Mishra et al. *Journal of Inequalities and Applications* 2013, **2013**:440
http://www.journalofinequalitiesandapplications.com/content/2013/1/440

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

L_r-Approximation of signals (functions) belonging to weighted $W(L_r, \xi(t))$-class by $C^1 \cdot N_p$ summability method of conjugate series of its Fourier series

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Abstract

Recently, Lal (Appl. Math. Comput. 209:346-350, 2009) has determined the degree of approximation of a function belonging to Lip α and weighted $W(L_r, \xi(t))$-classes using product $C^1 \cdot N_p$ summability with non-increasing weights $\{p_n\}$. In this paper, we determine the degree of approximation of function $\tilde{f}$, conjugate to a 2π-periodic function f belonging to weighted $W(L_r, \xi(t))$-class by dropping the monotonicity on the generating sequence $\{p_n\}$ with a new (proper) set of conditions, which in turn generalizes the results of Mishra *et al.* (Bull. Math. Anal. Appl., 2013) on Lip$(\xi(t), r)$-class and rectifies (removes) the errors of Mishra *et al.* (Mat. Vesn., 2013). Few examples and applications are also highlighted in this manuscript.

MSC: Primary 42B05; 42B08; 40G05; 41A10

Keywords: generalized Lipschitz $W(L_r, \xi(t))$ $(r \geq 1)$-class of functions; conjugate Fourier series; degree of approximation; C^1 means; N_p means; product summability $C^1 \cdot N_p$ transform

1 Introduction

Approximation by trigonometric polynomials is at the heart of approximation theory. The most important trigonometric polynomials used in the approximation theory are obtained by linear summation methods of Fourier series of 2π-periodic functions on the real line (*i.e.*, Cesàro means, Nörlund means and Product Cesàro-Nörlund means, *etc.*). Much of the advance in the theory of trigonometric approximation is due to the periodicity of the functions. The method of summability considered here was first introduced by Woronoi [1]. Summability techniques were also applied on some engineering problems like, Chen and Jeng [2] implemented the Cesàro sum of order $(C,1)$ and $(C,2)$, in order to accelerate the convergence rate to deal with the Gibbs phenomenon, for the dynamic response of a finite elastic body subjected to boundary traction. Chen *et al.* [3] applied regularization with Cesàro sum technique for the derivative of the double layer potential. Similarly, Chen and Hong [4] used Cesàro sum regularization technique for hyper singularity of dual integral equation. The degree of approximation of functions belonging to Lip α, Lip(α, r), Lip$(\xi(t), r)$ and $W(L_r, \xi(t))$ $(r \geq 1)$-classes by Nörlund (N_p) matrices and general summability matrices has been proved by various investigators like Govil [5], Khan [6], Qureshi

[7], Mohapatra and Chandra [8], Leindler [9], Rhoades *et al.* [10], Guven and Israfilov [11], Bhardwaj and Gupta [12] and Mishra *et al.* [13–20]. Here, Lal [21] has assumed monotonicity on the generating sequence $\{p_n\}$ to prove their theorems. The approximation of function $\tilde{f}$, conjugate to a periodic function $f \in W(L_r, \xi(t))$ $(r \geq 1)$ using product $(C^1 \cdot N_p)$-summability has not been studied so far. In this paper, we obtain a new theorem on the degree of approximation of function $\tilde{f}$, conjugate to a periodic function $f \in W(L_r, \xi(t))$-class using semi-monotonicity on the generating sequence $\{p_n\}$ and a proper set of the conditions.

A bidiagonal matrix is a matrix with non-zero entries along the main diagonal and *either* the diagonal above or the diagonal below. This means there are exactly two non-zero diagonals in the matrix.

When the diagonal above the main diagonal has the non-zero entries, the matrix is upper bidiagonal. When the diagonal below the main diagonal has the non-zero entries, the matrix is lower bidiagonal.

For example, the following matrix is upper bidiagonal:

$$\begin{pmatrix} 1 & 4 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

and the following matrix is lower bidiagonal:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 3 \end{pmatrix}.$$

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with the sequence of nth partial sums $\{s_n\}$. Let $\{p_n\}$ be a non-negative sequence of constants, real or complex, and let us write

$$P_n = \sum_{k=0}^{n} p_k \neq 0 \quad \forall n \geq 0, p_{-1} = 0 = P_{-1} \text{ and } P_n \to \infty \text{ as } n \to \infty.$$

The sequence to sequence transformation $t_n^N = \sum_{\nu=0}^{n} p_{n-\nu} s_\nu / P_n$ defines the sequence $\{t_n^N\}$ of Nörlund means of the sequence $\{s_n\}$, generated by the sequence of coefficients $\{p_n\}$. The series $\sum_{n=0}^{\infty} a_n$ is said to be summable N_p to the sum s if $\lim_{n\to\infty} t_n^N$ exists and is equal to s. In the special case in which

$$p_n = \binom{n+\alpha-1}{\alpha-1} = \frac{(n+\alpha)}{(n+1)(\alpha)} \quad (\alpha > 0).$$

The Nörlund summability N_p reduces to the familiar C^α summability.

The product of C^1 summability with an N_p summability defines $C^1 \cdot N_p$ summability. Thus, the $C^1 \cdot N_p$ mean is given by $t_n^{CN} = \frac{1}{n+1} \sum_{k=0}^{n} P_k^{-1} \sum_{\nu=0}^{k} p_{k-\nu} s_\nu$. If $t_n^{CN} \to s$ as $n \to \infty$, then the infinite series $\sum_{n=0}^{\infty} a_n$ or the sequence $\{s_n\}$ is said to be summable $C^1 \cdot N_p$ to the sum s if $\lim_{n\to\infty} t_n^{CN}$ exists and is equal to s.

$$\begin{aligned} s_n \to s \quad &\Rightarrow \quad N_p(s_n) = t_n^N = P_n^{-1}\sum_{\nu=0}^{n} p_{n-\nu}s_\nu \to s, \quad \text{as } n\to\infty,\ N_p \text{ method is regular,} \\ &\Rightarrow \quad C^1\bigl(N_p(s_n)\bigr) = t_n^{CN} \to s, \quad \text{as } n\to\infty,\ C^1 \text{ method is regular,} \\ &\Rightarrow \quad C^1\cdot N_p \text{ method is regular.} \end{aligned}$$

Let $f(x)$ be a 2π-periodic function and Lebesgue integrable. The Fourier series of $f(x)$ is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) \equiv \sum_{n=0}^{\infty} A_n(x) \tag{1.1}$$

with nth partial sum $s_n(f;x)$.

