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A kind of extension of the famous Young inequality

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Abstract

Young inequality, extended in (Geometry of Orlicz Spaces, 1986; Geometry of Orlicz Spaces, 1996), has extensive use and great effort in mathematical analysis. By the kind of extended Young inequality, we can get the famous Holder inequality and the Minkowski inequality. But until now, we have not found its strict proof of analysis. In (Geometry of Orlicz Spaces, 1986; Geometry of Orlicz Spaces, 1996), only the probable pattern description was found. In this paper, we will get the strict proof of analysis of a kind of extension of Young inequality with the approximation method.

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Keywords: Young inequality; *N*-function; strictly convex function

1 Introduction

The original Young inequality [1] has been proposed in an integral form by Young in 1912. Suppose that $f(x)$ is a strictly increasing and continuous function defined in $[0, c]$, $f^{-1}(x)$ is the inverse function of $f(x)$, $f(0) = 0$, $a \in [0, c]$, $b \in [f(0), f(c)]$. Then

$$\int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx \geq ab,$$

where the equality holds if and only if $b = f(a)$.

Young inequality has an extensive use and a great effort in mathematical analysis. Now Young inequality was extended as follows.

Let $M(u)$ and $N(v)$ be complementary N-function with each other (see Definition 2.1 and Definition 2.2), then the kind of Young inequality $uv \leq M(u) + N(v)$ holds, and the equality holds if and only if $u = q(|v|) \operatorname{sign} v$ or $v = p(|u|) \operatorname{sign} u$ for all $u, v \in (-\infty, +\infty)$.

By the kind of Young inequality, we can get the famous Holder inequality and the Minkowski inequality (see references [2] and [3]). But until now, we have not found its strict proof of analysis. In references [2] and [3], only the probable pattern description was found. Some other decisions can be found in [4–8].

In this paper, we will get its strict proof of analysis with the approximation method.

2 Preliminaries

Definition 2.1 [2] The mapping $M : (-\infty, +\infty) \to (-\infty, +\infty)$ is called an N-function if it has the following properties:

(i) $M(u)$ is even, continuous, convex and $M(0) = 0$.

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(ii) $M(u) > 0$ for all $u \neq 0$.

(iii) $\lim_{u \to 0} \frac{M(u)}{u} = 0$ and $\lim_{u \to \infty} \frac{M(u)}{u} = \infty$.

Lemma 2.1 [2] *$M(u)$ is an N-function if and only if there exists $p(u) : [0, +\infty) \to [0, +\infty)$ with the following properties*:

(i) *$p(u)$ is right-continuous and nondecreasing;*

(ii) *$p(u) > 0$ whenever $u \neq 0$;*

(iii) *$p(0) = 0$ and $p(\infty) = \infty$, $M(u) = \int_0^{|u|} p(t)\,dt$.*

Record 2.1 [2] *$p(u)$ is the right-derivative of N-function $M(u)$.*

Lemma 2.2 *Let $p_-(u)$ be the left-derivative of N-function $M(u)$, then $p_-(u) = \lim_{h \to 0^+} p(u - h)$, and $\int_0^{|u|} p_-(t)\,dt = M(u)$.*

Proof From the proof process of Theorem 1.4 in reference [2], we know $p_-(u)$ is left continuous, and for all $0 < u < v$, $p(u) \leq p_-(v) \leq p(v)$.

Hence, for $h > 0$, we have $p(v - h) \leq p_-(v)$.

Therefore,

$$\lim_{h \to 0^+} p(v - h) \leq p_-(v).$$

On the other hand, since $p_-(v) \leq p(v)$ and $p_-(v)$ is left continuous, we get

$$p_-(v) = \lim_{h \to 0^+} p_-(v - h) \leq \lim_{h \to 0^+} p(v - h).$$

Therefore, we have

$$p_-(v) = \lim_{h \to 0^+} p(v - h).$$

Since for all $h > 0$, $p(v - h) \leq p_-(v) \leq p(v)$, then we have

$$\int_0^{|u|} p(t - h)\,dt \leq \int_0^{|u|} p_-(t)\,dt \leq \int_0^{|u|} p(t)\,dt = M(u).$$

