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On variational inequalities with vanishing zero term

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Abstract

In this paper, we are concerned with variational inequalities (VIs), where the 'discount factor' (*i.e.*, the zero-order term) is set to zero. Especially, we introduce a new method for studying the finite element approximation, based on an algorithm of Bensoussan-Lions type and the concept of subsolutions.

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Keywords: variational inequality; Bensoussan-Lions algorithm; finite element; subsolutions; L^∞- error estimate

1 Introduction

Ergodic control problems may be solved by considering the variational inequality (VI)

$$\begin{cases} -\mathcal{A}_\alpha u_\alpha \le f, \qquad u_\alpha \le \psi & \text{in } \Omega, \\ (u_\alpha - \psi)(-\mathcal{A}_\alpha u_\alpha - f) = 0 & \text{in } \Omega, \\ \dfrac{\partial u_\alpha}{\partial n} = 0 & \text{on } \Gamma \end{cases} \tag{1.1}$$

as α tends to 0^+.

Here Ω is a bounded domain of R^N with smooth boundary Γ, f is a positive right-hand side in $L^\infty(\Omega)$, ψ is a positive obstacle in $W^{2,\infty}(\Omega)$ such that $\partial\psi/\partial n \ge 0$ on Γ, where n is the outward normal, and

$$-\mathcal{A}_\alpha = -\Delta + \alpha I. \tag{1.2}$$

Such problems play a fundamental role in the solution of problems of stochastic control with ergodic control type payoffs (see [1] and the references therein).

For $\alpha > 0$, there exists a unique solution u_α of (1.1) which belongs to $W^{2,p}(\Omega)$, $2 \le p < \infty$ (see [2]).

Let $(\cdot,\cdot)$ denote the inner product in $L^2(\Omega)$, let $a(\cdot,\cdot)$ be the bilinear form

$$a(u,v) = (\nabla u.\nabla v),$$

and

$$\mathbb{K} = \{v \in H^1(\Omega) \text{ such that } v \le \psi \text{ a.e. in } \Omega\}. \tag{1.3}$$

The weak formulation of (1.1) being

$$a(u_\alpha, v - u_\alpha) + \alpha(u_\alpha, v - u_\alpha) \geq (f, v - u_\alpha) \quad \forall v \in \mathbb{K}, \tag{1.4}$$

it can be shown that u_α converges uniformly in $C(\bar{\Omega})$, as $\alpha \to 0^+$, to u_0, the solution of the VI

$$a(u_0, v - u_0) \geq (f, v - u_0) \quad \forall v \in \mathbb{K}. \tag{1.5}$$

Also, denoting by $\mathbb{V}_h$ the finite element space consisting of continuous piecewise linear functions, r_h, the usual interpolation operator, and by

$$\mathbb{K}_h = \{v \in \mathbb{V}_h \text{ such that } v \leq r_h \psi\}, \tag{1.6}$$

it can be proved that the solution $u_{\alpha h} \in \mathbb{K}_h$ of the discrete VI

$$a(u_{\alpha h}, v - u_{\alpha h}) + \alpha(u_{\alpha h}, v - u_{\alpha h}) \geq (f, v - u_{\alpha h}) \quad \forall v \in \mathbb{K}_h \tag{1.7}$$

converges uniformly in $C(\bar{\Omega})$, as $\alpha \to 0^+$, to u_{0h}, the solution of the VI

$$a(u_{0h}, v - u_{0h}) \geq (f, v - u_{0h}) \quad \forall v \in \mathbb{K}. \tag{1.8}$$

In this paper, our primary aim is to study the finite element approximation in the L^∞ norm for VIs (1.4) and (1.5). More precisely, we establish the following optimal L^∞ error estimates:

$$\|u_\alpha - u_{\alpha h}\|_\infty \leq Ch^2 |\ln h|^2 \tag{1.9}$$

and, as $\alpha \to 0^+$,

$$\|u_0 - u_{0h}\|_\infty \leq Ch^2 |\ln h|^2, \tag{1.10}$$

where C is a constant independent of both α and h and $\|\cdot\|_\infty$ denotes the L^∞-norm.

The finite element approximation of variational inequalities with vanishing zero-order term was first studied in [3] and L^∞ error estimates were derived by means of the concept of subsolutions. In [4], a quasi-optimal convergence order was derived by adapting an algorithmic approach due to [5] for quasi-variational of impulse control problems. This method combines the approximation of both the continuous and discrete solutions of VIs (1.4) and (1.7) by monotone geometrically convergent iterative schemes of Bensoussan-Lions type and an estimation of the error in the maximum norm between the nth iterate of the iterative scheme and its finite element counterpart.

In this paper, this algorithmic approach is improved and optimal convergence order is derived. Besides the optimal convergence order (1.9) and (1.10), the other important novelty in this paper is the optimal L^∞ error estimate that is established between the nth iterate of the iterative scheme and its finite element counterpart, which we achieve by employing the concept of subsolutions.

It is worth mentioning that the approach introduced in this paper is new and different from the one employed in [3]. Moreover, it also has the merit of providing a basic computational scheme for the solution of (1.8). It may also be extended to the quasi-variational inequality of ergodic impulse control problems studied by Lions and Perthame [6].

The paper is organized as follows. In Section 2, we construct a monotone iterative scheme and establish its geometrical convergence to the solution of VI (1.4). In Section 3, the same study is reproduced for the discrete problems. Section 4 is devoted to the finite element error analysis and proofs of the main results of this paper.

2 The continuous problem

Let α be *fixed* in the open interval $(0,1)$, $\lambda = 1 - \alpha$. Then (1.4) is equivalent to the VI

$$b(u_\alpha, v - u_\alpha) \geq (f + \lambda u_\alpha, v - u_\alpha) \quad \forall v \in \mathbb{K} \tag{2.1}$$

with

$$b(u,v) = a(u,v) + (u,v). \tag{2.2}$$

Notice that the bilinear form (2.2) is independent of α, as its zero-order term is equal to 1.