The conjugate series of Fourier series (1.1) is given by

$$\sum_{n=1}^{\infty}(b_n\cos nx - a_n\sin nx) \equiv \sum_{n=1}^{\infty} B_n(x). \tag{1.2}$$

A function $f(x) \in \operatorname{Lip}\alpha$ if

$$\bigl|f(x+t) - f(x)\bigr| = \mathrm{O}\bigl(|t|^\alpha\bigr) \quad \text{for } 0<\alpha\le 1, t>0$$

and $f(x) \in \operatorname{Lip}(\alpha, r)$, [6] for $0 \le x \le 2\pi$, if

$$\bigl\|f(x+t) - f(x)\bigr\|_r = \left(\int_0^{2\pi} \bigl|f(x+t) - f(x)\bigr|^r\,dx\right)^{1/r} = \mathrm{O}\bigl(|t|^\alpha\bigr), \quad 0<\alpha\le 1, r\ge 1, t>0.$$

$f(x) \in \operatorname{Lip}(\xi(t), r)$ if

$$\bigl\|f(x+t) - f(x)\bigr\|_r = \left(\int_0^{2\pi} \bigl|f(x+t) - f(x)\bigr|^r\,dx\right)^{1/r} = \mathrm{O}\bigl(\xi(t)\bigr), \quad r\ge 1, t>0.$$

$f \in W(L_r, \xi(t))$, [18, 20] if

$$\begin{aligned} \omega_r(t;f) &= \bigl\|\bigl[f(x+t) - f(x)\bigr]\sin^\beta(x/2)\bigr\|_r = \left(\int_0^{2\pi} \bigl|f(x+t) - f(x)\bigr|^r \sin^{\beta r}(x/2)\,dx\right)^{1/r} \\ &= \mathrm{O}\bigl(\xi(t)\bigr), \quad \beta\ge 0, r\ge 1, t>0, \end{aligned}$$

where $\xi(t)$ is positive increasing function of t.

If $\beta = 0$, then $W(L_r, \xi(t))$ reduces to the class $\operatorname{Lip}(\xi(t), r)$, if $\xi(t) = t^\alpha$ $(0<\alpha\le 1)$, then $\operatorname{Lip}(\xi(t), r)$ class coincides with the class $\operatorname{Lip}(\alpha, r)$, and if $r\to\infty$, then $\operatorname{Lip}(\alpha, r)$ reduces to the class $\operatorname{Lip}\alpha$.

L_∞-norm of a function $f: R\to R$ is defined by $\|f\|_\infty = \sup\{|f(x)| : x\in R\}$.

L_r-norm of f is defined by $\|f\|_r = (\int_0^{2\pi} |f(x)|^r\,dx)^{1/r}$, $r\ge 1$.

The degree of approximation of a function $f : R \to R$ by trigonometric polynomial t_n of order n under sup norm $\| \|_\infty$ is defined by [22]

$$\|t_n - f\|_\infty = \sup\{|t_n(x) - f(x)| : x \in R\},$$

and $E_n(f)$ of a function $f \in L_r$ is given by $E_n(f) = \min_n \|t_n - f\|_r$.

The conjugate function $\tilde{f}(x)$ is defined for almost every x by

$$\tilde{f}(x) = -\frac{1}{2\pi}\int_0^\pi \psi(t)\cot t/2\,dt = \lim_{h\to 0}\left(-\frac{1}{2\pi}\int_h^\pi \psi(t)\cot t/2\,dt\right).$$

We note that t_n^N and t_n^{CN} are also trigonometric polynomials of degree (or order) n.

Abel's transformation: The formula

$$\sum_{k=m}^{n} u_k v_k = \sum_{k=m}^{n-1} U_k(v_k - v_{k+1}) - U_{m-1}v_m + U_n v_n, \tag{1.3}$$

where $0 \le m \le n$, $U_k = u_0 + u_1 + u_2 + \cdots + u_k$, if $k \ge 0$, $U_{-1} = 0$, which can be verified, is known as Abel's transformation and will be used extensively in what follows.

If $v_m, v_{m+1}, \ldots, v_n$ are non-negative and non-increasing, the left-hand side of (1.3) does not exceed $2v_m \max_{m-1\le k\le n} |U_k|$ in absolute value. In fact,

$$\begin{aligned}\left|\sum_{k=m}^{n} u_k v_k\right| &\le \max|U_k|\left\{\sum_{k=m}^{n-1}(v_k - v_{k+1}) + v_m + v_n\right\}\\ &= 2v_m \max|U_k|.\end{aligned} \tag{1.4}$$

We write throughout

$$\begin{aligned}&\psi(t) = f(x+t) - f(x-t), \qquad \phi(t) = f(x+t) - 2f(x) + f(x-t),\\ &V_n = \frac{1}{2\pi(n+1)}\sum_{k=0}^{n} P_k^{-1}\sum_{\nu=0}^{k}(\nu+1)|p_\nu - p_{\nu-1}|,\\ &\tilde{M}_n(t) = \frac{1}{2\pi(n+1)}\sum_{k=0}^{n} P_k^{-1}\sum_{\nu=0}^{k} p_\nu \frac{\cos(k-\nu+1/2)t}{\sin t/2}\end{aligned} \tag{1.5}$$

$\tau = [1/t]$, where τ denotes the greatest integer not exceeding $1/t$. Furthermore, C denotes an absolute positive constant, not necessarily the same at each occurrence.