That is,

$$M\big(|u| - h\big) - M(-h) \leq \int_0^{|u|} p_-(t)\,dt \leq M(u).$$

Let $h \to 0$, by the property (i) of $M(u)$ in Definition 2.1, we have

$$\int_0^{|u|} p_-(t)\,dt = M(u). \qquad \square$$

Definition 2.2 [2] Suppose that $M(u)$ is an N-function. Let $p(t)$ be the right derivative of $M(u)$. Let $q(s) = \sup_{p(t) \leq s} t = \inf_{p(t) > s} t$, called the right-inverse function of $p(t)$. By Theorem 1.5 in reference [2], we know that $q(s)$ also satisfies the three properties of Lemma 2.1, and $N(v) = \int_0^{|v|} q(s)\,ds$ is called the complementary N-function of $M(u)$. It is obvious, the left derivative $q_-(s)$ of $N(v)$ satisfies $q_-(s) = \sup_{p(t) < s} t = \inf_{p(t) \geq s} t$.

Lemma 2.3 [2] $q(p(t)) \geq t$, $p(q(s)) \geq s$; $q(p(t) - \varepsilon) \leq t$, $p(q(s) - \varepsilon) \leq s$.

Lemma 2.4 [2] *$M(u)$ is strictly convex if and only if $p(t)$ is strictly increasing, that is, $q(s)$ is continuous.*

Lemma 2.5 [2] *For any N-function $M(u)$ and $\varepsilon > 0$, there exists a strictly convex N-function $M_1(u)$, such that*

$$(1 - \varepsilon)p(t) \leq p_1(t) \leq (1 + \varepsilon)p(t), \qquad (1 - \varepsilon)M(u) \leq M_1(u) \leq (1 + \varepsilon)M(u),$$

where $p(t)$ and $p_1(t)$ are the right derivatives of $M(u)$ and $M_1(u)$, respectively.

Record 2.2 Lemma 2.5 is Theorem 1.10 in reference [2], but it reverses the old conclusion '$M(u) \leq M_1(u) \leq (1 + \varepsilon)M(u)$,' for the new conclusion '$(1 - \varepsilon)M(u) \leq M_1(u) \leq (1 + \varepsilon)M(u)$.' From the construction process of $p_1(t)$, in the proof in reference [3], we know if $p(t)$ is continuous, then $p_1(t)$ is also continuous.

Lemma 2.6 *Suppose that $u \geq 0$ and $v \geq 0$, then $u = q(v)$ or $v = p(u)$ if and only if $u \in [q_-(v), q(v)]$. By the symmetry, we get another necessary and sufficient condition, that is, $v \in [p_-(u), p(u)]$.*

Proof Sufficiency.

Suppose that $u \in [q_-(v), q(v)]$.

(i) If $q_-(v) = q(v)$, it is clear that $u = q_-(v) = q(v)$.

(ii) If $q_-(v) \neq q(v)$, then $q_-(v) < q(v)$. If $u = q(v)$, then the conclusion holds.

If $q_-(v) \leq u < q(v)$, we need only to prove that $p(u) = v$.

From Definition 2.2, we have

$$q_-(v) = \sup_{p(t)<v} t = \inf_{p(t) \geq v} t.$$

Since $q_-(v) \leq u \Rightarrow \sup_{p(t)<v} t \leq u$, then for any $\frac{1}{n}$, we get

$$p\left(u + \frac{1}{n}\right) \geq v.$$

Let $n \to \infty$, since $p(t)$ is right continuous, then we have

$$p(u) \geq v.$$

On the other hand, from $u < q(v) = \sup_{p(t) \leq v} t = \inf_{p(t) \geq v} t$, we get

$$p(u) \leq v.$$

So, we have

$$p(u) = v.$$

Necessity.

If $u = q(v)$, it is clearly established.

If $p(u) = v$, then from

$$q_-(v) = \inf_{p(t) \geq v} t \leq u$$

and

$$q(v) = \sup_{p(t) \leq v} t \geq u.$$

We have

$$u \in \left[q_-(v), q(v)\right]. \hspace{4cm} \square$$

The next two lemmas are about the change of variable of integral and distribute integral.

