |c_k| [2k(k-1)^2\alpha + 2k^3\beta r + k^2\alpha\beta + k^2\beta^2 r] r^{k-1}$, $M_{r,2}^{(\alpha,\beta)}(f) = \sum_{k=1}^{\infty} |c_k| \frac{k(k-1)(\alpha^2+\beta^2 r^2)}{2} r^{k-2}$.

In conclusion, we obtain

$$\begin{aligned}
&\left| M_n^{(\alpha,\beta)}(f;z) - f(z) - \frac{\alpha - (1+\beta)z}{n} f'(z) - \frac{z(1-z)}{n} f''(z) \right| \\
&\quad \leq |I_1| + |I_2| \leq \frac{M_r(f)}{n^2} + \frac{M_{r,1}^{(\alpha,\beta)}(f)}{n(n+\beta)} + \frac{M_{r,2}^{(\alpha,\beta)}(f)}{(n+\beta)^2}. \qquad \square
\end{aligned}$$

In the following theorem, we obtain the exact order of approximation.

Theorem 3 *Let $0 \leq \alpha \leq \beta$, $R > 1$, $D_R = \{z \in \mathbf{C} : |z| < R\}$. Suppose that $f : D_R \to \mathbf{C}$ is analytic in D_R. If f is not a polynomial of degree 0, then for any $r \in [1, R)$, we have*

$$\left\| M_n^{(\alpha,\beta)}(f;\cdot) - f \right\|_r \geq \frac{C_r^{(\alpha,\beta)}(f)}{n}, \quad n \in \mathbf{N},$$

where $\|f\|_r = \max\{|f(z)|; |z| \leq r\}$ and the constant $C_r^{(\alpha,\beta)}(f) > 0$ depends on f, r and α, β, but it is independent of n.

Proof Define $e_1(z) = z$ and

$$H_n^{(\alpha,\beta)}(f;z) = M_n^{(\alpha,\beta)}(f;z) - f(z) - \frac{\alpha - (1+\beta)z}{n} f'(z) - \frac{z(1-z)}{n} f''(z).$$

For all $z \in D_R$ and $n \in \mathbf{N}$, we have

$$\begin{aligned} &M_n^{(\alpha,\beta)}(f;z) - f(z) \\ &\quad = \frac{1}{n}\left\{\bigl[\alpha - (1+\beta)z\bigr]f'(z) + z(1-z)f''(z) + \frac{1}{n}\bigl[n^2 H_n^{(\alpha,\beta)}(f;z)\bigr]\right\}. \end{aligned}$$

In view of the property $\|F + G\|_r \ge |\|F\|_r - \|G\|_r| \ge \|F\|_r - \|G\|_r$, it follows

$$\begin{aligned} &\bigl\|M_n^{(\alpha,\beta)}(f;\cdot) - f\bigr\|_r \\ &\quad \ge \frac{1}{n}\left\{\bigl\|\bigl[\alpha - (1+\beta)e_1\bigr]f' + e_1(1-e_1)f''\bigr\|_r - \frac{1}{n}\bigl[n^2\bigl\|H_n^{(\alpha,\beta)}(f;\cdot)\bigr\|_r\bigr]\right\}. \end{aligned}$$

Considering the hypothesis that f is not a polynomial of degree 0 in D_R, we have

$$\bigl\|\bigl[\alpha - (1+\beta)e_1\bigr]f' + e_1(1-e_1)f''\bigr\|_r > 0.$$

Indeed, supposing the contrary, it follows that

$$\bigl[\alpha - (1+\beta)z\bigr]f'(z) + z(1-z)f''(z) = 0 \quad \text{for all } z \in \overline{D}_r.$$

Defining $y(z) = f'(z)$ and looking for the analytic function $y(z)$ under the form $y(z) = \sum_{k=0}^{\infty} a_k z^k$, after replacement in the differential equation, the coefficients identification method immediately leads to $a_k = 0$ for all $k \in \mathbf{N} \cup \{0\}$. This implies that $y(z) = 0$ for all $z \in \overline{D}_r$ and therefore f is constant on $\overline{D}_r$, a contradiction with the hypothesis.

Using inequality (4), we get

$$n^2\bigl\|H_n^{(\alpha,\beta)}(f;\cdot)\bigr\|_r \le N_r^{(\alpha,\beta)}(f), \tag{5}$$

where $N_r^{(\alpha,\beta)}(f) = M_r(f) + M_{r,1}^{(\alpha,\beta)}(f) + M_{r,2}^{(\alpha,\beta)}(f)$.

