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The estimates of diagonally dominant degree and eigenvalues distributions for the Schur complements of matrices

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Abstract

By applying the properties of Schur complement and some inequality techniques, some new estimates of diagonally dominant degree on the Schur complement of matrices are obtained, which improve the main results of Liu (SIAM J. Matrix Anal. Appl. 27:665-674, 2005) and Liu (Linear Algebra Appl. 432:1090-1104, 2010). As an application, we present some new distribution theorems for eigenvalues of the Schur complement. Finally, we give a numerical example to illustrate the theory results.

MSC: 15A45; 15A48

Keywords: matrix; Schur complement; diagonally dominant degree; eigenvalue distribution

1 Introduction

Let $\mathbb{C}^{n\times n}$ denote the set of all $n\times n$ complex matrices, $N=\{1,2,\dots,n\}$ and $A=(a_{ij})\in\mathbb{C}^{n\times n}$ $(n\geq 2)$. Denote

$$R_i(A)=\sum_{j\neq i}|a_{ij}|,\qquad C_i(A)=\sum_{j\neq i}|a_{ji}|,\quad i\in N,$$

$$N_r(A)=\{i:|a_{ii}|>R_i(A),i\in N\},\qquad N_c(A)=\{i:|a_{ii}|>C_i(A),i\in N\}.$$

We call A a strictly diagonally dominant matrix (abbreviated SD_n) if

$$|a_{ii}|>R_i(A),\quad \forall i\in N.$$

A is called an Ostrowski matrix (abbreviated OS_n) (see [1]) if

$$|a_{ii}||a_{jj}|>R_i(A)R_j(A),\quad \forall i,j\in N,i\neq j.$$

As in [2], for all $i\in N$ and $\alpha\in[0,1]$, we call $|a_{ii}|-R_i(A)$, $|a_{ii}|-\alpha R_i(A)-(1-\alpha)C_i(A)$ and $|a_{ii}|-[R_i(A)]^\alpha[C_i(A)]^{1-\alpha}$ the ith dominant degree, α-dominant degree and product α-dominant degree of A, respectively.

For $\beta\subseteq N$, denote by $|\beta|$ the cardinality of β and $\overline{\beta}=N/\beta$. If $\beta,\gamma\subseteq N$, then $A(\beta,\gamma)$ is the submatrix of A lying in the rows indexed by β and the columns indicated by γ. In particular, $A(\beta,\beta)$ is abbreviated to $A(\beta)$. If $A(\beta)$ is nonsingular, then

$$A/\beta = A/A(\beta) = A(\overline{\beta}) - A(\overline{\beta},\beta)\big[A(\beta)\big]^{-1}A(\beta,\overline{\beta})$$

is called the Schur complement of A with respect to $A(\beta)$.

The comparison matrix of A, $\mu(A) = (\alpha_{ij})$ is defined by

$$\alpha_{ij} = \begin{cases} |a_{ij}|, & \text{if } i = j, \\ -|a_{ij}|, & \text{if } i \neq j. \end{cases}$$

A matrix A is called an M-matrix if there exist a nonnegative matrix B and a real number $s > \rho(B)$ such that $A = sI - B$, where $\rho(B)$ is the spectral radius of B. It is well known that A is an H-matrix if and only if $\mu(A)$ is an M-matrix, and if A is an M-matrix, then the Schur complement of A is also an M-matrix and $\det A > 0$ (see [3]). We denote by H_n and M_n the sets of H-matrices and M-matrices, respectively.

The Schur complement has been proved to be a useful tool in many fields such as control theory, statistics and computational mathematics, and many works have been done on it (see [4–8]). Meanwhile, studying the locations of eigenvalues of the Schur complement of matrices is of great significance as shown in [2, 3, 9–14]. In this paper, we present some new estimates of diagonally dominant degree on the Schur complement of matrices and use them to study the distributions for the eigenvalues of the Schur complement of matrices.

The paper is organized as follows. In Section 2, we give several new estimates of the diagonally dominant degree, the α-diagonally dominant degree and product α-diagonally dominant degree on the Schur complement of matrices. In Section 3, several new distribution theorems for eigenvalues of the Schur complements are obtained. In Section 4, we present a numerical example to illustrate the theory results.