We note that the series, conjugate to a Fourier series, is not necessarily a Fourier series. Hence a separate study of conjugate series is desirable and attracted the attention of researchers.

2 Known theorem

Lal [21] has obtained the degree of approximation of the functions belonging to $W(L_r, \xi(t))$-class by $C^1 \cdot N_p$ means with monotonicity on the generating sequence $\{p_n\}$. He proved the following.

Theorem 2.1 *If $f(x)$ is a 2π-periodic function and Lebesgue integrable on* $[0, 2\pi]$ *and is belonging to $W(L^r, \xi(t))$-class, then its degree of approximation by $C^1 \cdot N_p$ means of its Fourier series* (1.1) *is given by*

$$\left\| t_n^{CN} - f \right\|_r = O\left((n+1)^{\beta+1/r}\xi\left(\frac{1}{n+1}\right)\right), \tag{2.1}$$

provided $\xi(t)$ satisfies the following conditions:

$$\{\xi(t)/t\} \text{ be a decreasing sequence}, \tag{2.2}$$

$$\left(\int_0^{1/(n+1)} \left(\frac{t|\phi(t)|}{\xi(t)}\right)^r \sin^{\beta r} t\,dt\right)^{1/r} = O\left((n+1)^{-1}\right), \tag{2.3}$$

$$\left(\int_{1/(n+1)}^{\pi} \left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^r dt\right)^{1/r} = O\left((n+1)^{\delta}\right), \tag{2.4}$$

where δ is an arbitrary number such that $s(1-\delta) - 1 > 0$, $r^{-1} + s^{-1} = 1$, $1 \le r \le \infty$, conditions (2.3) *and* (2.4) *hold uniformly in x.*

Remark 2.2 The condition $1/\sin^{\beta}(t) = O(1/t^{\beta})$, $1/(n+1) \le t \le \pi$ used by Lal [21, pp.349-350] in writing the proof of Theorem 2.1 is not valid since $\sin t \to 0$ as $t \to \pi$.

Remark 2.3 There is a fatal error in the proof of Theorem 2.1 of Lal [21, p.349], in calculating

$$\left[\frac{t^{-\beta s - s + 1}}{-\beta s - s + 1}\right]_{\epsilon}^{1/(n+1)},$$

note that $-\beta s - s + 1 < 0$. Therefore, one has $\frac{1}{\beta s + s - 1}[\frac{1}{\epsilon^{\beta s + s - 1}} - (n+1)^{\beta s + s - 1}]$, which need not be $O((n+1)^{\beta s + s - 1})$, since ϵ might be $O(1/n^{\gamma})$ for some $\gamma > 1$.

3 Main theorem

It is well known that the theory of approximations, *i.e.*, TFA, which originated from a well-known theorem of Weierstrass, has become an exciting interdisciplinary field of study for the last 130 years. These approximations have assumed important new dimensions due to their wide applications in signal analysis [23], in general and in digital signal processing [24] in particular, in view of the classical Shannon sampling theorem. Mittal *et al.* [25, 26] have obtained many interesting results on TFA using summability methods without monotonicity on the rows of the matrix T: a digital filter. Broadly speaking, signals are treated as functions of one variable, and images are represented by functions of two variables. Very recently, Mishra *et al.* [13, 18] obtained the degree of approximation of conjugate of a function using $C^1 \cdot N_p$ product summability method of its conjugate series of its Fourier series in Lipschitz and weighted spaces, respectively. Therefore, the purpose of the present paper is to generalize the results of Mishra *et al.* [13] on the degree of approximation of a function $\tilde{f}(x)$, conjugate to a 2π-periodic function f belonging to weighted $W(L_r, \xi(t))$-class by $C^1 \cdot N_p$ means of conjugate series of its Fourier series by dropping the monotonicity condition on the generating sequence $\{p_n\}$ with the help of a new (proper) set of conditions to rectify the errors of Mishra *et al.* [18]. More precisely, we state our main theorem as follows.

Theorem 3.1 *If $\tilde{f}(x)$, conjugate to a 2π-periodic function f belonging to $W(L_r, \xi(t))$-class, then its degree of approximation by $C^1 \cdot N_p$ means of conjugate series of Fourier series* (1.2) *is given by*

$$\|\tilde{t}_n^{CN} - \tilde{f}\|_r = \mathrm{O}\left((n)^{\beta/2+r/2}\xi\left(\frac{1}{\sqrt{n}}\right)\right), \tag{3.1}$$

provided $\{p_n\}$ *satisfies*

$$V_n < C, \tag{3.2}$$

and $\xi(t)$ satisfies the following conditions:

$$\{\xi(t)/t\} \text{ is non-increasing in } t, \tag{3.3}$$

$$\left(\int_0^{\pi/\sqrt{n}} \left(\frac{|\psi(t)|}{\xi(t)}\right)^r \sin^{\beta r}(t/2)\,dt\right)^{1/r} = \mathrm{O}(1), \tag{3.4}$$

$$\left(\int_{\pi/\sqrt{n}}^{\pi} \left(\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right)^r dt\right)^{1/r} = \mathrm{O}\big((\sqrt{n})^{\delta}\big), \tag{3.5}$$

where δ is an arbitrary number such that $s(\beta-\delta)-1>0$, $r^{-1}+s^{-1}=1$, $1 \le r \le \infty$, conditions (3.4) *and* (3.5) *hold uniformly in x.*

Note 3.2 $\xi(\frac{\pi}{\sqrt{n}}) \le \pi\xi(\frac{1}{\sqrt{n}})$, for $(\frac{\pi}{\sqrt{n}}) \ge (\frac{1}{\sqrt{n}})$.

Note 3.3 Condition $V_n < C \Rightarrow np_n < CP_n$, [27].