2.1 Construction of monotone sequences for VI (1.4)

Let us first consider the following mapping:

$$\begin{aligned} &T : L^\infty(\Omega) \longrightarrow L^\infty(\Omega) \\ &w \rightarrow Tw = \zeta, \end{aligned} \tag{2.3}$$

where $\zeta \in \mathbb{K}$ is the unique solution of the following VI

$$b(\zeta, v - \zeta) \geq (f + \lambda w, v - \zeta) \quad \forall v \in \mathbb{K}. \tag{2.4}$$

So, we obviously have

$$u_\alpha = Tu_\alpha. \tag{2.5}$$

Let $\hat{u}_0$ be the solution of the equation

$$b(\hat{u}_0, v) = (f + \psi, v) \quad \forall v \in H^1(\Omega). \tag{2.6}$$

Thanks to [2], problem (2.6) has a unique solution which belongs to $W^{2,p}(\Omega)$, $2 \leq p < \infty$. Moreover, $\hat{u}_0$ is independent of α, as the bilinear form (2.2) is itself independent of α.

Lemma 1 *Let* $\mathbb{C} = \{w \in L^\infty(\Omega)$ *such that* $0 \leq w \leq \hat{u}_0\}$. *Then the mapping* T *is increasing, concave, and satisfies* $Tw \leq \hat{u}_0$, $\forall w \in \mathbb{C}$.

Proof It is an easy adaptation of [2]. □

Now, starting from $\hat{u}_0$, the solution of (2.6), and from $\check{u}_0 = 0$, we define the sequences

$$\hat{u}_\alpha^n = T\hat{u}_\alpha^{n-1}, \quad \forall n \geq 1, \tag{2.7}$$

and

$$\check{u}_\alpha^n = T\check{u}_\alpha^{n-1}, \quad \forall n \geq 1, \tag{2.8}$$

respectively.

As a result of Lemma 1, it is clear that both sequences $(\hat{u}_\alpha^n)$ and $(\check{u}_\alpha^n)$ are well defined in $\mathbb{C}$. Moreover, they are monotone decreasing and increasing, respectively.

Next, we shall establish the geometrical convergence of these sequences.

2.2 Geometrical convergence

Lemma 2 *The solution u_α of VI* (1.4) *or* (2.1) *satisfies* $0 < u_\alpha \leq \hat{u}_0$.

Proof First, notice that $u_\alpha > 0$ as f and ψ are both positive. On the other hand, as $u_\alpha \leq \psi$, by taking $v = u_\alpha - (u_\alpha - \hat{u}_0)^+$ in (2.1) and $v = (u_\alpha - \hat{u}_0)^+$ in (2.6), we obtain by addition

$$b\big(\hat{u}_0 - \hat{u}_\alpha, (u_\alpha - \hat{u}_0)^+\big) \geq \big((\psi - \lambda u_\alpha), (u_\alpha - \hat{u}_0)^+\big) \geq 0.$$

So,

$$b\big((u_\alpha - \hat{u}_0)^+, (u_\alpha - \hat{u}_0)^+\big) \leq 0,$$

which, thanks to the coercivity of the bilinear form $b(\cdot,\cdot)$, implies

$$(u_\alpha - \hat{u}_0)^+ = 0.$$

Thus,

$$u_\alpha \leq \hat{u}_0.$$ □

Lemma 3 *Let* $\beta = \inf \psi > 0$ *on* $\bar{\Omega}$, *and assume that* $f \geq f_0 > 0$, *where* f_0 *is a positive constant, and*

$$w - \tilde{w} \leq \gamma w \quad \forall w, \tilde{w} \in \mathbb{C}, \gamma \in [0,1]. \tag{2.9}$$

Then

$$Tw - T\tilde{w} \leq \gamma(1-\mu)Tw, \tag{2.10}$$

where

$$0 < \mu < \min\left(\frac{\beta}{\|\hat{u}_0\|_\infty}, \frac{f_0}{\|\psi\|_\infty + f_0}\right). \tag{2.11}$$

Proof Let us first show that

$$\check{u}_1 = T(0) \geq \mu\hat{u}_0.$$

Indeed, in view of the choice of μ in (2.11), it is clear that $\mu\hat{u}^0$ can be taken as a test function for the VI whose $\check{u}_1$ is a solution. So, $\check{u}_1 + (\check{u}_1 - \mu\hat{u}_0)^-$ is also a test function for that VI, and we then have

$$b\big(\check{u}_1, (\check{u}_1 - \mu\hat{u}_0)^-\big) \geq \big(f, (\check{u}_1 - \mu\hat{u}_0)^-\big). \tag{2.12}$$

Also, taking $v = (\check{u}_1 - \mu\hat{u}_0)^-$ as a test function in equation (2.6), we get

$$b\big(\hat{u}_0, (\check{u}_1 - \mu\hat{u}_0)^-\big) = \big(f + \psi, (\check{u}_1 - \mu\hat{u}_0)^-\big), \tag{2.13}$$

which, multiplied by $-\mu$, yields

$$b\big(-\mu\hat{u}_0, (\check{u}_1 - \mu\hat{u}_0)^-\big) = \big(-\mu f - \mu\psi, (\check{u}_1 - \mu\hat{u}_0)^-\big). \tag{2.14}$$

So, by addition, we obtain

$$\begin{aligned} b\big(\check{u}_1 - \mu\hat{u}_0, (\check{u}_1 - \mu\hat{u}_0)^-\big) &\geq \big(f(1-\mu) - \mu\psi, (\check{u}_1 - \mu\hat{u}_0)^-\big) \\ &\geq \big(f_0(1-\mu) - \psi, (\check{u}_1 - \mu\hat{u}_0)^-\big) \end{aligned}$$

because $\psi > 0$ and $0 < \mu < 1$.