Lemma 2.7 [9] *Suppose that $f(x)$ and $g(x)$ are defined on the interval $[a, b]$, and the Stieltjes integral of $f(x)$ about $g(x)$ exists. Suppose that $x(t)$ is a strictly increasing and continuous function on the interval $[\alpha, \beta]$, and $x(\alpha) = a$ and $x(\beta) = b$, then*

$$\int_\alpha^\beta f\big(x(t)\big) \, dg\big(x(t)\big) = \int_a^b f(x) \, dg(x).$$

Lemma 2.8 [9] *Suppose that $f(x)$ and $g(x)$ are defined on the interval $[a, b]$, and the Stieltjes integral of $f(x)$ about $g(x)$ exists, then*

$$\int_a^b f(x) \, dg(x) + \int_a^b g(x) \, df(x) = f(b)g(b) - f(a)g(a).$$

3 Main result

Theorem 3.1 *Suppose that $M(u)$ is an N-function, and $N(v)$ is the complementary N-function of $M(u)$, then Young inequality $uv \leq M(u) + N(v)$ holds, and $uv = M(u) + N(v)$ holds if and only if $u = q(|v|) \operatorname{sign} v$ or $v = p(|u|) \operatorname{sign} u$.*

Proof Suppose that $u \geq 0$ and $v \geq 0$.

Firstly, we will prove the necessity of the equality.

Suppose that there exist $u_0 \geq 0$ and $v_0 \geq 0$ satisfying

$$M(u_0) + N(v_0) = u_0 v_0.$$

Let

$$F(u, v) = M(u) + N(v) - uv.$$

From Young inequality, we have learned that for all u and v, $F(u, v) \geq 0$.

From $M(u_0) + N(v_0) = u_0 v_0$, we have $F(u, v_0) = M(u) + N(v_0) - uv_0$ and we can get the minimum 0 in u_0.

If $u_0 = 0$, from $M(u_0) + N(v_0) = u_0 v_0$, we get that $v_0 = 0$, then $u_0 = q(v_0) = 0$ or $v_0 = p(u_0) = 0$, that is, the necessity of the equality holds.

If $u_0 \neq 0$, then $F(u_0, v_0)$ is the minimum of the $F(u, v_0)$ on the interval $(0, +\infty)$.

Therefore, the left derivative of $F(u, v_0)$ is less than or equal to zero on the point u_0, and the right derivative of $F(u, v_0)$ is more than or equal to zero on the point u_0.

That is,

$$p_-(u_0) - v_0 \leq 0, p(u_0) - v_0 \geq 0.$$

Then

$$v_0 \in \left[p_-(u_0), p(u_0) \right].$$

From Lemma 2.6, we get $u_0 = q(v_0)$ or $v_0 = p(u_0)$.

That is, the necessity of the equality holds.

Secondly, we will get the proof of the Young inequality and the sufficiency of the equality in three steps.

Step I. Suppose that $M(u)$ and $N(v)$ are all strictly convex. From Lemma 2.4, the right derivative $p(t)$ and $q(s)$ are all strictly increasing, continuous, and are the right inverse-function of each other. From the reference [9], we have that the Stieltjes integral $\int_0^{q(v)} t \, dp(t)$ exists.

From Lemma 2.7 and Lemma 2.8, we have

$$
\begin{aligned}
M(u) &+ N(v) \\
&= \int_0^u p(t) \, dt + \int_0^v q(s) \, ds \\
&= \int_0^u p(t) \, dt + \int_0^{q(v)} t \, dp(t) \\
&= \int_0^u p(t) \, dt + vq(v) - \int_0^{q(v)} p(t) \, dt \\
&= vq(v) + \int_{q(v)}^u p(t) \, dt.
\end{aligned}
\tag{1}
$$

(i) If $u > q(v)$, then $p(u) > v$.

Hence, by expression (1), we have

$$
\begin{aligned}
M(u) &+ N(v) \\
&= \int_{q(v)}^u p(t) \, dt + vq(v) \\
&= \int_{q(v)}^u p(t) \, dt + uv - v\big(u - q(v)\big) \\
&\geq p\big(q(v)\big)\big(u - q(v)\big) + uv - v\big(u - q(v)\big) \\
&= uv.
\end{aligned}
$$

(ii) If $u < q(v)$, then $p(u) \leq v$.

Hence, by expression (1), we have

$$M(u) + N(v)$$

$$= vq(v) - \int_u^{q(v)} p(t)\,dt$$

$$= uv + v\big(q(v) - u\big) - \int_u^{q(v)} p(t)\,dt$$

$$\geq uv + v\big(q(v) - u\big) - p\big(q(v)\big)\big(q(v) - u\big)$$

$$= uv.$$

(iii) If $u = q(v)$, then $v = p(u)$.

From expression (1), we have $uv = M(u) + N(v)$.