Note 3.4 The product transform $C^1 \cdot N_p$ plays an important role in signal theory as a double digital filter [14] and theory of machines in mechanical engineering [14].

4 Lemmas

We need the following lemmas for the proof of our theorem.

Lemma 4.1 $|\tilde{M}_n(t)| = \mathrm{O}[1/t]$ *for* $0 < t \le \pi/\sqrt{n}$.

Proof For $0 < t \le \pi/\sqrt{n}$, $\sin(t/2) \ge (t/\pi)$ and $|\cos nt| \le 1$.

$$\begin{aligned}
|\tilde{M}_n(t)| &= \left|\frac{1}{2\pi(n+1)}\sum_{k=0}^{n} P_k^{-1}\sum_{\nu=0}^{k} p_\nu \frac{\cos(k-\nu+1/2)t}{\sin t/2}\right| \\
&\le \frac{1}{2\pi(n+1)}\sum_{k=0}^{n} P_k^{-1}\sum_{\nu=0}^{k} p_\nu \frac{|\cos(k-\nu+1/2)t|}{|\sin t/2|} \\
&\le \frac{1}{2t(n+1)}\sum_{k=0}^{n} P_k^{-1}\sum_{\nu=0}^{k} p_\nu = \frac{1}{2t(n+1)}\sum_{k=0}^{n} P_k^{-1}P_k \\
&= \mathrm{O}[\tau].
\end{aligned}$$

This completes the proof of Lemma 4.1. □

Lemma 4.2 *Let* $\{p_n\}$ *be a non-negative sequence and satisfy* (3.2), *then*

$$\left|\tilde{M}_n(t)\right| = \mathrm{O}(\tau) + \mathrm{O}\left(\frac{\tau^2}{n}\right)\left(\sum_{k=\tau}^{n} P_k^{-1} \sum_{\nu=0}^{k-1} |\Delta p_\nu|\right) \quad \textit{uniformly in } 0 < t \le \pi. \tag{4.1}$$

Proof We have

$$\begin{aligned}
\tilde{M}_n(t) &= \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} P_k^{-1} \sum_{\nu=0}^{k} p_\nu \frac{\cos(k-\nu+1/2)t}{\sin t/2} \\
&= \frac{1}{2\pi(n+1)} \left(\sum_{k=0}^{\tau-1} + \sum_{k=\tau}^{n}\right) P_k^{-1} \sum_{\nu=0}^{k} p_\nu \frac{\cos(k-\nu+1/2)t}{\sin t/2} \\
&= \left(\tilde{M}_n(t)\right)_1 + \left(\tilde{M}_n(t)\right)_2, \quad \text{say,}
\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}
\left|\left(\tilde{M}_n(t)\right)_1\right| &= \left|\frac{1}{2\pi(n+1)} \sum_{k=0}^{\tau-1} P_k^{-1} \sum_{\nu=0}^{k} p_\nu \frac{\cos(k-\nu+1/2)t}{\sin t/2}\right| \\
&\le \frac{1}{2\pi(n+1)} \sum_{k=0}^{\tau-1} P_k^{-1} \sum_{\nu=0}^{k} p_\nu \frac{|\cos(k-\nu+1/2)t|}{|\sin t/2|} \\
&\le \frac{1}{2t(n+1)} \sum_{k=0}^{\tau-1} P_k^{-1} \sum_{\nu=0}^{k} p_\nu \\
&= \mathrm{O}\left(\frac{\tau^2}{(n+1)}\right),
\end{aligned} \tag{4.3}$$

and using Abel's transformation and $\sin(t/2) \ge (t/\pi)$, for $0 < t \le \pi$, we get

$$\begin{aligned}
\left|\left(\tilde{M}_n(t)\right)_2\right| &= \left|\frac{1}{2\pi(n+1)} \sum_{k=\tau}^{n} P_k^{-1} \sum_{\nu=0}^{k} p_\nu \frac{\cos(k-\nu+1/2)t}{\sin t/2}\right| \\
&\le \frac{1}{2t(n+1)} \sum_{k=\tau}^{n} P_k^{-1} \left\{\sum_{\nu=0}^{k-1} |\Delta p_\nu| \left|\left(\sum_{\gamma=0}^{\nu} \cos(k-\gamma+1/2)t\right)\right|\right. \\
&\quad \left. + \left|\left(\sum_{\gamma=0}^{k} \cos(k-\gamma+1/2)t\right)\right| p_k\right\} \\
&= \frac{\mathrm{O}(t^{-1})}{2t(n+1)} \left(\sum_{k=\tau}^{n} P_k^{-1} \sum_{\nu=0}^{k-1} |\Delta p_\nu| + \sum_{k=\tau}^{n} P_k^{-1} p_k\right)
\end{aligned}$$

by virtue of the fact that $\sum_{k=\lambda}^{\mu} \exp(-ikt) = \mathrm{O}(t^{-1})$, $0 \le \lambda \le k \le \mu$.

$$\begin{aligned}
\left|\left(\tilde{M}_n(t)\right)_2\right| &= \mathrm{O}\left(\frac{\tau^2}{(n+1)}\right)\left(\sum_{k=\tau}^{n} P_k^{-1} \sum_{\nu=0}^{k-1} |\Delta p_\nu| + \sum_{k=\tau}^{n} P_k^{-1} p_k \frac{k}{k}\right) \\
&= \mathrm{O}\left(\frac{\tau^2}{(n+1)}\right)\left(\sum_{k=\tau}^{n} P_k^{-1} \sum_{\nu=0}^{k-1} |\Delta p_\nu| + \frac{(n+1)}{\tau}\right)
\end{aligned}$$

$$= \mathrm{O}(\tau) + \mathrm{O}\left(\frac{\tau^2}{(n+1)}\right)\sum_{k=\tau}^{n} P_k^{-1}\sum_{\nu=0}^{k-1}|\Delta p_\nu|,$$

$$\left|\left(\tilde{M}_n(t)\right)_2\right| = \mathrm{O}(\tau) + \mathrm{O}\left(\frac{\tau^2}{n}\right)\sum_{k=\tau}^{n} P_k^{-1}\sum_{\nu=0}^{k-1}|\Delta p_\nu| \tag{4.4}$$

in view of Note 3.3. Combining (4.2), (4.3) and (4.4) yields (4.1).