But in view of the choice of μ, we have

$$f_0(1-\mu) > \mu\|\psi\|_\infty \geq \mu\psi.$$

So,

$$f_0(1-\mu) - \mu\psi \geq 0,$$

and therefore

$$b\big((\check{u}_1 - \mu\hat{u}_0)^-, (\check{u}_1 - \mu\hat{u}_0)^-\big) \leq 0.$$

Thus, by the coercivity of $b(\cdot,\cdot)$, we get

$$(\check{u}_1 - \mu\hat{u}_0)^- = 0$$

and hence

$$\check{u}_1 \geq \mu\hat{u}_0.$$

We are now in a position to prove (2.10). Indeed, (2.9) implies

$$(1-\gamma)w \leq \tilde{w},$$

and since T is nondecreasing, we have

$$T\tilde{w} \geq T(1-\gamma)w.$$

So, using the concavity of T, we get

$$\begin{aligned} T\tilde{w} &\geq T\big((1-\gamma)w + \gamma 0\big) \\ &\geq (1-\gamma)Tw + \gamma T(0) \\ &\geq Tw - \gamma Tw + \gamma\mu\hat{u}_0, \end{aligned}$$

and since $Tw \leq \hat{u}_0$, we have

$$\begin{aligned} Tw - T\tilde{w} &\leq \gamma Tw - \gamma\mu\hat{u}_0 \\ &\leq \gamma Tw - \gamma\mu Tw \\ &\leq \gamma(1-\mu)Tw, \end{aligned}$$

which completes the proof. □

Remark 1 The constant μ defined in (2.11) is independent of α as ψ, f, and $\hat{u}_0$ are themselves independent of α.

Theorem 1 *The sequences defined in* (2.7) *and* (2.8) *converge geometrically to* u_α, *the unique solution of VI* (1.4), *that is*,

$$\|\hat{u}_\alpha^n - u_\alpha\|_\infty \leq (1-\mu)^n \|\hat{u}_0\|_\infty, \tag{2.15}$$

$$\|\check{u}_\alpha^n - u_\alpha\|_\infty \leq (1-\mu)^n \|\hat{u}_0\|_\infty. \tag{2.16}$$

Proof The proof will be carried out by induction. We shall give only the proof of (2.15) as that of (2.16) is similar. Indeed, we clearly have

$$0 \leq \hat{u}_0 - u_\alpha \leq \hat{u}_0.$$

Then, using (2.10) with $\gamma = 1$ and the fact that $u_\alpha = Tu_\alpha$, we get

$$T\hat{u}_0 - Tu_\alpha \leq (1-\mu)T\hat{u}_0$$

or

$$0 \leq \hat{u}_\alpha^1 - u_\alpha \leq (1-\mu)\hat{u}_\alpha^1 \leq (1-\mu)\hat{u}_0.$$

Now, assume that

$$0 \leq \hat{u}_\alpha^n - u_\alpha \leq (1-\mu)^n \hat{u}_0.$$

Then, making use of (2.10) with $\gamma = (1-\mu)^n$, we obtain

$$0 \le T\hat{u}_\alpha^n - Tu_\alpha \le (1-\mu)(1-\mu)^n T\hat{u}_\alpha^n$$

or

$$0 \le \hat{u}_\alpha^{n+1} - u_\alpha \le (1-\mu)^{n+1}\hat{u}_\alpha^{n+1} \le (1-\mu)^{n+1}\hat{u}_0.$$

Thus, (2.15) follows. □

Next, we shall give the existence and uniqueness for VI (1.5).

Theorem 2 *The solution u_α of VI* (1.4) *converges uniformly in $C(\bar{\Omega})$ and strongly in $H^1(\Omega)$, $\alpha \to 0^+$, to u_0, the unique solution of VI* (1.5).

Proof For uniqueness, see [1]. Let us give the existence.

First, set $g = f + \lambda u_\alpha$ in (2.1). Since $0 < u_\alpha \le \psi$, then g is uniformly bounded in $L^\infty(\Omega)$, *i.e.*,

$$\|g\|_\infty \le \|f + \lambda u_\alpha\|_\infty \le \|f\|_\infty + \lambda\|\psi\|_\infty.$$

So, using Lewy-Stampacchia inequality [2] associated with the operator

$$\mathcal{B}\varphi = -\triangle\varphi + \varphi,$$

we get

$$\mathcal{B}\psi \wedge (f + \lambda u_\alpha) \le \mathcal{B}u_\alpha \le f + \lambda u_\alpha,$$

where

$$f \wedge g = \inf(f, g).$$

Hence,

$$\|\mathcal{B}u_\alpha\|_\infty \le C \quad \text{(independent of } \alpha\text{)},$$

and thus

$$\|u_\alpha\|_{W^{2,p}(\Omega)} \le C \quad \text{(independent of } \alpha\text{)}, 2 \le p < \infty. \tag{2.17}$$

Consequently,

$$u_\alpha \to u_0 \quad \text{uniformly in } C(\bar{\Omega}), \tag{2.18}$$

and from (2.17), we also have

$$u_\alpha \to u_0 \quad \text{weakly in } H^1(\Omega). \tag{2.19}$$

Let us now pass to the limit. Indeed, since

$$a(u_\alpha, v - u_\alpha) + \alpha(u_\alpha, v - u_\alpha) \geq (f, v - u_\alpha),$$

then

$$a(u_\alpha, v) - a(u_\alpha, u_\alpha) + \alpha(u_\alpha, v) - \alpha(u_\alpha, u_\alpha) \geq (f, v) - (f, u_\alpha).$$

So, combining (2.18) and (2.19), we get

$$a(u_0, v) \geq \liminf_{\alpha} a(u_\alpha, u_\alpha) + (f, v - u_0) \geq a(u_0, u_0) + (f, v - u_0),$$

that is, u_0 solves (1.5). In addition, choosing $v = u_0$, we obtain

$$a(u_\alpha, u_\alpha) \to a(u_0, u_0),$$

and thus

$$\nabla u_\alpha \to \nabla u_0$$

proving the strong convergence in $H^1(\Omega)$. □

3 The discrete problem

We assume that Ω is polyhedral. The extension to the general case can be set up by the usual techniques (see [7]). Let τ_h be a regular and quasi-uniform triangulation of Ω consisting of triangles of diameter less than h. Let also $\{\varphi_i\}$, $i = 1, \dots, m(h)$, be the basis functions of $\mathbb{V}_h$, and $[b(\varphi_i, \varphi_j)]$ be the stiffness matrix associated with the bilinear form $b(\cdot,\cdot)$.