2 Estimates of diagonally dominant degree for the Schur complement

In this section, we present several new estimates for the diagonally dominant degree, the α-diagonally dominant degree and product α-diagonally dominant degree of the Schur complement of matrices.

Lemma 1 [5] *If $A \in H_n$, then $[\mu(A)]^{-1} \geq |A^{-1}|$.*

Lemma 2 [5] *If A is an SD_n or A is an OS_n, then $A \in H_n$, i.e., $\mu(A) \in M_n$.*

Lemma 3 [3] *If A is an SD_n or A is an OS_n and $\beta \subseteq N$, then the Schur complement of A is an $SD_{|\overline{\beta}|}$ or an $OS_{|\overline{\beta}|}$, where $\overline{\beta} = N - \beta$ is the complement of β in N, and $|\overline{\beta}|$ is the cardinality of $\overline{\beta}$.*

Lemma 4 [2] *Let $a > b$, $c > b$, $b > 0$ and $0 \leq \alpha \leq 1$. Then*

$$a^{\alpha}c^{1-\alpha} \geq (a-b)^{\alpha}(c-b)^{1-\alpha} + b.$$

Theorem 1 *Let* $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $\beta = \{i_1, i_2, \ldots, i_k\} \subseteq N_r(A) \neq \phi$, $\overline{\beta} = \{j_1, j_2, \ldots, j_l\}$, $k < n$, *and let* $A/\beta = (a'_{ts})$. *Then for all* $1 \le t \le l$,

$$\left|a'_{tt}\right| - R_t(A/\beta) \ge |a_{j_tj_t}| - R_{j_t}(A) + \delta_{j_t} \ge |a_{j_tj_t}| - R_{j_t}(A), \tag{1}$$

and

$$\left|a'_{tt}\right| + R_t(A/\beta) \le |a_{j_tj_t}| + R_{j_t}(A) - \delta_{j_t} \le |a_{j_tj_t}| + R_{j_t}(A), \tag{2}$$

where

$$\delta_{j_t} = \min_{1\le u\le k} \frac{|a_{i_ui_u}| - P_{i_u}(A)}{|a_{i_ui_u}|} \sum_{v=1}^{k} |a_{j_ti_v}|, \qquad r = \max_{1\le u\le k} \frac{\sum_{v=1}^{l} |a_{i_uj_v}|}{|a_{i_ui_u}| - \sum_{v\neq u}^{k} |a_{i_ui_v}|},$$

$$P_{i_u}(A) = r \sum_{v\neq u}^{k} |a_{i_ui_v}| + \sum_{v=1}^{l} |a_{i_uj_v}|, \quad i_u \in \beta, 1 \le u \le k.$$

Proof Since $\beta \subseteq N_r(A) \neq \phi$, then $A(\beta) \in H_k$, $\mu(A(\beta)) \in M_k$. From Lemma 1 and Lemma 2, we have

$$\left[\mu\left(A(\beta)\right)\right]^{-1} \ge \left|\left[A(\beta)\right]^{-1}\right|.$$

Thus, for $1 \le t \le l$,

$$\begin{aligned}
&\left|a'_{tt}\right| - R_t(A/\beta) \\
&\quad = \left|a'_{tt}\right| - \sum_{s\neq t}^{l} \left|a'_{ts}\right| \\
&\quad \ge |a_{j_tj_t}| - \sum_{s\neq t}^{l} |a_{j_tj_s}| - \sum_{s=1}^{l} \left(|a_{j_ti_1}|, \ldots, |a_{j_ti_k}|\right)\left[\mu\left(A(\beta)\right)\right]^{-1} \begin{pmatrix} |a_{i_1j_s}| \\ \vdots \\ |a_{i_kj_s}| \end{pmatrix} \\
&\quad = |a_{j_tj_t}| - R_{j_t}(A) + \sum_{v=1}^{k} |a_{j_ti_v}| + \delta_{j_t} - \delta_{j_t} - \sum_{s=1}^{l} \left(|a_{j_ti_1}|, \ldots, |a_{j_ti_k}|\right)\left[\mu\left(A(\beta)\right)\right]^{-1} \begin{pmatrix} |a_{i_1j_s}| \\ \vdots \\ |a_{i_kj_s}| \end{pmatrix} \\
&\quad = |a_{j_tj_t}| - R_{j_t}(A) + \delta_{j_t} \\
&\qquad + \frac{1}{\det[\mu(A(\beta))]} \det \begin{pmatrix} \sum_{v=1}^{k} |a_{j_ti_v}| - \delta_{j_t} & -|a_{j_ti_1}| & \cdots & -|a_{j_ti_k}| \\ -\sum_{s=1}^{l} |a_{i_1j_s}| & & & \\ \vdots & & \mu(A(\beta)) & \\ -\sum_{s=1}^{l} |a_{i_kj_s}| & & & \end{pmatrix} \\
&\quad \stackrel{\text{def.}}{=} |a_{j_tj_t}| - R_{j_t}(A) + \delta_{j_t} + \frac{1}{\det[\mu(A(\beta))]} \det B.
\end{aligned}$$