That is, the sufficiency of the equality holds.

Step II. Suppose that $M(u)$ is strictly convex, then from Lemma 2.4, the right derivative $p(t)$ is strictly increasing, and the right-inverse function $q(s)$ is continuous and nondecreasing.

From Lemma 2.5 and Record 2.2, $\forall 0 < \varepsilon < \frac{1}{2}$, we can construct a function strictly increasing and continuous $q_1(s)$ such that

$$(1 - \varepsilon)q(s) \leq q_1(s) \leq (1 + \varepsilon)q(s).$$

Hence,

$$\frac{1}{1+\varepsilon}q_1(s) \leq q(s) \leq \frac{1}{1-\varepsilon}q_1(s). \tag{2}$$

Let $p_1(t)$ be the right-inverse function of $q_1(s)$, then $p_1(t)$ is strictly increasing and continuous.

In the following, we will get the relation of $p_1(t)$ and $p(t)$.

In expression (2), let $s = p_1(t)$, we have

$$\frac{1}{1+\varepsilon}q_1\big(p_1(t)\big) \leq q\big(p_1(t)\big) \leq \frac{1}{1-\varepsilon}q_1\big(p_1(t)\big).$$

That is,

$$\frac{t}{1+\varepsilon} \leq q\big(p_1(t)\big) \leq \frac{t}{1-\varepsilon}. \tag{3}$$

From Lemma 2.3 and expression (3), we get

$$q\left(p\left(\frac{t}{1-2\varepsilon}\right)\right) \geq \frac{t}{1-2\varepsilon} > \frac{t}{1-\varepsilon} \geq q\big(p_1(t)\big). \tag{4}$$

Since $q(s)$ is nondecreasing, by expression (4), we get

$$p\left(\frac{t}{1-2\varepsilon}\right) > p_1(t). \tag{5}$$

From the result in Step I, we get

$$uv \leq M_1(u) + N_1(v) = \int_0^u p_1(t)\,dt + \int_0^v q_1(s)\,ds.$$

Therefore,

$$
\begin{aligned}
M(u) &+ N(v) \\
&= \int_0^u p(t)\,dt + \int_0^v q(s)\,ds \\
&\geq \int_0^u p(t)\,dt + \int_0^v \frac{1}{1+\varepsilon} q_1(s)\,ds \quad \text{(by (2))} \\
&\geq \int_0^u p(t)\,dt + \frac{1}{1+\varepsilon}\left(uv - \int_0^u p_1(t)\,dt\right) \\
&> \frac{1}{1+\varepsilon}uv + \int_0^u p(t)\,dt - \frac{1}{1+\varepsilon}\int_0^u p\left(\frac{t}{1-2\varepsilon}\right)dt \quad \text{(by (5))} \\
&= \frac{1}{1+\varepsilon}uv + \int_0^u p(t)\,dt - \frac{1-2\varepsilon}{1+\varepsilon}\int_0^{\frac{u}{1-2\varepsilon}} p(t)\,dt.
\end{aligned}
$$

Let $\varepsilon \to 0$, we have

$$M(u) + N(v) \geq uv.$$

In the following, we will prove the sufficiency of the equality.
If $v = p(u)$, from Lemma 2.3 and expression (3), for $0 < \varepsilon < \frac{1}{2}$ above, we have

$$q\left(p\left(\frac{t}{1+2\varepsilon}\right) - \varepsilon\right) \leq \frac{t}{1+2\varepsilon} < \frac{t}{1+\varepsilon} \leq q(p_1(t)) \quad \Rightarrow \quad p\left(\frac{t}{1+2\varepsilon}\right) - \varepsilon < p_1(t). \qquad (6)$$

In expression (6), let $\varepsilon \to 0$, by Lemma 2.2, we get

$$p_-(t) \leq \lim_{\varepsilon \to 0} p_1(t). \qquad (7)$$

On the other hand, in expression (5), let $\varepsilon \to 0$, we get

$$\lim_{\varepsilon \to 0} p_1(t) \leq p(t).$$

Therefore,

$$p_-(t) \leq \lim_{\varepsilon \to 0} p_1(t) \leq p(t). \qquad (8)$$

By Lemma 2.2, we get

$$\int_0^u p_-(t)\,dt = M(u) = \int_0^u \lim_{\varepsilon \to 0} p_1(t)\,dt = \int_0^u p(t)\,dt. \qquad (9)$$