Therefore, there exists an index n_0, depending only on f, r and α, β, such that for all $n \ge n_0$, we have

$$\begin{aligned} &\bigl\|\bigl[\alpha - (1+\beta)e_1\bigr]f' + e_1(1-e_1)f''\bigr\|_r - \frac{1}{n}\bigl[n^2\bigl\|H_n^{(\alpha,\beta)}(f;\cdot)\bigr\|_r\bigr] \\ &\quad \ge \frac{1}{2}\bigl\|\bigl[(1+\alpha) - (2+\beta)e_1\bigr]f' + e_1(1-e_1)f''\bigr\|_r, \end{aligned}$$

which implies

$$\bigl\|M_n^{(\alpha,\beta)}(f;\cdot) - f\bigr\|_r \ge \frac{1}{2n}\bigl\|\bigl[\alpha - (1+\beta)e_1\bigr]f' + e_1(1-e_1)f''\bigr\|_r \quad \text{for all } n \ge n_0.$$

For $n \in \{1, 2, \ldots, n_0 - 1\}$, we have

$$\left\| M_n^{(\alpha,\beta)}(f;\cdot) - f \right\|_r \geq \frac{W_{r,n}^{(\alpha,\beta)}(f)}{n},$$

where $W_{r,n}^{(\alpha,\beta)}(f) = n\|M_n^{(\alpha,\beta)}(f;\cdot) - f\|_r > 0$.

As a conclusion, we have

$$\left\| M_n^{(\alpha,\beta)}(f;\cdot) - f \right\|_r \geq \frac{C_r^{(\alpha,\beta)}(f)}{n} \quad \text{for all } n \in \mathbf{N},$$

where

$$C_r^{(\alpha,\beta)}(f) = \min\left\{ W_{r,1}^{(\alpha,\beta)}(f), W_{r,2}^{(\alpha,\beta)}(f), \ldots, W_{r,n_0-1}^{(\alpha,\beta)}(f), \right.$$
$$\left. \frac{1}{2}\left\| \left[\alpha - (1+\beta)e_1\right] f' + e_1(1-e_1)f'' \right\|_r \right\},$$

this completes the proof. □

Combining Theorem 3 with Theorem 1, we get the following result.

Corollary 1 *Let* $0 \leq \alpha \leq \beta$, $R > 1$, $D_R = \{z \in \mathbf{C} : |z| < R\}$. *Suppose that* $f : D_R \to \mathbf{C}$ *is analytic in* D_R. *If* f *is not a polynomial of degree* 0, *then for any* $r \in [1, R)$, *we have*

$$\left\| M_n^{(\alpha,\beta)}(f;\cdot) - f \right\|_r \asymp \frac{1}{n}, \quad n \in \mathbf{N},$$

where $\|f\|_r = \max\{|f(z)|; |z| \leq r\}$ *and the constants in the equivalence depend on* f, r *and* α, β, *but they are independent of* n.

Theorem 4 *Let* $0 \leq \alpha \leq \beta$, $R > 1$, $D_R = \{z \in \mathbf{C} : |z| < R\}$. *Suppose that* $f : D_R \to \mathbf{C}$ *is analytic in* D_R. *Also, let* $1 \leq r < r_1 < R$ *and* $p \in \mathbf{N}$ *be fixed. If* f *is not a polynomial of degree* $\leq p - 1$, *then we have*

$$\left\| \left(M_n^{(\alpha,\beta)}(f;\cdot)\right)^{(p)} - f^{(p)} \right\|_r \asymp \frac{1}{n}, \quad n \in \mathbf{N},$$

where $\|f\|_r = \max\{|f(z)|; |z| \leq r\}$ *and the constants in the equivalence depend on* f, r, r_1, p, α *and* β, *but they are independent of* n.