Similarly to the proof of Lemma 4 in [13], we can prove that $\det B \ge 0$. Thus, we obtain Inequation (1). Similarly, we can prove Inequation (2). □

Remark 1 Note that

$$\frac{P_{i_u}(A)}{|a_{i_u i_u}|} \le r \le \frac{R_{i_u}(A)}{|a_{i_u i_u}|}, \quad 1 \le u \le k.$$

This shows that Theorem 1 improves Theorem 1 of [13].

Theorem 2 *Let* $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $\beta = \{i_1, i_2, \ldots, i_k\} \subseteq N_r(A) \cap N_c(A) \neq \phi$, $\overline{\beta} = \{j_1, j_2, \ldots, j_l\}$, $k < n$, *and let* $A/\beta = (a'_{ts})$. *Then for all* $1 \le t \le l$, $0 \le \alpha \le 1$,

$$\left|a'_{tt}\right| - \left(R_t(A/\beta)\right)^{\alpha}\left(C_t(A/\beta)\right)^{1-\alpha} \ge |a_{j_t j_t}| - \left(R_{j_t}(A) - \delta_t\right)^{\alpha}\left(C_{j_t}(A) - \delta_t^T\right)^{1-\alpha}, \tag{3}$$

and

$$\left|a'_{tt}\right| + \left(R_t(A/\beta)\right)^{\alpha}\left(C_t(A/\beta)\right)^{1-\alpha} \le |a_{j_t j_t}| + \left(R_{j_t}(A) - \delta_t\right)^{\alpha}\left(C_{j_t}(A) - \delta_t^T\right)^{1-\alpha}, \tag{4}$$

where

$$\delta_t = \min_{1\le\omega\le k} \frac{|a_{i_\omega i_\omega}| - P_{i_\omega}(A)}{|a_{i_\omega i_\omega}|} \sum_{v=1}^{k} |a_{j_t i_v}|, \qquad \eta = \max_{1\le\omega\le k} \frac{\sum_{v=1}^{l} |a_{i_\omega j_v}|}{|a_{i_\omega i_\omega}| - \sum_{v\neq\omega}^{k} |a_{i_\omega i_v}|},$$

$$\delta_t^T = \min_{1\le\omega\le k} \frac{|a_{i_\omega i_\omega}| - Q_{i_\omega}(A)}{|a_{i_\omega i_\omega}|} \sum_{v=1}^{k} |a_{i_v j_t}|, \qquad \xi = \max_{1\le\omega\le k} \frac{\sum_{v=1}^{l} |a_{j_v i_\omega}|}{|a_{i_\omega i_\omega}| - \sum_{v\neq\omega}^{k} |a_{i_v i_\omega}|},$$

$$P_{i_\omega}(A) = \eta \sum_{v\neq\omega}^{k} |a_{i_\omega i_v}| + \sum_{v=1}^{l} |a_{i_\omega j_v}|, \qquad Q_{i_\omega}(A) = \xi \sum_{v\neq\omega}^{k} |a_{i_v i_\omega}| + \sum_{v=1}^{l} |a_{j_v i_\omega}|.$$

Proof Since $\beta \subseteq N_r(A) \neq \phi$, then $A(\beta) \in H_k$, $\mu(A(\beta)) \in M_k$. From Lemma 1 and Lemma 2, we have

$$\left[\mu\left(A(\beta)\right)\right]^{-1} \ge \left|\left[A(\beta)\right]^{-1}\right|.$$