The discrete maximum principle assumption (DMP) We assume that the matrix $\mathbb{B} = (b(\varphi_i, \varphi_j))$ is an M-Matrix [8, 9].

It is not hard to see that VI (1.7) is equivalent to the VI: find $u_{\alpha h} \in \mathbb{K}_h$ such that

$$b(u_{\alpha h}, v - u_{\alpha h}) \geq (f + \lambda u_{\alpha h}, v - u_{\alpha h}) \quad \forall v \in \mathbb{K}_h. \tag{3.1}$$

As in the continuous case, we shall construct two discrete sequences and prove their geometrical convergence to the solution of VI (1.7).

3.1 Construction of monotone sequences for VI (1.7)

Indeed, consider the mapping

$$\begin{aligned} &T_h : L^\infty(\Omega) \longrightarrow \mathbb{V}_h, \\ &w \to T_h w = \zeta_h, \end{aligned} \tag{3.2}$$

where $\zeta_h \in \mathbb{K}_h$ is the unique solution of the following VI:

$$b(\zeta_h, v - \zeta_h) \geq (f + \lambda w, v - \zeta_h) \quad \forall v \in \mathbb{K}_h. \tag{3.3}$$

So, clearly,

$$u_{\alpha h} = T_h u_{\alpha h}. \tag{3.4}$$

Let $\hat{u}^0_{\alpha h}$ be the solution of

$$b(\hat{u}^0_{\alpha h}, v) = (f + \psi, v) \quad \forall v \in \mathbb{V}_h. \tag{3.5}$$

Lemma 4 *Let $\mathbb{C}_h = \{w \in L^\infty(\Omega)$ such that $0 \leq w \leq \hat{u}_{0h}\}$. Then, under the DMP, the mapping T_h is increasing, concave, and satisfies $0 \leq T_h w \leq \hat{u}_{0h}$, $\forall w \in \mathbb{C}_h$.*

Now, starting from $\hat{u}^0_{\alpha h}$ and $\check{u}_{0h} = 0$, we define the sequences

$$\hat{u}^n_{\alpha h} = T_h \hat{u}^{n-1}_{\alpha h}, \quad \forall n \geq 1, \tag{3.6}$$

and

$$\check{u}^n_{\alpha h} = T_h \check{u}^{n-1}_{\alpha h}, \quad \forall n \geq 1, \tag{3.7}$$

respectively.

Thanks to Lemma 4, the sequences $(\hat{u}^n_{\alpha h})$ and $(\check{u}^n_{\alpha h})$ are well defined in $\mathbb{C}_h$. Moreover, they are monotone decreasing and increasing, respectively.

3.1.1 Geometrical convergence

As in the continuous case, in order to establish the geometrical convergence of sequences (3.6) and (3.7), we shall need the following lemmas. Their proofs will be omitted as they are very similar to those of their respective continuous counterparts.

Lemma 5 *Let the DMP hold. Then the solution $u_{\alpha h}$ of VI* (1.7) *or* (3.1) *satisfies $0 < u_{\alpha h} \leq \hat{u}_{0h}$.*

Lemma 6 *Let $\mathbb{C}_h = \{w \in L^\infty(\Omega)$ such that $0 \leq w \leq \hat{u}_{0h}\}$. Then, under the DMP, the mapping T_h is increasing, concave, and satisfies $0 \leq T_h w \leq \hat{u}_{0h}$, $\forall w \in \mathbb{C}_h$.*

Remark 2 Thanks to Lemma 6, sequences (3.6) and (3.7) are well defined in $\mathbb{C}_h$. Moreover, they are monotone decreasing and increasing, respectively.

Lemma 7 *Assume that $f \geq f_0 > 0$, where f_0 is a positive constant, and*

$$w - \tilde{w} \leq \gamma w, \quad \forall w, \tilde{w} \in \mathbb{C}_h. \tag{3.8}$$

Then

$$T_h w - T_h \tilde{w} \leq \gamma(1 - \mu) T_h w, \tag{3.9}$$

where

$$0 < \mu < \min\left(\frac{\beta}{\|\hat{u}_{0h}\|_\infty}, \frac{f_0}{\|\psi\|_\infty + f_0}\right). \tag{3.10}$$

Remark 3 The constant μ is independent of α as ψ, f, and $\hat{u}_{0h}$ are themselves independent of α.

Theorem 3 *Sequences* (3.6) *and* (3.7) *converge, geometrically, to* $u_{\alpha h}$, *the unique solution of VI* (1.7), *that is,*

$$\left\| \hat{u}_{\alpha h}^{n} - u_{\alpha h} \right\|_{\infty} \leq (1-\mu)^{n} \|\hat{u}_{0h}\|_{\infty}, \tag{3.11}$$

$$\left\| \hat{u}_{\alpha h}^{n} - u_{\alpha h} \right\|_{\infty} \leq (1-\mu)^{n} \|\hat{u}_{0h}\|_{\infty}. \tag{3.12}$$

4 L^{∞}-Error estimates

This section is devoted to proving the main results of this paper. For that, let us recall some useful properties enjoyed by elliptic variational inequalities of obstacle type.