Now we need to prove that

$$\int_{p_1(u)}^{p(u)} q(s)\,ds = u\big(p(u) - p_1(u)\big). \tag{10}$$

In fact, if $s = p(u)$, from Definition 2.2, since $p(u)$ is strictly increasing, then we have $q(s) = \sup_{p(t)\leq s} t = \sup_{p(t)\leq p(u)} t = u$. If $s \in [p_-(u), p(u))$, from Lemma 2.6, we get $q(s) = u$. Therefore, we have $\int_{p_1(u)}^{p(u)} q(s)\,ds = \int_{p_1(u)}^{p(u)} u\,ds = u(p(u) - p_1(u))$.

By the result in Step I, we have

$$up_1(u) = M_1(u) + N_1\big(p_1(u)\big) = \int_0^u p_1(t)\,dt + \int_0^{p_1(u)} q_1(s)\,ds. \tag{11}$$

From expressions (9) and (11), we get

$$M(u) + N\big(p(u)\big)$$

$$= \int_0^u p(t)\,dt + \int_0^{p(u)} q(s)\,ds$$

$$= \int_0^u p(t)\,dt + \int_0^{p_1(u)} q(s)\,ds + \int_{p_1(u)}^{p(u)} q(s)\,ds$$

$$\leq \int_0^u p(t)\,dt + \int_0^{p_1(u)} \frac{1}{1-\varepsilon} q_1(s)\,ds + \int_{p_1(u)}^{p(u)} q(s)\,ds \quad \text{(by (2))}$$

$$= \int_0^u p(t)\,dt + \left(\frac{1}{1-\varepsilon}\right)\left(up_1(u) - \int_0^u p_1(t)\,dt\right) + \int_{p_1(u)}^{p(u)} q(s)\,ds \quad \text{(by (11))}$$

$$= \int_0^u p(t)\,dt - \frac{1}{1-\varepsilon}\int_0^u p_1(t)\,dt + \frac{1}{1-\varepsilon} up_1(u) + u\big(p(u) - p_1(u)\big) \quad \text{(by (10))}$$

$$= M(u) - \frac{1}{1-\varepsilon}\int_0^u p_1(t)\,dt + \left(\frac{u}{1-\varepsilon} - u\right)p_1(u) + up(u).$$

Let $\varepsilon \to 0$, we have

$$M(u) + N\big(p(u)\big) \leq up(u). \tag{12}$$

On the other hand, we have got the inequality $uv \leq M(u) + N(v)$.
Let $v = p(u)$, we have

$$up(u) \leq M(u) + N\big(p(u)\big).$$

Therefore, together with expression (12), we have

$$M(u) + N\big(p(u)\big) = up(u).$$

That is, the sufficiency of the equality holds.

Step III for any N-function $M(u)$, suppose that its complementary N-function is $N(v)$, $p(t)$ is the right-inverse function of $M(u)$, and $q(s)$ is the right-inverse function of $N(v)$.

From Lemma 2.5, for $0 < \varepsilon < \frac{1}{2}$ above, we can find a strictly convex N-function $M_1(u)$ and its right-derivative $p_1(t)$ such that

$$(1 - \varepsilon)p(t) \leq p_1(t) \leq (1 + \varepsilon)p(t), \qquad (1 - \varepsilon)M(u) \leq M_1(u) \leq (1 + \varepsilon)M(u). \tag{13}$$

Suppose that $N_1(v)$ is the complementary N-function of $M_1(u)$, $q_1(s)$ is the right derivative of $N_1(v)$.

In the following, we will get the relation of $q_1(t)$ and $q(t)$ for $0 < \varepsilon < \frac{1}{2}$ above. In expression (13), let $t = q_1(s) - \varepsilon$, we have

$$(1 - \varepsilon)p\big(q_1(s) - \varepsilon\big) \leq p_1\big(q_1(s) - \varepsilon\big) \leq (1 + \varepsilon)p\big(q_1(s) - \varepsilon\big). \tag{14}$$

From Lemma 2.3, we have that

$$p_1\big(q_1(s) - \varepsilon\big) \leq s, \quad p_1\big(q_1(s)\big) \geq s. \tag{15}$$

Therefore, by expressions (14) and (15), we have

$$p\big(q_1(s) - \varepsilon\big) \leq \frac{s}{1 - \varepsilon}, \qquad p\big(q_1(s)\big) \geq \frac{s}{1 + \varepsilon}. \tag{16}$$