Thus, for all $1 \le t \le l$, $0 \le \alpha \le 1$,

$$\begin{aligned}
&\left|a'_{tt}\right| - \left(R_t(A/\beta)\right)^{\alpha}\left(C_t(A/\beta)\right)^{1-\alpha} \\
&\quad \ge |a_{j_t j_t}| - \left(|a_{j_t i_1}|, \ldots, |a_{j_t i_k}|\right)\left[\mu\left(A(\beta)\right)\right]^{-1} \begin{pmatrix} |a_{i_1 j_t}| \\ \vdots \\ |a_{i_k j_t}| \end{pmatrix} \\
&\qquad - \left(\sum_{s\neq t}^{l} \left[|a_{j_t j_s}| + \left(|a_{j_t i_1}|, \ldots, |a_{j_t i_k}|\right)\left[\mu\left(A(\beta)\right)\right]^{-1} \begin{pmatrix} |a_{i_1 j_s}| \\ \vdots \\ |a_{i_k j_s}| \end{pmatrix} \right]\right)^{\alpha} \\
&\qquad \times \left(\sum_{s\neq t}^{l} \left[|a_{j_s j_t}| + \left(|a_{j_s i_1}|, \ldots, |a_{j_s i_k}|\right)\left[\mu\left(A(\beta)\right)\right]^{-1} \begin{pmatrix} |a_{i_1 j_t}| \\ \vdots \\ |a_{i_k j_t}| \end{pmatrix} \right]\right)^{1-\alpha}.
\end{aligned}$$

Let

$$\zeta = \left(|a_{j_t i_1}|, \ldots, |a_{j_t i_k}|\right)\left[\mu\left(A(\beta)\right)\right]^{-1}\begin{pmatrix}|a_{i_1 j_t}|\\ \vdots \\ |a_{i_k j_t}|\end{pmatrix}.$$

By the proof of Theorem 1, we have

$$\sum_{s\neq t}^{l}\left[|a_{j_t j_s}| + \left(|a_{j_t i_1}|, \ldots, |a_{j_t i_k}|\right)\left[\mu\left(A(\beta)\right)\right]^{-1}\begin{pmatrix}|a_{i_1 j_s}|\\ \vdots \\ |a_{i_k j_s}|\end{pmatrix}\right] \leq R_{j_t}(A) - \delta_t - \zeta.$$

Similarly,

$$\sum_{s\neq t}^{l}\left[|a_{j_s j_t}| + \left(|a_{j_s i_1}|, \ldots, |a_{j_s i_k}|\right)\left[\mu\left(A(\beta)\right)\right]^{-1}\begin{pmatrix}|a_{i_1 j_t}|\\ \vdots \\ |a_{i_k j_t}|\end{pmatrix}\right] \leq C_{j_t}(A) - \delta_t^T - \zeta.$$

By Lemma 4, we have

$$\begin{aligned}
&\left|a'_{tt}\right| - \left(R_t(A/\beta)\right)^{\alpha}\left(C_t(A/\beta)\right)^{1-\alpha}\\
&\quad\geq |a_{j_t j_t}| - \zeta - \left(R_{j_t}(A) - \delta_t - \zeta\right)^{\alpha}\left(C_{j_t}(A) - \delta_t^T - \zeta\right)^{1-\alpha}\\
&\quad\geq |a_{j_t j_t}| - \zeta - \left[\left(R_{j_t}(A) - \delta_t\right)^{\alpha}\left(C_{j_t}(A) - \delta_t^T\right)^{1-\alpha} - \zeta\right]\\
&\quad= |a_{j_t j_t}| - \left(R_{j_t}(A) - \delta_t\right)^{\alpha}\left(C_{j_t}(A) - \delta_t^T\right)^{1-\alpha}.
\end{aligned}$$

