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Nonlinear integral inequalities with delay for discontinuous functions and their applications

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Abstract

This paper investigates integral inequalities with delay for discontinuous functions involving two nonlinear terms. We do not require the classes $\wp$ and $\jmath$ in Gallo and Piccirillo's paper (Bound. Value Probl. 2009:808124, 2009). Our main results can be applied to generalize Gallo and Piccirillo's results and Iovane's results (Nonlinear Anal., Theory Methods Appl. 66:498-508, 2007). Examples to show the bounds of solutions of an impulsive differential equation are also given, which can not be estimated by Gallo and Piccirillo's results.

MSC: 26D15; 26D20

Keywords: integral inequalities; discontinuous functions; nonlinear; impulsive differential equations

1 Introduction

The Gronwall-Bellman integral inequalities and their various linear and nonlinear generalizations, involving continuous or discontinuous functions, play very important roles in investigating different qualitative characteristics of solutions for differential equations and impulsive differential equations such as existence, uniqueness, continuation, boundedness, continuous dependence of parameters, stability, attraction, practical stability. The literature on inequalities for continuous functions and their applications is vast (see [1–8]). Recently, more attention has been paid to generalizations of Gronwall-Bellman's results for discontinuous functions (see [9–17]) and their applications (see [11, 18, 19]). Among them, one of the important things is that Samoilenko and Perestyuk [17] studied the following inequality

$$u(x) \le c + \int_{x_0}^{x} f(s)u(s)\,ds + \sum_{x_0<x_i<x} \beta_i u(x_i - 0) \tag{1.1}$$

about the nonnegative piecewise continuous function $u(x)$, where c, β_i are nonnegative constants, $f(s)$ is a positive function, and x_i are the first kind discontinuity points of the function $u(x)$. Then Borysenko [20] considered

$$u(x) \le c + \int_{x_0}^{x} f(s)u^m(s)\,ds + \sum_{x_0<x_i<x} \beta_i u(x_i - 0), \quad m > 0, m \ne 1,$$

$$u(x) \le c + \int_{x_0}^{x} f(s)u^m(s)\,ds + \sum_{x_0<x_i<x} \beta_i u^m(x_i - 0), \quad m > 0, m \ne 1. \tag{1.2}$$

He replaced the constant c by a positive monotonously nondecreasing function $a(x)$, and also estimated the inequalities

$$u(x) \le a(x) + \int_{x_0}^{x} f(s)u(s)\,ds + \sum_{x_0<x_i<x} \beta_i u^m(x_i - 0), \quad m > 0,$$
$$u(x) \le a(x) + \int_{x_0}^{x} f(s)u^m(s)\,ds + \sum_{x_0<x_i<x} \beta_i u^m(x_i - 0), \quad m > 0, m \ne 1. \tag{1.3}$$

In 2005, he [18] generalized the inequalities above from one integral to two integrals with a form

$$u(x) \le c + \int_{x_0}^{x} q(s)u(s)\,ds + \int_{x_0}^{x} q(s)\int_{x_0}^{s} g(\tau)u^m(\tau)\,d\tau\,ds + \sum_{x_0<x_i<x} \beta_i u^m(x_i - 0), \quad m > 0. \tag{1.4}$$

In 2007, Iovane [21] investigated the inequalities with delay

$$u(x) \le a(x) + \int_{x_0}^{x} f(s)u\bigl(b(s)\bigr)\,ds + \sum_{x_0<x_i<x} \beta_i u^m(x_i - 0), \quad m > 0,$$
$$u(x) \le a(x) + \int_{x_0}^{x} f(s)u^m\bigl(b(s)\bigr)\,ds + \sum_{x_0<x_i<x} \beta_i u^m(x_i - 0), \quad m > 0, m \ne 1. \tag{1.5}$$