4.1 Elliptic variational inequality

Let g in $L^{\infty}(\Omega)$, ψ in $W^{1,\infty}(\Omega)$ be such that $\partial\psi/\partial n \geq 0$ on Γ, let $b(\cdot,\cdot)$ be the bilinear form defined in (2.2), and let ω be the solution of the following variational inequality:

$$b(\omega, v-\omega) \geqq (g, v-\omega) \quad \forall v \in \mathbb{K}. \tag{4.1}$$

Definition 1 $w \in K$ is said to be a subsolution for VI (4.1) if

$$b(w, v) \leq (g, v) \quad \forall v \in H^{1}(\Omega), v \geq 0. \tag{4.2}$$

Theorem 4 [2] *Let X denote the set of continuous subsolutions. Then the solution* ω *of VI* (4.1) *is the least upper bound of the set X.*

Theorem 5 [2] *Let g and* $\tilde{g}$ *in* $W^{1,\infty}(\Omega)$ *and* ω *and* $\tilde{\omega}$ *be the corresponding solutions to* (4.1). *Then*

$$\|\omega - \tilde{\omega}\|_{\infty} \leq \|g - \tilde{g}\|_{\infty}.$$

Similarly, let $\omega_h \in K_h$ be the finite element counterpart of ω, that is,

$$b(\omega_h, v-\omega_h) \geqq (g, v-\omega_h) \quad \forall v \in \mathbb{K}_h. \tag{4.3}$$

Definition 2 $w \in K_h$ is said to be a subsolution for VI (4.1) if

$$b(w, \varphi_i) \leq (g, \varphi_i) \quad \forall \varphi_i, i = 1, 2, \ldots, m(h).$$

Theorem 6 *Let* X_h *denote the set of discrete subsolutions. Then, under the DMP, the solution* ω_h *of VI* (4.3) *is the least upper bound of the set* X_h.

Theorem 7 *Let g and* $\tilde{g}$ *be in* $W^{1,\infty}(\Omega)$, *and let* ω_h *and* $\tilde{\omega}_h$ *be the corresponding solutions to* (4.1). *Then*

$$\|\omega_h - \tilde{\omega}_h\|_{\infty} \leq \|g - \tilde{g}\|_{\infty}.$$

Let us now introduce two auxiliary variational inequalities.

4.2 Two auxiliary sequences of variational inequalities

We define the sequence $\{\bar{u}_\alpha^n\}_{n\geq 1}$ such that $\bar{u}_\alpha^n$ solves the continuous VI

$$b(\bar{u}_\alpha^n, v - \bar{u}_\alpha^n) \geq (f + \lambda \hat{u}_{\alpha h}^{n-1}, v - \bar{u}_\alpha^n) \quad \forall v \in \mathbb{K}, \tag{4.4}$$

where $\hat{u}_{\alpha h}^{n-1}$ is defined in (3.6), and the sequence $\{\bar{u}_{\alpha h}^n\}_{n\geq 1}$ is such that $\bar{u}_{\alpha h}^n$ solves the discrete VI

$$b(\bar{u}_{\alpha h}^n, v - \bar{u}_{\alpha h}^n) \geq (f + \lambda \hat{u}_\alpha^{n-1}, v - \bar{u}_{\alpha h}^n) \quad \forall v \in \mathbb{K}_h, \tag{4.5}$$

where $\hat{u}_\alpha^{n-1}$ is defined in (2.7).

Lemma 8 *There exists a constant C independent of* α, h, *and* n *such that*

$$\|\bar{u}_\alpha^n - \hat{u}_{\alpha h}^n\|_\infty \leq Ch^2|\ln h|^2 \tag{4.6}$$

and

$$\|\hat{u}_\alpha^n - \bar{u}_{\alpha h}^n\|_\infty \leq Ch^2|\ln h|^2. \tag{4.7}$$

Proof Since $\|\hat{u}_{\alpha h}^{n-1}\|_\infty \leq C$ (independent of α, h, and n) and $\|\hat{u}_{\alpha h}^{n-1}\|_\infty \leq C$ (independent of α, h, and n), making use of [10], we get both (4.6) and (4.7). □

4.3 Optimal L^∞-error estimates

Next, we shall estimate the error in the maximum norm between the nth iterates $\hat{u}_\alpha^n$ and $\hat{u}_{\alpha h}^n$ defined in (2.7) and (3.6), respectively.

Theorem 8

$$\|\hat{u}_\alpha^n - \hat{u}_{\alpha h}^n\|_\infty \leq Ch^2|\ln h|^2. \tag{4.8}$$

In order to prove Theorem 8, we need the following lemma.

Lemma 9 *There exists a sequence of continuous subsolutions* $(\beta^n)_{n\geq 1}$ *such that*

$$\beta^n \leq \hat{u}_\alpha^n, \quad \forall n \geq 1,$$

and

$$\|\beta^n - \hat{u}_{\alpha h}^n\|_\infty \leq Ch^2|\ln h|^2$$

and a sequence of discrete subsolution $(\gamma_h^n)_{n\geq 1}$ *such that*

$$\gamma_h^n \leq \hat{u}_{\alpha h}^n, \quad \forall n \geq 1,$$

and

$$\|\gamma_h^n - \hat{u}_\alpha^n\|_\infty \leq Ch^2|\ln h|^2.$$

Proof Consider the VI

$$b(\bar{u}_\alpha^1, v - \bar{u}_\alpha^1) \geq (f + \lambda \hat{u}_h^0, v - \bar{u}_\alpha^1) \quad \forall v \in \mathbb{K}.$$