Then, by Lemma 2.3, together with expression (16), we have

$$
\begin{aligned}
p\big(q_1(s) - \varepsilon\big) &\leq \frac{s}{1 - \varepsilon} < \frac{s}{1 - 2\varepsilon} \leq p\bigg(q\bigg(\frac{s}{1 - 2\varepsilon}\bigg)\bigg), \\
p\big(q_1(s)\big) &\geq \frac{s}{1 + \varepsilon} > \frac{s}{1 + 2\varepsilon} \geq p\bigg(q\bigg(\frac{s}{1 + 2\varepsilon}\bigg) - \varepsilon\bigg).
\end{aligned}
\tag{17}
$$

Since $p(t)$ is nondecreasing, then by expression (17), we get

$$q\bigg(\frac{s}{1 + 2\varepsilon}\bigg) - \varepsilon < q_1(s) < q\bigg(\frac{s}{1 - 2\varepsilon}\bigg) + \varepsilon. \tag{18}$$

From the result in Step II, we get

$$
\begin{aligned}
uv &\leq M_1(u) + N_1(v) \\
&\leq (1 + \varepsilon)M(u) + \int_0^v q_1(s)\,ds \\
&< (1 + \varepsilon)M(u) + \int_0^v \bigg(q\bigg(\frac{s}{1 - 2\varepsilon}\bigg) + \varepsilon\bigg)\,ds \quad \text{(by (18))} \\
&= (1 + \varepsilon)M(u) + \varepsilon v + (1 - 2\varepsilon)\int_0^{\frac{v}{1-2\varepsilon}} q(s)\,ds \\
&= (1 + \varepsilon)M(u) + \varepsilon v + (1 - 2\varepsilon)N\bigg(\frac{v}{1 - 2\varepsilon}\bigg).
\end{aligned}
$$

Let $\varepsilon \to 0$, we have

$$uv \leq M(u) + N(v).$$

In the following, we will prove sufficiency of the equality.

By the result in Step II, we have

$$M_1(u) + N_1\big(p_1(u)\big) = up_1(u). \tag{19}$$

Therefore,

$$
\begin{aligned}
&M(u) + N\big(p(u)\big) \\
&\leq \frac{1}{1-\varepsilon} M_1(u) + N\big(p(u)\big) \\
&= \left(\frac{1}{1-\varepsilon} - 1\right) M_1(u) + M_1(u) + N_1\big(p_1(u)\big) + N\big(p(u)\big) - N_1\big(p_1(u)\big) \\
&= \left(\frac{1}{1-\varepsilon} - 1\right) M_1(u) + up_1(u) + N\big(p(u)\big) - N_1\big(p_1(u)\big) \quad \text{(by (19))} \\
&\leq \left(\frac{1}{1-\varepsilon} - 1\right) M_1(u) + up_1(u) + N\big(p(u)\big) - \int_0^{p_1(u)} \left(q\left(\frac{s}{1+2\varepsilon}\right) - \varepsilon\right) ds \quad \text{(by (18))} \\
&= \left(\frac{1}{1-\varepsilon} - 1\right) M_1(u) + up_1(u) + \int_0^{p(u)} q(s)\, ds - (1+2\varepsilon) \int_0^{\frac{p_1(u)}{1+2\varepsilon}} q(s)\, ds + \varepsilon p_1(u) \\
&= \left(\frac{1}{1-\varepsilon} - 1\right) M_1(u) + up_1(u) + N\big(p(u)\big) - (1+2\varepsilon) N\left(\frac{p_1(u)}{1+2\varepsilon}\right) + \varepsilon p_1(u).
\end{aligned}
$$

Let $\varepsilon \to 0$, together with expression (13), we get $p_1(u) \to p(u)$, $M_1(u) \to M(u)$, and $N(\frac{p_1(u)}{1+2\varepsilon}) \to N(p(u))$ since $N(v)$ is continuous.

Therefore,

$$M(u) + N\big(p(u)\big) \leq up(u). \tag{20}$$

On the other hand, we have got the inequality $uv \leq M(u) + N(v)$.

Let $v = p(u)$, we have

$$up(u) \leq M(u) + N\big(p(u)\big).$$

Therefore, together with expression (20), we have

$$M(u) + N\big(p(u)\big) = up(u).$$

That is, the sufficiency of the equality holds. □

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