Later, Gallo and Piccirillo [22] further discussed

$$u(x) \le a(x) + h(x)\int_{x_0}^{x} f(s)w\bigl(u\bigl(b(s)\bigr)\bigr)\,ds + \sum_{x_0<x_i<x} \beta_i u^m(x_i - 0) \tag{1.6}$$

with a general nonlinear term $w(u)$ of u. They assumed that $w \in \wp$ or $w \in \jmath$, where the class $\wp$ consists of all nonnegative, nondecreasing and continuous functions $w(u)$ on $[0,\infty)$ such that $w(0) = 0$ and $w(\alpha u) \le w(\alpha)w(u)$ for all $\alpha > 0$ and $u \ge 0$, and the class $\jmath$ consists of all positive, nondecreasing and continuous functions $w(u)$ on $(0,\infty)$ such that $w(0) = 0$ and $w(\alpha^{-1}u) > \alpha^{-1}w(u)$ for all $\alpha \ge 1$ and $u > 0$. The classes $\wp$ and $\jmath$ allow a reduction of $a(t)$ to the case of a constant a_0 by dividing $a(x)$ if $a(x)$ is a positive and nondecreasing function. Actually, when we study behaviors of solutions of impulsive differential equations, $a(x)$ may not be a nondecreasing function, and w may not satisfy the condition $w \in \wp$ or $w \in \jmath$. For example, $w(u) = e^u$ does not belong to the class $\wp$ and $\jmath$ for any $\alpha > 1$ and large $u > 0$. Thus, it is interesting to avoid such conditions.

Motivated by this observation, in this paper, we consider the following much more general inequality

$$u(x) \le a(x) + \int_{b_1(x_0)}^{b_1(x)} f_1(x,s)w_1\bigl(u(s)\bigr)\,ds + \int_{b_2(x_0)}^{b_2(x)} f_2(x,s)w_2\bigl(u(s)\bigr)\,ds + \sum_{x_0<x_i<x} \beta_i u^m(x_i - 0), \quad m > 0 \tag{1.7}$$

with two nonlinear terms $w_1(u)$ and $w_2(u)$ of u, where we do not restrict w_1 and w_2 to the class $\wp$ or the class $\jmath$. We also show that many integral inequalities for discontinuous functions such as (1.3), (1.4) and (1.6) can be reduced to the form of (1.7). Our main result is applied to estimate the bounds of solutions of an impulsive ordinary differential equation.

2 Main results

Consider (1.7), and assume that

(C_1) $w_1(x)$ and $w_2(x)$ are continuous and nondecreasing functions on $[0,\infty)$ and are positive on $(0,\infty)$ such that $\frac{w_2(x)}{w_1(x)}$ is nondecreasing;
(C_2) $a(x)$ is defined on $[x_0,\infty)$ and $a(x_0)\neq 0$; β_i is a nonnegative constant for any positive integer i;
(C_3) $f_1(x,s)$ and $f_2(x,s)$ are continuous and nonnegative functions on $[x_0,\infty)\times[x_0,\infty)$;
(C_4) $b_1(x)$ and $b_2(x)$ are continuously differentiable and nondecreasing such that $x_0 \le b_1(x)\le x$ and $x_0\le b_2(x)\le x$ on $[x_0,\infty)$;
(C_5) For $x\in[x_0,\infty)$, $u(x)$ is nonnegative and piecewise-continuous with the first kind of discontinuities at the points $x_i : x_0 < x_1 < \cdots$, where i is a nonnegative integer and $\lim_{i\to\infty} x_i = \infty$.

Let $W_j(u) = \int_{\tilde{u}_j}^{u}\frac{dz}{w_j(z)}$ for $u\ge\tilde{u}_j$ and $j = 1,2$, where $\tilde{u}_j$ is a given positive constant. Clearly, W_j is strictly increasing so its inverse W_j^{-1} is well defined, continuous and increasing in its corresponding domain.

Theorem 2.1 *Suppose that* (C_k) ($k = 1,\dots,5$) *hold, and* $u(x)$ *satisfies* (1.7) *for a positive constant* m. *Let* $u_i(x) = u(x)$ *for* $x\in[x_i,x_{i+1})$. *Then the estimate of* $u(x)$ *is recursively given by for* $x\in[x_i,x_{i+1})$, $i = 0,1,2,\dots$,

$$u_i(x)\le W_2^{-1}\left[W_2\circ W_1^{-1}\left(W_1\left(r_{i+1}(x)\right)+\int_{b_1(x_i)}^{b_1(x)}\tilde{f}_1(x,s)\,ds\right)+\int_{b_2(x_i)}^{b_2(x)}\tilde{f}_2(x,s)\,ds\right], \tag{2.1}$$