Then, as $\bar{u}_\alpha^1$ is solution to a VI, it is also a subsolution, *i.e.*,

$$\begin{aligned} b(\bar{u}_\alpha^1, v) &\leq (f + \lambda \hat{u}_h^0, v) \quad \forall v \in H^1(\Omega), v \geq 0, \\ &\leq b(\bar{u}_\alpha^1, v) \leq (f + \lambda \hat{u}_h^0 - \lambda \hat{u}_0 + \lambda \hat{u}_0, v) \quad \forall v \in H^1(\Omega), v \geq 0. \end{aligned}$$

But

$$\|\hat{u}_h^0 - \hat{u}_0\|_\infty \leq Ch^2 |\ln h| \quad \text{(see [11])}.$$

Then

$$\begin{aligned} b(\bar{u}_\alpha^1, v) &\leq (f + \lambda \|\hat{u}_h^0 - \hat{u}_0\|_\infty + \lambda \hat{u}_0, v) \quad \forall v \in H^1(\Omega), v \geq 0, \\ &\leq (f + Ch^2 |\ln h| + \lambda \hat{u}_0, v) \quad \forall v \in H^1(\Omega), v \geq 0. \end{aligned}$$

So, $\bar{u}_\alpha^1$ is a subsolution for the VI whose solution is $\bar{U}_\alpha^1 = \partial(f + Ch^2 |\ln h| + \lambda \hat{u}_0)$. Then, as $\hat{u}_\alpha^1 = \partial(f + \lambda \hat{u}_0)$, making use of Theorem 5, we have

$$\begin{aligned} \|\bar{U}_\alpha^1 - \hat{u}_\alpha^1\|_\infty &\leq \|f + Ch^2 |\ln h| + \lambda \hat{u}_0 - (f + \lambda \hat{u}_0)\|_\infty \\ &\leq Ch^2 |\ln h|. \end{aligned}$$

Hence, making use of Theorem 4, we have

$$\bar{u}_\alpha^1 \leq \bar{U}_\alpha^1 \leq \hat{u}_\alpha^1 + Ch^2 |\ln h|.$$

Putting

$$\beta^1 = \bar{u}_\alpha^1 - Ch^2 |\ln h|,$$

we get

$$\beta_1^{(h)} \leq \hat{u}_\alpha^1 \tag{4.9}$$

and

$$\begin{aligned} \|\beta^1 - \hat{u}_{\alpha h}^1\|_\infty &\leq \|\bar{u}_\alpha^1 - Ch^2 |\ln h| - \hat{u}_{\alpha h}^1\|_\infty \\ &\leq \|\bar{u}_\alpha^1 - \hat{u}_{\alpha h}^1\|_\infty + Ch^2 |\ln h| \\ &\leq Ch^2 |\ln h|^2 + Ch^2 |\ln h| \\ &\leq Ch^2 |\ln h|^2. \end{aligned} \tag{4.10}$$

Consider now the discrete VI

$$b(\bar{u}^1_{\alpha h}, v - \bar{u}^1_{\alpha h}) \geq (f + \lambda \hat{u}_0, v - \bar{u}^1_{\alpha h}) \quad \forall v \in \mathbb{K}_h.$$

Then

$$b(\bar{u}^1_{\alpha h}, \varphi_i) \leq (f + \lambda \hat{u}_0, \varphi_i) \quad \forall \varphi_i$$

or

$$\begin{aligned} b(\bar{u}^1_{\alpha h}, \varphi_i) &\leq (f + \lambda \hat{u}_0 - \lambda \hat{u}^0_h + \lambda \hat{u}^0_h, \varphi_i) \quad \forall \varphi_i \\ b(\bar{u}^1_{\alpha h}, \varphi_i) &\leq (f + \lambda \|\hat{u}_0 - \hat{u}^0_h\|_\infty + \lambda \hat{u}^0_h, \varphi_i) \quad \forall \varphi_i \\ &\leq (f + \lambda C h^2 |\ln h| + \lambda \hat{u}^0_h, \varphi_i) \quad \forall \varphi_i. \end{aligned}$$

So, $\bar{u}^1_{\alpha h}$ is a subsolution for the VI whose solution is $\bar{U}^1_{\alpha h} = \partial_h(f + \lambda C h^2 |\ln h| + \lambda \hat{u}^0_h)$. And, as $\hat{u}^1_{\alpha h} = \partial_h(f + \lambda \hat{u}^0_h)$, making use of Theorem 7, we get

$$\begin{aligned} \|\bar{U}^1_{\alpha h} - \hat{u}^1_{\alpha h}\|_\infty &\leq \|f + \lambda C h^2 |\ln h| + \lambda \hat{u}^0_h - (f + \lambda \hat{u}_{0h})\|_\infty \\ &\leq C h^2 |\ln h|, \end{aligned}$$

and, making use of Theorem 6, we have

$$\bar{u}^1_{\alpha h} \leq \bar{U}^1_{\alpha h} \leq \hat{u}^1_{\alpha h} + C h^2 |\ln h|.$$

Now, taking

$$\gamma^1_h = \bar{u}^1_{\alpha h} - C h^2 |\ln h|,$$

we have

$$\gamma^1_h \leq \hat{u}^1_{\alpha h} \tag{4.11}$$

and

$$\begin{aligned} \|\gamma^1_h - \hat{u}^1_\alpha\|_\infty &\leq \|\bar{u}^1_{\alpha h} - \hat{u}^1_\alpha\|_\infty + C h^2 |\ln h| \\ &\leq C h^2 |\ln h|^2 + C h^2 |\ln h| \\ &\leq C h^2 |\ln h|^2. \end{aligned} \tag{4.12}$$

Thus, combining (4.9), (4.10) and (4.11), (4.12), we obtain

$$\begin{aligned} \hat{u}^1_\alpha &\leq \gamma^1_h + C h^2 |\ln h|^2 \\ &\leq \hat{u}^1_{\alpha h} + C h^2 |\ln h|^2 \\ &\leq \beta^{(h)}_1 + C h^2 |\ln h|^2 \\ &\leq \hat{u}^1_\alpha + C h^2 |\ln h|^2. \end{aligned}$$