Proof Taking into account the upper estimate in Theorem 1, it remains to prove the lower estimate only. Denoting by Γ the circle of radius $r_1 > r$ and center 0, by Cauchy's formula, it follows that for all $|z| \leq r$ and $n \in \mathbf{N}$, we have

$$\left(M_n^{(\alpha,\beta)}(f;z)\right)^{(p)} - f^{(p)}(z) = \frac{p!}{2\pi i} \int_\Gamma \frac{M_n^{(\alpha,\beta)}(f;v) - f(v)}{(v-z)^{p+1}}\, dv.$$

Keeping the notation there for $H_n^{(\alpha,\beta)}(f;z)$, for all $n \in \mathbf{N}$, we have

$$\begin{aligned}
&M_n^{(\alpha,\beta)}(f;z) - f(z) \\
&\quad = \frac{1}{n}\left\{\left[\alpha - (1+\beta)z\right]f'(z) + z(1-z)f''(z) + \frac{1}{n}\left[n^2 H_n^{(\alpha,\beta)}(f;z)\right]\right\}.
\end{aligned}$$

By using Cauchy's formula, for all $\nu \in \Gamma$, we get

$$\begin{aligned}
\left(M_n^{(\alpha,\beta)}(f;z)\right)^{(p)} - f^{(p)}(z) = \frac{1}{n}\Big\{&\left[\left(\alpha - (1+\beta)z\right)f'(z) + z(1-z)f''(z)\right]^{(p)} \\
&+ \frac{1}{n}\cdot\frac{p!}{2\pi i}\int_\Gamma \frac{n^2 H_n^{(\alpha,\beta)}(f;\nu)}{(\nu - z)^{p+1}}\,d\nu\Big\}.
\end{aligned}$$

Passing now to $\|\cdot\|_r$ and denoting $e_1(z) = z$, it follows

$$\begin{aligned}
\left\|\left(M_n^{(\alpha,\beta)}(f;\cdot)\right)^{(p)} - f^{(p)}\right\|_r \geq \frac{1}{n}\Bigg[&\left\|\left[\left(\alpha - (1+\beta)e_1\right)f' + e_1(1-e_1)f''\right]^{(p)}\right\|_r \\
&- \frac{1}{n}\left\|\frac{p!}{2\pi i}\int_\Gamma \frac{n^2 H_n^{(\alpha,\beta)}(f;\nu)}{(\nu - \cdot)^{p+1}}\,d\nu\right\|_r\Bigg].
\end{aligned}$$

Since for any $|z| \leq r$ and $\upsilon \in \Gamma$ we have $|\upsilon - z| \geq r_1 - r$, so, by inequality (5), we get

$$\begin{aligned}
\left\|\frac{p!}{2\pi i}\int_\Gamma \frac{n^2 H_n^{(\alpha,\beta)}(f;\nu)}{(\nu - \cdot)^{p+1}}\,d\nu\right\|_r &\leq \frac{p!}{2\pi}\cdot\frac{2\pi r_1 n^2 \|H_n^{(\alpha,\beta)}(f;\cdot)\|_{r_1}}{(r_1 - r)^{p+1}} \\
&\leq \frac{N_{r_1}^{(\alpha,\beta)}(f)p!r_1}{(r_1 - r)^{p+1}},
\end{aligned}$$

where $N_{r_1}^{(\alpha,\beta)}(f) = M_{r_1}(f) + M_{r_1,1}^{(\alpha,\beta)}(f) + M_{r_1,2}^{(\alpha,\beta)}(f)$.

Taking into account that the function f is analytic in D_R, by following exactly the lines in Gal [5], seeing also the book Gal [6, pp.77-78] (where it is proved that $\|[(\alpha - \beta e_1)f' + \frac{e_1(1-e_1)}{2}f'']^{(p)}\|_r > 0$), we have

$$\left\|\left[\left(\alpha - (1+\beta)e_1\right)f' + e_1(1-e_1)f''\right]^{(p)}\right\|_r > 0.$$

In continuation, reasoning exactly as in the proof of Theorem 3, we can get the desired conclusion. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Acknowledgements

The authors are most grateful to the editor and anonymous referee for careful reading of the manuscript and valuable suggestions which helped in improving an earlier version of this paper. This work is supported by the National Natural Science Foundation of China (Grant no. 61170324), the Class A Science and Technology Project of Education Department of Fujian Province of China (Grant no. JA12324), and the Natural Science Foundation of Fujian Province of China (Grant no. 2013J01017).

Received: 29 April 2013 Accepted: 23 August 2013 Published: 23 September 2013

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doi:10.1186/1029-242X-2013-442
Cite this article as: Ren et al.: **Approximation by complex Durrmeyer-Stancu type operators in compact disks.** *Journal of Inequalities and Applications* 2013 **2013**:442.