Thus, we obtain Inequation (3). Similarly, we can prove Inequation (4). □

Remark 2 Note that

$$\delta_t = \min_{1\leq\omega\leq k}\frac{|a_{i_\omega i_\omega}| - P_{i_\omega}(A)}{|a_{i_\omega i_\omega}|}\sum_{\nu=1}^{k}|a_{j_t i_\nu}| \geq \min_{1\leq\omega\leq k}\frac{|a_{i_\omega i_\omega}| - R_{i_\omega}(A)}{|a_{i_\omega i_\omega}|}\sum_{\nu=1}^{k}|a_{j_t i_\nu}|,$$

$$\delta_t^T = \min_{1\leq\omega\leq k}\frac{|a_{i_\omega i_\omega}| - Q_{i_\omega}(A)}{|a_{i_\omega i_\omega}|}\sum_{\nu=1}^{k}|a_{i_\nu j_t}| \geq \min_{1\leq\omega\leq k}\frac{|a_{i_\omega i_\omega}| - C_{i_\omega}(A)}{|a_{i_\omega i_\omega}|}\sum_{\nu=1}^{k}|a_{i_\nu j_t}|.$$

This shows that Theorem 2 improves Theorem 2 of [2].

Similarly to the proof of Theorem 2, we can prove the following theorem.

Theorem 3 *Let* $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $\beta = \{i_1, i_2, \ldots, i_k\} \subseteq N_r(A) \cap N_c(A) \neq \phi$, $\overline{\beta} = \{j_1, j_2, \ldots, j_l\}$, $k < n$ *and* $A/\beta = (a'_{ts})$. *Then for all* $1 \leq t \leq l$, $0 \leq \alpha \leq 1$,

$$\begin{aligned}
&\left|a'_{tt}\right| - \alpha R_t(A/\beta) - (1-\alpha)C_t(A/\beta)\\
&\quad\geq |a_{j_t j_t}| - \alpha R_{j_t}(A) - (1-\alpha)C_{j_t}(A) + \alpha\delta_t + (1-\alpha)\delta_t^T\\
&\quad\geq |a_{j_t j_t}| - \alpha R_{j_t}(A) - (1-\alpha)C_{j_t}(A),
\end{aligned}$$

and

$$\begin{aligned}
&\left|a'_{tt}\right| + \alpha R_t(A/\beta) + (1-\alpha)C_t(A/\beta) \\
&\quad \leq |a_{j_tj_t}| + \alpha R_{j_t}(A) + (1-\alpha)C_{j_t}(A) - \alpha\delta_t - (1-\alpha)\delta_t^T \\
&\quad \leq |a_{j_tj_t}| + \alpha R_{j_t}(A) + (1-\alpha)C_{j_t}(A).
\end{aligned}$$

3 Distribution for eigenvalues of the Schur complement

In this section, we present two new distribution theorems for eigenvalues of the Schur complement.

Lemma 5 [2] *Let* $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ *and* $0 \leq \alpha \leq 1$. *Then for any eigenvalue* λ *of* A, *there exists* $1 \leq t \leq n$ *such that*

$$|\lambda - a_{tt}| \leq \big(R_t(A)\big)^{\alpha}\big(C_t(A)\big)^{1-\alpha}.$$

Theorem 4 *Let* $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $\beta = \{i_1, i_2, \ldots, i_k\} \subseteq N_r(A) \neq \phi$, $\overline{\beta} = \{j_1, j_2, \ldots, j_l\}$, *and let* $A/\beta = (a'_{ts})$. *Then for any eigenvalue* λ *of* A/β, *there exists* $1 \leq t \leq l$ *such that*

$$|\lambda - a_{j_tj_t}| \leq R_{j_t}(A) - \delta_{j_t} \leq R_{j_t}(A). \tag{5}$$

Proof Let λ be an eigenvalue of A/β. From the famous Gerschgorin circle theorem, we know that there exists $1 \leq t \leq l$ such that $|\lambda - a'_{tt}| \leq R_t(A/\beta)$. Hence