That is,

$$\|\hat{u}_\alpha^1 - \hat{u}_{\alpha h}^1\|_\infty \le Ch^2|\ln h|^2.$$

Step n. Let us now assume that

$$\|\hat{u}_\alpha^{n-1} - \hat{u}_{\alpha h}^{n-1}\|_\infty \le Ch^2|\ln h|^2 \tag{4.13}$$

and prove that

$$\|\hat{u}_\alpha^n - \hat{u}_{\alpha h}\|_\infty \le Ch^2|\ln h|^2.$$

For that, consider the VI

$$b(\bar{u}_\alpha^n, v - \bar{u}_\alpha^n) \ge (f + \lambda\hat{u}_{\alpha h}^{n-1}, v - \bar{u}_\alpha^n) \quad \forall v \in \mathbb{K}.$$

Then

$$b(\bar{u}_\alpha^n, v) \le (f + \lambda\hat{u}_{\alpha h}^{n-1}, v) \quad \forall v \in H^1(\Omega), v \ge 0,$$

or

$$\begin{aligned} b(\bar{u}_\alpha^n, v) &\le (f + \lambda\hat{u}_{\alpha h}^{n-1} - \lambda\hat{u}_\alpha^{n-1} + \lambda\hat{u}_\alpha^{n-1}, v) \quad \forall v \in H^1(\Omega), v \ge 0 \\ &\le (f + \lambda\|\hat{u}_{\alpha h}^{n-1} - \hat{u}_\alpha^{n-1}\|_\infty + \lambda\hat{u}_\alpha^{n-1}, v) \quad \forall v \in H^1(\Omega), v \ge 0. \end{aligned}$$

So, using (4.13), we get

$$b(\bar{u}_\alpha^n, v) \le (f + Ch^2|\ln h|^2 + \lambda\hat{u}_\alpha^{n-1}, v) \quad \forall v \in H^1(\Omega), v \ge 0.$$

Hence, $\bar{u}_\alpha^n$ is a subsolution for the VI whose solution is $\bar{U}_\alpha^n = \partial(f + Ch^2|\ln h|^2 + \lambda\hat{u}_\alpha^{n-1})$. Then, as $\hat{u}_\alpha^n = \partial(f + \lambda\hat{u}_\alpha^{n-1})$, making use of Theorem 5, we have

$$\begin{aligned} \|\bar{U}_\alpha^n - \hat{u}_\alpha^n\|_\infty &\le \|f + Ch^2|\ln h|^2 + \lambda\hat{u}_\alpha^{n-1} - (f + \lambda\hat{u}_\alpha^{n-1})\|_\infty \\ &\le Ch^2|\ln h|^2. \end{aligned}$$

Hence, applying Theorem 4, we get

$$\bar{u}_\alpha^n \le \bar{U}_\alpha^n \le \hat{u}_\alpha^n + Ch^2|\ln h|^2.$$

Putting

$$\beta^n = \bar{u}_\alpha^n - Ch^2|\ln h|^2,$$

we get

$$\beta^n \le \hat{u}_\alpha^n \tag{4.14}$$

and

$$\begin{aligned}\|\beta^n - \hat{u}^n_{\alpha h}\|_\infty &\le \|\bar{u}^n_\alpha - Ch^2|\ln h|^2 - \hat{u}^n_{\alpha h}\|_\infty \\ &\le \|\bar{u}^n_\alpha - \hat{u}^n_{\alpha h}\|_\infty + Ch^2|\ln h|^2 \\ &\le Ch^2|\ln h|^2 + Ch^2|\ln h|^2 \\ &\le Ch^2|\ln h|^2. \end{aligned} \tag{4.15}$$

Consider now the discrete VI

$$b(\bar{u}^n_{\alpha h}, v - \bar{u}^n_{\alpha h}) \ge (f + \lambda\hat{u}^{n-1}_\alpha, v - \bar{u}^n_{\alpha h}) \quad \forall v \in \mathbb{K}_h.$$

Then

$$b(\bar{u}^n_{\alpha h}, \varphi_i) \le (f + \lambda\hat{u}^{n-1}_\alpha, \varphi_i) \quad \forall\varphi_i,$$

and, making use of (4.13), we obtain

$$\begin{aligned} b(\bar{u}^n_{\alpha h}, \varphi_i) &\le (f + \lambda\hat{u}^{n-1}_\alpha - \lambda\hat{u}^{n-1}_{\alpha h} + \lambda\hat{u}^{n-1}_{\alpha h}, \varphi_i) \quad \forall\varphi_i \\ &\le (f + \lambda\|\hat{u}^{n-1}_\alpha - \hat{u}^{n-1}_{\alpha h}\|_\infty + \lambda\hat{u}^{n-1}_{\alpha h}, \varphi_i) \quad \forall\varphi_i \\ &\le (f + \lambda Ch^2|\ln h|^2 + \lambda\hat{u}^{n-1}_{\alpha h}, \varphi_i) \quad \forall\varphi_i. \end{aligned}$$

So, $\bar{u}^1_{\alpha h}$ is a subsolution for the VI whose solution is $\bar{U}^n_{\alpha h} = \partial_h(f + \lambda Ch^2|\ln h|^2 + \lambda\hat{u}^{n-1}_{\alpha h})$. And, as $\hat{u}^n_{\alpha h} = \partial_h(f + \lambda\hat{u}^{n-1}_{\alpha h})$, making use of Theorem 7, we get

$$\begin{aligned}\|\bar{U}^n_{\alpha h} - \hat{u}^n_{\alpha h}\|_\infty &\le \|f + \lambda Ch^2|\ln h|^2 + \lambda\hat{u}^{n-1}_{\alpha h} - (f + \lambda\hat{u}^{n-1}_{\alpha h})\|_\infty \\ &\le Ch^2|\ln h|^2, \end{aligned}$$

and, making use of Theorem 6, we have

$$\bar{u}^n_{\alpha h} \le \bar{U}^n_{\alpha h} \le \hat{u}^n_{\alpha h} + Ch^2|\ln h|^2.$$