This completes the proof of Lemma 4.2. □

5 Proof of theorem

Let $\tilde{s}_n(f;x)$ denote the partial sum of series (1.2), we have

$$\tilde{s}_n(f;x) - \tilde{f}(x) = \frac{1}{2\pi}\int_0^{\pi}\psi(t)\frac{\cos(n+1/2)t}{\sin t/2}\,dt.$$

Denoting $C^1 \cdot N_p$ means of $\tilde{s}_n(f;x)$ by $\tilde{t}_n^{CN}$, we write

$$\begin{aligned}
\tilde{t}_n^{CN}(x) - \tilde{f}(x) &= \int_0^{\pi}\psi(t)\frac{1}{2\pi(n+1)}\sum_{k=0}^{n} P_k^{-1}\sum_{\nu=0}^{k} p_\nu\frac{\cos(k-\nu+1/2)t}{\sin t/2}\,dt \\
&= \int_0^{\pi}\psi(t)\tilde{M}_n(t)\,dt \\
&= \left[\int_0^{\pi/\sqrt{n}} + \int_{\pi/\sqrt{n}}^{\pi}\right]\psi(t)\tilde{M}_n(t)\,dt \\
&= I_1 + I_2 \quad \text{(say)}.
\end{aligned} \tag{5.1}$$

Clearly, $|\psi(x+t) - \psi(t)| \le |f(u+x+t) - f(u+x)| + |f(u-x-t) - f(u-x)|$.

Hence, by Minkowski's inequality,

$$\begin{aligned}
&\left\{\int_0^{2\pi}\left|\left(\psi(x+t) - \psi(t)\right)\sin^{\beta}(x/2)\right|^r dx\right\}^{1/r} \\
&\quad\le \left\{\int_0^{2\pi}\left|\left(f(u+x+t) - f(u+x)\right)\sin^{\beta}(x/2)\right|^r dx\right\}^{1/r} \\
&\qquad + \left\{\int_0^{2\pi}\left|\left(f(u-x-t) - f(u-x)\right)\sin^{\beta}(x/2)\right|^r dx\right\}^{1/r} = \mathrm{O}\left(\xi(t)\right).
\end{aligned}$$

Then $f \in W(L_r, \xi(t)) \Rightarrow \psi \in W(L_r, \xi(t))$.

Using Hölder's inequality, $\psi(t) \in W(L_r, \xi(t))$, condition (3.4), $\sin(t/2) \ge (t/\pi)$, for $0 < t \le \pi$, Lemma 4.1, Note 3.2 and second mean value theorem for integrals, we have

$$\begin{aligned}
|I_1| &\le \left[\int_0^{\pi/\sqrt{n}}\left(\frac{|\psi(t)\sin^{\beta}(t/2)|}{\xi(t)}\right)^r dt\right]^{1/r}\left[\int_0^{\pi/\sqrt{n}}\left(\frac{\xi(t)|\tilde{M}_n(t)|}{\sin^{\beta}(t/2)}\right)^s dt\right]^{1/s} \\
&= \mathrm{O}(1)\left[\int_0^{\pi/\sqrt{n}}\left(\frac{\xi(t)}{t^{1+\beta}}\right)^s dt\right]^{1/s} = \mathrm{O}\left\{\xi\left(\frac{\pi}{\sqrt{n}}\right)\right\}\left[\int_h^{\pi/\sqrt{n}}\left(\frac{1}{t^{1+\beta}}\right)^s dt\right]^{1/s}, \quad \text{as } h \to 0 \\
&= \mathrm{O}\left((n)^{\beta/2+r/2}\xi\left(\frac{1}{\sqrt{n}}\right)\right), \quad r^{-1} + s^{-1} = 1.
\end{aligned} \tag{5.2}$$

Using Lemma 4.2, we have

$$|I_2| = \mathrm{O}\left[\int_{\pi/\sqrt{n}}^{\pi} \frac{|\psi(t)|}{t}\,dt\right] + \mathrm{O}\left[\int_{\pi/\sqrt{n}}^{\pi} \frac{|\psi(t)|}{tn}\left(\tau\sum_{k=\tau}^{n} P_k^{-1}\sum_{\nu=0}^{k-1}|p_\nu|\right)dt\right] = \mathrm{O}(I_{21}) + \mathrm{O}(I_{22}).$$

Using Hölder's inequality, conditions (3.3) and (3.5), $|\sin t| \le 1$, $\sin(t/2) \ge (t/\pi)$, for $0 < t \le \pi$, Note 3.2 and second mean value theorem for integrals, we have

$$\begin{aligned}
|I_{21}| &\le \left[\int_{\pi/\sqrt{n}}^{\pi}\left(\frac{t^{-\delta}|\psi(t)|\sin^{\beta}(t/2)}{\xi(t)}\right)^{r}dt\right]^{1/r}\left[\int_{\pi/\sqrt{n}}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+1}\sin^{\beta}(t/2)}\right)^{s}dt\right]^{1/s} \\
&= \mathrm{O}\big((\sqrt{n})^{\delta}\big)\left[\int_{\pi/\sqrt{n}}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+1+\beta}}\right)^{s}dt\right]^{1/s} = \mathrm{O}\big\{(\sqrt{n})^{\delta}\big\}\left[\int_{1/\pi}^{\sqrt{n}/\pi}\left(\frac{\xi(1/y)}{y^{\delta-1-\beta}}\right)^{s}\frac{dy}{y^{2}}\right]^{1/s} \\
&= \mathrm{O}\left((\sqrt{n})^{\delta}\frac{\xi(\pi/\sqrt{n})}{\pi/\sqrt{n}}\right)\left[\int_{1/\pi}^{\sqrt{n}/\pi}\left(\frac{dy}{y^{(\delta-\beta)s+2}}\right)\right]^{1/s} \\
&= \mathrm{O}\left((\sqrt{n})^{\delta+1}\xi\left(\frac{1}{\sqrt{n}}\right)\right)\left(\frac{(\sqrt{n})^{(\beta-\delta)s-1}-(\pi)^{(-\beta+\delta)s+1}}{(\beta-\delta)s-1}\right)^{1/s} \\
&= \mathrm{O}\left((n)^{\beta/2+r/2}\xi\left(\frac{1}{\sqrt{n}}\right)\right).
\end{aligned} \tag{5.3}$$