$$\begin{aligned}
0 &\geq \left|\lambda - a'_{tt}\right| - R_t(A/\beta) \\
&= \left|\lambda - a_{j_tj_t} + (a_{j_ti_1}, \ldots, a_{j_ti_k})\big[A(\beta)\big]^{-1}\begin{pmatrix} a_{i_1j_t} \\ \vdots \\ a_{i_kj_t}\end{pmatrix}\right| \\
&\quad - \sum_{s=1,\neq t}^{l}\left|a_{j_tj_s} - (a_{j_ti_1}, \ldots, a_{j_ti_k})\big[A(\beta)\big]^{-1}\begin{pmatrix} a_{i_1j_s} \\ \vdots \\ a_{i_kj_s}\end{pmatrix}\right| \\
&\geq |\lambda - a_{j_tj_t}| - \sum_{s=1,\neq t}^{l}|a_{j_tj_s}| - \sum_{s=1}^{l}\big(|a_{j_ti_1}|, \ldots, |a_{j_ti_k}|\big)\big[\mu\big(A(\beta)\big)\big]^{-1}\begin{pmatrix} |a_{i_1j_s}| \\ \vdots \\ |a_{i_kj_s}|\end{pmatrix} \\
&= |\lambda - a_{j_tj_t}| - R_{j_t}(A) + \sum_{v=1}^{k}|a_{j_ti_v}| + \delta_{j_t} - \delta_{j_t} \\
&\quad - \sum_{s=1}^{l}\big(|a_{j_ti_1}|, \ldots, |a_{j_ti_k}|\big)\big[\mu\big(A(\beta)\big)\big]^{-1}\begin{pmatrix} |a_{i_1j_s}| \\ \vdots \\ |a_{i_kj_s}|\end{pmatrix} \\
&\geq |\lambda - a_{j_tj_t}| - R_{j_t}(A) + \delta_{j_t},
\end{aligned}$$

that is

$$|\lambda - a_{j_tj_t}| \le R_{j_t}(A) - \delta_{j_t} \le R_{j_t}(A).$$

Thus, Inequation (5) holds. □

Theorem 5 *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $\beta = \{i_1, i_2, \ldots, i_k\} \subseteq N_r(A) \cap N_c(A) \neq \phi$, $\overline{\beta} = \{j_1, j_2, \ldots, j_l\}$, $k < n$, and let $A/\beta = (a'_{ts})$. Then for any $0 \le \alpha \le 1$ and every eigenvalue λ of A/β, there exists $1 \le t \le l$ such that*

$$|\lambda - a_{j_tj_t}| \le \left(R_{j_t}(A) - \delta_t\right)^{\alpha} \left(C_{j_t}(A) - \delta_t^T\right)^{1-\alpha} \le \left(R_{j_t}(A)\right)^{\alpha} \left(C_{j_t}(A)\right)^{1-\alpha}. \tag{6}$$

Proof Let λ be an eigenvalue of A/β. By Lemma 5, we know that there exists $1 \le t \le l$ such that

$$\left|\lambda - a'_{tt}\right| \le \left(R_t(A/\beta)\right)^{\alpha} \left(C_t(A/\beta)\right)^{1-\alpha}, \quad 0 \le \alpha \le 1.$$

Thus,

$$\begin{aligned}
0 &\ge \left|\lambda - a'_{tt}\right| - \left(R_t(A/\beta)\right)^{\alpha} \left(C_t(A/\beta)\right)^{1-\alpha} \\
&= \left|\lambda - a_{j_tj_t} - (a_{j_ti_1}, \ldots, a_{j_ti_k})\left[A(\beta)\right]^{-1} \begin{pmatrix} a_{i_1j_t} \\ \vdots \\ a_{i_kj_t} \end{pmatrix}\right| \\
&\quad - \left[\sum_{s=1,\neq t}^{l} \left|a_{j_tj_s} + (a_{j_ti_1}, \ldots, a_{j_ti_k})\left[A(\beta)\right]^{-1} \begin{pmatrix} a_{i_1j_s} \\ \vdots \\ a_{i_kj_s} \end{pmatrix}\right|\right]^{\alpha} \\
&\quad \times \left[\sum_{s=1,\neq t}^{l} \left|a_{j_sj_t} + (a_{j_si_1}, \ldots, a_{j_si_k})\left[A(\beta)\right]^{-1} \begin{pmatrix} a_{i_1j_t} \\ \vdots \\ a_{i_kj_t} \end{pmatrix}\right|\right]^{1-\alpha} \\
&\ge |\lambda - a_{j_tj_t}| - \left|(a_{j_ti_1}, \ldots, a_{j_ti_k})\left[A(\beta)\right]^{-1} \begin{pmatrix} a_{i_1j_t} \\ \vdots \\ a_{i_kj_t} \end{pmatrix}\right| \\
&\quad - \left(\sum_{s=1,\neq t}^{l} \left[|a_{j_tj_s}| + \left|(a_{j_ti_1}, \ldots, a_{j_ti_k})\left[A(\beta)\right]^{-1} \begin{pmatrix} a_{i_1j_s} \\ \vdots \\ a_{i_kj_s} \end{pmatrix}\right|\right]\right)^{\alpha} \\
&\quad \times \left(\sum_{s=1,\neq t}^{l} \left[|a_{j_sj_t}| + \left|(a_{j_si_1}, \ldots, a_{j_si_k})\left[A(\beta)\right]^{-1} \begin{pmatrix} a_{i_1j_t} \\ \vdots \\ a_{i_kj_t} \end{pmatrix}\right|\right]\right)^{1-\alpha}.
\end{aligned}$$