Now, taking

$$\gamma^n_h = \bar{u}^n_{\alpha h} - Ch^2|\ln h|^2,$$

we have

$$\gamma^n_h \le \hat{u}^n_{\alpha h} \tag{4.16}$$

and

$$\begin{aligned}\|\gamma^n_h - \hat{u}^n_\alpha\|_\infty &\le \|\bar{u}^n_{\alpha h} - \hat{u}^n_\alpha\|_\infty + Ch^2|\ln h|^2 \\ &\le Ch^2|\ln h|^2. \end{aligned} \tag{4.17}$$

Thus, combining (4.14), (4.15) and (4.16), (4.17), we obtain

$$\begin{aligned}\hat{u}_\alpha^n &\le \gamma_h^n + Ch^2|\ln h|^2\\ &\le \hat{u}_{\alpha h}^n + Ch^2|\ln h|^2\\ &\le \beta_n^{(h)} + Ch^2|\ln h|^2\\ &\le \hat{u}_\alpha^n + Ch^2|\ln h|^2.\end{aligned}$$

That is,

$$\left\|\hat{u}_\alpha^n - \hat{u}_{\alpha h}^n\right\|_\infty \le Ch^2|\ln h|. \qquad \square$$

Theorem 9 *There exists a constant C independent of both α and h such that*

$$\|u_\alpha - u_{\alpha h}\|_\infty \le Ch^2|\ln h|^2. \tag{4.18}$$

Proof Indeed, combining estimates (2.15), (3.11), and (4.8), we get

$$\begin{aligned}\|u_\alpha - u_{\alpha h}\|_\infty &\le \left\|u_\alpha - \hat{u}_\alpha^n\right\|_\infty + \left\|\hat{u}_\alpha^n - \hat{u}_{\alpha h}^n\right\|_\infty + \left\|\hat{u}_{\alpha h}^n - u_{\alpha h}\right\|_\infty\\ &\le \left\|u_\alpha - \hat{u}_\alpha^n\right\|_\infty + Ch^2|\ln h|^2 + \left\|\hat{u}_{\alpha h}^n - u_{\alpha h}\right\|_\infty\\ &\le (1-\mu)^n\|\hat{u}_{0h}\|_\infty + Ch^2|\ln h|^2 + (1-\mu)^n\|\hat{u}_{0h}\|_\infty.\end{aligned}$$

So, passing to the limit, as $n \to \infty$, we get

$$\|u_\alpha - u_{\alpha h}\|_\infty \le Ch^2|\ln h|^2. \qquad \square$$

Theorem 10 *The solution $u_{\alpha h}$ of VI* (1.7) *converges, as $\alpha \to 0^+$, uniformly in $C(\bar{\Omega})$ to u_{0h}, the solution of discrete VI* (1.8).

Proof Since $0 \le u_{\alpha h} \le r_h\psi$, then taking $v = 0$ as a trial function in the VI

$$b(u_{\alpha h}, v - u_{\alpha h}) + (u_{\alpha h}, v - u_{\alpha h}) \ge (f + \lambda u_{\alpha h}, v - u_{\alpha h}) \quad \forall v \in \mathbb{K}_h,$$

we get

$$\begin{aligned}\|u_{\alpha h}\|_{H^1(\Omega)}^2 &\le b(u_{\alpha h}, u_{\alpha h}) + (u_{\alpha h}, u_{\alpha h})\\ &\le (f + \lambda u_{\alpha h}, u_{\alpha h}) \le C\|u_{\alpha h}\|_{H^1(\Omega)}.\end{aligned}$$

That is,

$$\|u_{\alpha h}\|_{H^1(\Omega)} \le C.$$

On the other hand, we have from the theorem

$$\|u_\alpha - u_{\alpha h}\|_\infty \le Ch^2|\ln h|^2.$$

Then, by the inverse inequality, we have

$$\|u_\alpha - u_{\alpha h}\|_{W^{1,\infty}(\Omega)} \le Ch|\ln h|^2,$$

and therefore

$$\|u_{\alpha h}\|_{W^{1,\infty}} \le C \quad \text{(independent of } \alpha \text{ and } h).$$

The rest of the proof is similar to that of Theorem 2. □

Next, combining Theorems 2, 9, and 10, we are in a position to derive the main result of this paper.

Theorem 11 *There exists a constant independent of both α and h such that*

$$\|u_0 - u_{0h}\|_\infty \le Ch^2|\ln h|^2.$$

Proof Indeed, using estimate (4.18), we have

$$\begin{aligned}\|u_0 - u_{0h}\|_\infty &\le \|u_0 - u_\alpha\|_\infty + \|u_\alpha - u_{\alpha h}\|_\infty + \|u_{\alpha h} - u_{0h}\|_\infty \\ &\le \|u_0 - u_\alpha\|_\infty + Ch^2|\ln h|^2 + \|u_{\alpha h} - u_{0h}\|_\infty.\end{aligned}$$

So, passing to the limit, as $\alpha \to 0^+$, we get

$$\|u_0 - u_{0h}\|_\infty \le \lim_{\alpha\to 0}\|u_0 - u_\alpha\|_\infty + Ch^2|\ln h|^2 + \lim_{\alpha\to 0}\|u_{\alpha h} - u_{0h}\|_\infty.$$

Thus

$$\|u_0 - u_{0h}\|_\infty \le Ch^2|\ln h|^2.$$ □

Competing interests
The author did not provide this information.

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