Similarly, as conditions (3.2), (3.3) and (3.5) above, $|\sin t| \le 1$, $\sin(t/2) \ge (t/\pi)$, for $0 < t \le \pi$, Note 3.2 and second mean value theorem for integrals, we have

$$\begin{aligned}
|I_{22}| &\le \left[\int_{\pi/\sqrt{n}}^{\pi}\left(\frac{t^{-\delta}|\psi(t)|\sin^{\beta}(t/2)}{\xi(t)}\right)^{r}dt\right]^{1/r} \\
&\quad\times\left[\int_{\pi/\sqrt{n}}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+1}\sin^{\beta}(t/2)}\frac{1}{n}\left(\tau\sum_{k=\tau}^{n}P_k^{-1}\sum_{\nu=0}^{k-1}|p_\nu|\right)\right)^{s}dt\right]^{1/s} \\
&= \mathrm{O}\left(\frac{(\sqrt{n})^{\delta}}{n}\right)\left[\int_{\pi/\sqrt{n}}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+1+\beta}}\left(\tau\sum_{k=\tau}^{n}P_k^{-1}\sum_{\nu=0}^{k-1}|p_\nu|\right)\right)^{s}dt\right]^{1/s} \\
&= \mathrm{O}\left(\frac{(\sqrt{n})^{\delta}}{n}\right)\left[\int_{\pi/\sqrt{n}}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+1+\beta}}\left(\sum_{k=\tau}^{n}P_k^{-1}\sum_{\nu=0}^{k-1}(\nu+1)|p_\nu|\right)\right)^{s}dt\right]^{1/s} \\
&= \mathrm{O}\left(\frac{(\sqrt{n})^{\delta}}{n}\right)\left[\int_{\pi/\sqrt{n}}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+1+\beta}}\left(\sum_{k=0}^{n}P_k^{-1}\sum_{\nu=0}^{k}(\nu+1)|p_\nu|\right)\right)^{s}dt\right]^{1/s} \\
&= \mathrm{O}\left(\frac{(\sqrt{n})^{\delta}}{n}\right)\left[\int_{\pi/\sqrt{n}}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+1+\beta}}V_n 2\pi(n)\right)^{s}dt\right]^{1/s} \\
&= \mathrm{O}\big((\sqrt{n})^{\delta}\big)\left[\int_{\pi/\sqrt{n}}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+1+\beta}}\right)^{s}dt\right]^{1/s} \\
&= \mathrm{O}\big\{(\sqrt{n})^{\delta}\big\}\left[\int_{1/\pi}^{\sqrt{n}/\pi}\left(\frac{\xi(1/y)}{y^{\delta-1-\beta}}\right)^{s}\frac{dy}{y^{2}}\right]^{1/s} \\
&= \mathrm{O}\left((n)^{\beta/2+r/2}\xi\left(\frac{1}{\sqrt{n}}\right)\right).
\end{aligned} \tag{5.4}$$

Collecting (5.1)-(5.4), we have

$$\left|\tilde{t}_n^{CN} - \tilde{f}\right| = O\left((n)^{\beta/2+r/2}\xi\left(\frac{1}{\sqrt{n}}\right)\right). \tag{5.5}$$

Now, using the L_r-norm of a function, we get

$$\begin{aligned}
\left\|\tilde{t}_n^{CN} - \tilde{f}\right\|_r &= \left\{\int_0^{2\pi} \left|\tilde{t}_n^{CN}(x) - \tilde{f}(x)\right|^r dx\right\}^{1/r} \\
&= O\left(\int_0^{2\pi} \left((n)^{\beta/2+r/2}\xi\left(\frac{1}{\sqrt{n}}\right)\right)^r dx\right)^{1/r} \\
&= O\left((n)^{\beta/2+r/2}\xi\left(\frac{1}{\sqrt{n}}\right)\left(\int_0^{2\pi} dx\right)^{1/r}\right) \\
&= O\left((n)^{\beta/2+r/2}\xi\left(\frac{1}{\sqrt{n}}\right)\right).
\end{aligned}$$

This completes the proof of Theorem 3.1.

6 Applications

The theory of approximation is a very extensive field, which has various applications. As mentioned in [24], the Lp-space in general and L_2 and L_∞ in particular play an important role in the theory of signals and filters. From the point of view of the applications, the Sharper estimates of infinite matrices are useful to get bounds for the lattice norms (which occur in solid state physics) of matrix-valued functions and enable to investigate perturbations of matrix-valued functions and compare them. Some interesting applications of the Cesàro summability can be seen in [2–4]. The following corollaries can be derived from Theorem 3.1.