Similarly to the proof of Theorem 2, we can prove

$$\begin{aligned}
&\left|(a_{j_t i_1},\ldots,a_{j_t i_k})\big[A(\beta)\big]^{-1}\begin{pmatrix}a_{i_1 j_t}\\ \vdots\\ a_{i_k j_t}\end{pmatrix}\right| \\
&\quad + \left(\sum_{s=1,\neq t}^{l}\left[|a_{j_t j_s}| + \left|(a_{j_t i_1},\ldots,a_{j_t i_k})\big[A(\beta)\big]^{-1}\begin{pmatrix}a_{i_1 j_s}\\ \vdots\\ a_{i_k j_s}\end{pmatrix}\right|\right]\right)^{\alpha} \\
&\quad \times \left(\sum_{s=1,\neq t}^{l}\left[|a_{j_s j_t}| + \left|(a_{j_s i_1},\ldots,a_{j_s i_k})\big[A(\beta)\big]^{-1}\begin{pmatrix}a_{i_1 j_t}\\ \vdots\\ a_{i_k j_t}\end{pmatrix}\right|\right]\right)^{1-\alpha} \\
&\leq \big(R_{j_t}(A)-\delta_t\big)^{\alpha}\big(C_{j_t}(A)-\delta_t^T\big)^{1-\alpha}.
\end{aligned}$$

Therefore, we have

$$0 \geq \left|\lambda - a'_{tt}\right| - \big(R_t(A/\beta)\big)^{\alpha}\big(C_t(A/\beta)\big)^{1-\alpha} \geq |\lambda - a_{j_t j_t}| - \big(R_{j_t}(A)-\delta_t\big)\big(C_{j_t}(A)-\delta_t^T\big)^{1-\alpha}.$$

That is, Inequation (6) holds. □

4 A numerical example

In this section, we present a numerical example to illustrate the theory results.

Example 1 Let

$$A = \begin{pmatrix} 16 & 1 & 5 & 2 & 2\\ 3 & 15 & 2 & 4 & 3\\ 2 & 2 & 18 & 1 & 4\\ 5 & 3 & 5 & 8 & 2\\ 5 & 2 & 2 & 3 & 9 \end{pmatrix}, \qquad \beta = \{1,3\}.$$

By calculation with Matlab 7.1, we have that

$$\begin{array}{lllll}
R_1(A)=10; & R_2(A)=12; & R_3(A)=9; & R_4(A)=15; & R_5(A)=12;\\
C_1(A)=15; & C_2(A)=8; & C_3(A)=14; & C_4(A)=10; & C_5(A)=11;\\
\delta_2 = 2.7273; & \delta_4 = 5.4545; & \delta_5 = 3.8182; & & \\
\delta_2^T = 0.2143; & \delta_4^T = 0.2143; & \delta_5^T = 0.4286. & &
\end{array}$$

Since $\beta \in N_r(A)$, by Theorem 4, any eigenvalue λ of A/β satisfies

$$\lambda \in \big\{\lambda : |\lambda - 15| \leq 9.2727\big\} \cup \big\{\lambda : |\lambda - 8| \leq 9.5455\big\} \cup \big\{\lambda : |\lambda - 9| \leq 8.1818\big\} = \Gamma_1.$$

From Theorem 3 of [2], any eigenvalue λ of A/β satisfies

$$\lambda \in \big\{\lambda : |\lambda - 15| \leq 10.1250\big\} \cup \big\{\lambda : |\lambda - 8| \leq 11.2500\big\} \cup \big\{\lambda : |\lambda - 9| \leq 9.3750\big\} = \Gamma'_1.$$

Evidently, $\Gamma_1 \subset \Gamma'_1$, we use Figure 1 to show the fact.