Corollary 1 *If* $\beta = 0$, *then the generalized weighted Lipschitz* $W(L^r, \xi(t))$ $(r \geq 1)$*-class reduces to the class* $\mathrm{Lip}(\xi(t), r)$, *and the degree of approximation of a function* $\tilde{f}(x)$, *conjugate to a* 2π*-periodic function* f *belonging to the class* $\mathrm{Lip}(\xi(t), r)$, *is given by*

$$\left|\tilde{t}_n^{CN}(f;x) - \tilde{f}(x)\right| = O\left(n^{r/2}\xi(1/\sqrt{n})\right). \tag{6.1}$$

Proof The result follows by setting $\beta = 0$ in Theorem 3.1, we have

$$\left\|\tilde{t}_n^{CN}(f;x) - \tilde{f}(x)\right\|_r = \left\{\int_0^{2\pi} \left|\tilde{t}_n^{CN}(f;x) - \tilde{f}(x)\right|^r dx\right\}^{1/r} = O\left(n^{r/2}\xi(1/\sqrt{n})\right), \quad r \geq 1.$$

Thus, we get

$$\left|\tilde{t}_n^{CN}(f;x) - \tilde{f}(x)\right| \leq \left\{\int_0^{2\pi} \left|\tilde{t}_n^{CN}(x) - \tilde{f}(x)\right|^r dx\right\}^{1/r} = O\left(n^{r/2}\xi(1/\sqrt{n})\right), \quad r \geq 1.$$

This completes the proof of Corollary 1. □

Corollary 2 *If* $\beta = 0$, $\xi(t) = t^{\alpha}$, $0 < \alpha \leq 1$, *then the generalized weighted Lipschitz* $W(L^r, \xi(t))$ $(r \geq 1)$*-class reduces to the class* $\mathrm{Lip}(\alpha, r)$, $(1/r) < \alpha < 1$ *and the degree of approximation of a function* $\tilde{f}(x)$, *conjugate to a* 2π*-periodic function* f *belonging to the class* $\mathrm{Lip}(\alpha, r)$, *is given by*

$$\left|\tilde{t}_n^{CN}(f;x) - \tilde{f}(x)\right| = \mathrm{O}\bigl((n)^{-\alpha/2 + r/2}\bigr).$$

Proof Putting $\beta = 0$, $\xi(t) = t^{\alpha}$, $0 < \alpha \leq 1$ in Theorem 3.1, we have

$$\left\|\tilde{t}_n^{CN}(f;x) - \tilde{f}(x)\right\|_r = \left\{\int_0^{2\pi} \left|\tilde{t}_n^{CN}(f;x) - \tilde{f}(x)\right|^r dx\right\}^{1/r}$$

or,

$$\mathrm{O}\left(n^{\beta/2 + r/2}\xi\left(\frac{1}{\sqrt{n}}\right)\right) = \left\{\int_0^{2\pi} \left|\tilde{t}_n^{CN}(f;x) - \tilde{f}(x)\right|^r dx\right\}^{1/r}$$

or,

$$\mathrm{O}(1) = \left\{\int_0^{2\pi} \left|\tilde{t}_n^{CN}(f;x) - \tilde{f}(x)\right|^r dx\right\}^{1/r} \mathrm{O}\left(\frac{1}{n^{\beta/2 + r/2}\xi(\frac{1}{\sqrt{n}})}\right),$$

since otherwise, the right-hand side of the equation above will not be O(1).

Hence

$$\left|\tilde{t}_n^{CN}(f;x) - \tilde{f}(x)\right| = \mathrm{O}\bigl(n^{-\alpha/2} n^{r/2}\bigr) = \mathrm{O}\bigl(n^{-\alpha/2 + r/2}\bigr).$$

This completes the proof of Corollary 2. □

Corollary 3 *If* $\beta = 0$, $\xi(t) = t^{\alpha}$ *for* $0 < \alpha < 1$ *and* $r \to \infty$ *in* (3.1), *then* $f \in \mathrm{Lip}\,\alpha$. *In this case, the degree of approximation of a function* $\tilde{f}(x)$, *conjugate to a* 2π*-periodic function* f *belonging to the class* $\mathrm{Lip}\,\alpha$ $(0 < \alpha < 1)$ *is given by*

$$\left|\tilde{t}_n^{CN}(f;x) - \tilde{f}(x)\right| = \mathrm{O}\bigl((n)^{-\alpha/2}\bigr).$$

Proof For $r \to \infty$ in Corollary 2, we get

$$\left\|\tilde{t}_n^{CN}(f;x) - \tilde{f}(x)\right\|_\infty = \sup_{0 \leq x \leq 2\pi} \left|\tilde{t}_n^{CN}(f;x) - \tilde{f}(x)\right| = \mathrm{O}\bigl((n)^{-\alpha/2}\bigr).$$

Thus, we have

$$\begin{aligned}\left|\tilde{t}_n^{CN}(f;x) - \tilde{f}(x)\right| &\leq \left\|\tilde{t}_n^{CN}(f;x) - \tilde{f}(x)\right\|_\infty \\ &= \sup_{0 \leq x \leq 2\pi} \left|\tilde{t}_n^{CN}(f;x) - \tilde{f}(x)\right| = \mathrm{O}\bigl((n)^{-\alpha/2}\bigr).\end{aligned}$$

This completes the proof of Corollary 3. □

Examples (i) From Theorem 20 of Hardy's 'Divergent Series,' if a Nörlund method (N,p) has increasing weights $\{p_n\}$, then it is stronger than $(C,1)$.

Therefore, there is an easy way to find a sequence that is $(C,1)$ summable, but not convergent. One such sequence is

$$s_n := \sum_{k=0}^{n} (-1)^k.$$

Here $s_1 = 1$, $s_2 = 0$, $s_3 = 1$, $s_4 = 0, \ldots$.

For $(C,1)$ summability,

$$\sigma_n^1 = \frac{s_1 + s_2 + \cdots + s_n}{n}.$$

Case (i) If n is even, *i.e.*, $n = 2m$, then

$$\begin{aligned}\sigma_n^1 &= \frac{s_1 + s_2 + \cdots + s_{2m}}{2m} \\ &= \frac{(1 + 0 + 1 + \cdots + 1 + 0)}{2m} \\ &= \frac{m}{2m} = \frac{1}{2}.\end{aligned}$$

Case (ii) If n is odd, *i.e.*, $n = 2m + 1$, then

$$\begin{aligned}\sigma_n^1 &= \frac{s_1 + s_2 + \cdots + s_{2m+1}}{2m+1} \\ &= \frac{(1 + 0 + 1 + \cdots + 0 + 1)}{2m+1} \\ &= \frac{1}{2}.\end{aligned}$$

Thus, number $\frac{1}{2}$ is assigned to the infinite series $\sum (-1)^n$.

This sequence is summable $(C,1)$ to $1/2$, but is clearly not convergent.

Now, take (N,p) to be the Nörlund matrix generated by $p_n = n + 1$. Then this sequence is summable by (N,p) method.

For

$$N_n(f;x) = \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} (n - k + 1) s_k(f;x) = \begin{cases} 0, & k \text{ even}, \\ 1, & k \text{ odd}. \end{cases}$$

Since it is summable $(C,1)$, also, and (N,p) method is stronger than $(C,1)$ method. Therefore, it is then summable by the product $(C,1)(N,p)$ method.

(ii) In the first example, we discuss the case in which the Nörlund method is stronger than $(C,1)$. The product C_1N sums a divergent sequence, where N is the Nörlund method.

We know already that $(C,1)$ sums the sequence $s_n = \sum_{k=0}^{n} (-1)^k$. We will now show that CN is stronger than C, for any regular Nörlund matrix (with non-negative generating sequence).

The statement CN is stronger than C is equivalent to showing that the matrix CNC^{-1} is regular.

Since C^{-1} is a bidiagonal matrix $c^{-1}_{nn} = n+1$ and $c^{-1}_{n+1} = -(n-1)$, for $k < n$,

$$\begin{aligned}
(CNC^{-1})_{n,k} &= \sum_{j=k}^{n}(CN)_{nj}(C^{-1})_{jk} \\
&= (CN)_{nk}(C^{-1})_{kk} + (CN)_{n,k+1}(C^{-1})_{k+1,k} \\
&= (CN)_{nk}(k+1) + (CN)_{n,k+1}\bigl(-(k+1)\bigr) \\
&= (k+1)\bigl((CN)_{nk} - (CN)_{n,k+1}\bigr) \\
&= (k+1)\left[\sum_{j=k}^{n}\frac{1}{(n+1)}\frac{p_{j-k}}{P_j} - \sum_{j=k+1}^{n}\frac{1}{(n+1)}\frac{p_{j-k}}{P_j}\right] \\
&= \left(\frac{k+1}{n+1}\right)\frac{p_0}{P_n} > 0.
\end{aligned}$$

Also,

$$(CNC^{-1})_{nn} = \left(\frac{1}{n+1}\right)\frac{p_0}{P_n}(n+1) = \frac{p_0}{P_n} > 0.$$

Therefore,

$$\|CNC^{-1}\| = CNC^{-1}(e),$$

where e denotes the sequence of all ones. $C^{-1}(e) = e$, so that $CN(e) = C(N(e)) = Ce = e$. Thus, not only does CNC^{-1} have a finite norm, but it also has row sums equal to one.

Let $\{e^k\}$ denote the column vector with one in position k and zeros elsewhere. Then

$$b_n = C^{-1}e^k = \begin{cases} 0, & n < k, \\ -(k+1), & n = k, \\ (k+1), & n = k+1, \\ 0, & n > k+1. \end{cases}$$

Therefore, $\{b_n\}$ is a null sequence. Since N is a regular Nörlund matrix, with $b := \{b_n\}$, $t = N(b)$ is a null sequence. Since C is also regular. Ct is a null sequence, and CNC^{-1} has null columns. Therefore, it is regular, and CN is stronger than C.

7 Conclusion

Various results concerning the degree of approximation of periodic signals (functions) belonging to the generalized weighted class by matrix operator have been reviewed, and the condition of monotonicity on the weights $\{p_n\}$ has been relaxed (*i.e.*, weakening the conditions on the filter, we improve the quality of digital filter). Further, a proper (correct) set of conditions has been discussed to rectify the errors pointed out in Remarks 2.2 and 2.3. Some interesting application of the operator $(C^1 \cdot N_p)$ used in this paper is pointed out in Note 3.4. The theorem of this paper is an attempt to formulate the problem of approximation of the function $\tilde{f}$, conjugate to a periodic function $f \in W(L_r, \xi(t))$ $(r \geq 1)$-class through trigonometric polynomials generated by the product summability $(C^1 \cdot N_p)$-transform of conjugate series of Fourier series of f in a simpler manner by dropping monotonicity on the generating sequence $\{p_n\}$ with the help of proper (new) set of conditions. Few applications and examples on product summability $(C^1 \cdot N_p)$-transform of signals (functions) are also discussed in this manuscript.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
LNM computed lemmas and established the main theorem with a proper (a new) set of conditions to remove the errors in this direction. LNM studied examples (i) and (ii) in its depth. LNM and VNM conceived of the study and participated in its design and coordination. LNM, VNM contributed equally and significantly in writing this paper. All the authors, *i.e.*, VNM, VS and LNM drafted the manuscript, read and approved the final manuscript.

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Acknowledgements
The authors would like to thank the anonymous learned referees for several useful interesting comments and suggestions about the paper. Special thanks are due to our great master and friend academician Prof. Ravi P. Agarwal, Texas A&M University, Kingsville, TX, USA, for kind cooperation, smooth behavior during communication and for his efforts to send the reports of the manuscript timely.

Received: 23 January 2013 Accepted: 30 July 2013 Published: 18 September 2013

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doi:10.1186/1029-242X-2013-440
Cite this article as: Mishra et al.: L_r-**Approximation of signals (functions) belonging to weighted** $W(L_r, \xi(t))$**-class by** $C^1 \cdot N_p$ **summability method of conjugate series of its Fourier series.** *Journal of Inequalities and Applications* 2013 **2013**:440.