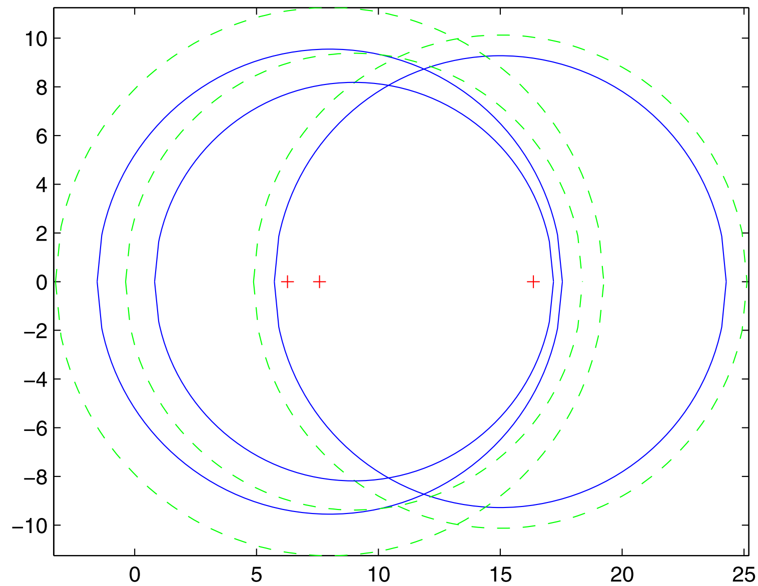

Figure 1 The blue solid line and the green dashed line denote the corresponding discs Γ_1 and Γ_1', respectively.

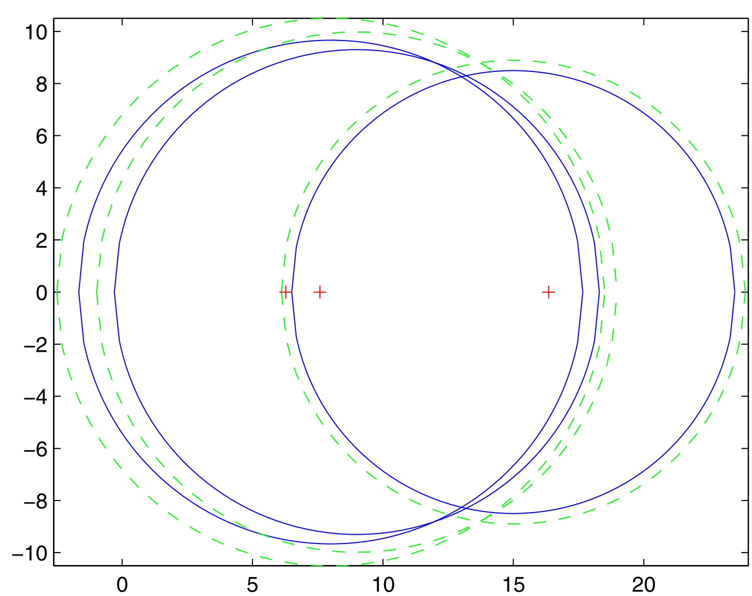

Figure 2 The blue solid line and the green dashed line denote the corresponding discs Γ_2 and Γ_2', respectively.

Meanwhile, since $\beta \in N_r(A) \cap N_c(A)$, by taking $\alpha = 0.5$ in Theorem 5, any eigenvalue λ of A/β satisfies

$$\lambda \in \{\lambda : |\lambda - 15| \leq 8.4968\} \cup \{\lambda : |\lambda - 8| \leq 9.6648\} \cup \{\lambda : |\lambda - 9| \leq 9.3002\} = \Gamma_2.$$

From Theorem 4 of [2], any eigenvalue λ of A/β satisfies

$$\lambda \in \{\lambda : |\lambda - 15| \leq 8.8939\} \cup \{\lambda : |\lambda - 8| \leq 10.5067\} \cup \{\lambda : |\lambda - 9| \leq 9.9804\} = \Gamma_2'.$$

Evidently, $\Gamma_2 \subset \Gamma_2'$, we use Figure 2 to show the fact.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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