where

$$\tilde{f}_j(x,s) = \max_{x_0\le\tau\le x} f_j(\tau,s),\quad j = 1,2,\qquad r_1(x) = \max_{x_0\le\tau\le x}\left|a(\tau)\right|,$$

$$\begin{aligned} r_{i+1}(x) = r_1(x) &+ \sum_{k=1}^{i}\int_{b_1(x_{k-1})}^{b_1(x_k)}\tilde{f}_1(x,s)w_1\left(u_{k-1}(s)\right)ds \\ &+ \sum_{k=1}^{i}\int_{b_2(x_{k-1})}^{b_2(x_k)}\tilde{f}_2(x,s)w_2\left(u_{k-1}(s)\right)ds + \sum_{k=1}^{i}\beta_k u_{k-1}^{m}(x_k-0), \end{aligned} \tag{2.2}$$

provided that

$$\begin{aligned} &W_1\left(r_{i+1}(x)\right)+\int_{b_1(x_i)}^{b_1(x)}\tilde{f}_1(x,s)\,ds\le\int_{\tilde{u}_1}^{\infty}\frac{dz}{w_1(z)}, \\ &W_2\circ W_1^{-1}\left(W_1\left(r_{i+1}(x)\right)+\int_{b_1(x_i)}^{b_1(x)}\tilde{f}_1(x,s)\,ds\right)+\int_{b_2(x_i)}^{b_2(x)}\tilde{f}_2(x,s)\,ds\le\int_{\tilde{u}_2}^{\infty}\frac{dz}{w_2(z)}. \end{aligned} \tag{2.3}$$

The proof is given in Section 3.

Remark 2.1 (1) If w_j satisfies $\int_{\tilde{u}_j}^{\infty}\frac{dz}{w_j(z)} = \infty$ for $j = 1,2$, then i in Theorem 2.1 can be any nonzero integer. [6] pointed out that different choices of $\tilde{u}_j$ in W_j do not affect our results for $j = 1,2$. If $a(x)\equiv 0$, then define $W_1(0) = 0$, and (2.1) is still true.

(2) Take $b_1(x) = x$, $a(x) = c$, $f_1(t,s) = f(s)$, $f_2(t,s) = 0$, $w_1(u) = u$ and $m = 1$. Hence, (1.7) becomes (1.1). It is easy to check that $W_1(u) = \ln \frac{u}{\tilde{u}_1}$ and $W_1^{-1}(u) = \tilde{u}_1 e^u$. From Theorem 2.1, we know that for $x \in [x_i, x_{i+1})$,

$$u_i(x) \le r_{i+1}(x) e^{\int_{x_i}^x f(s)\,ds}$$

with

$$r_{i+1}(x) = c + \sum_{k=1}^{i} \int_{x_{k-1}}^{x_k} f(s) u_{k-1}(s)\,ds + \sum_{k=1}^{i} \beta_k u_{k-1}(x_k - 0).$$

Hence,

$$\begin{aligned}
r_1(x) &= c, \qquad u_0(x) \le c e^{\int_{x_0}^x f(s)\,ds}, \\
r_2(x) &= c + \int_{x_0}^{x_1} f(s) u_0(s)\,ds + \beta_1 u_0(x_1 - 0) \\
&\le c + \int_{x_0}^{x_1} f(s) c e^{\int_{x_0}^s f(\tau)\,d\tau}\,ds + c\beta_1 e^{\int_{x_0}^{x_1} f(s)\,ds} \\
&= c + c e^{\int_{x_0}^s f(\tau)\,d\tau} \big|_{x_0}^{x_1} + c\beta_1 e^{\int_{x_0}^{x_1} f(s)\,ds} = c(1 + \beta_1) e^{\int_{x_0}^{x_1} f(s)\,ds}, \\
u_1(x) &\le c(1 + \beta_1) e^{\int_{x_0}^x f(s)\,ds}.
\end{aligned}$$

After recursive calculations, we have for $x \ge x_0$

$$u(x) \le c \Pi_{x_0 < x_k < x} (1 + \beta_k) e^{\int_{x_0}^x f(s)\,ds},$$

which is same as the one in [17].

(3) Clearly, (1.2) and (1.3) are special cases of (1.7). If $b'(x) > 0$ on $[x_0, \infty)$, then (1.6) can be rewritten as

$$u(x) \le a(x) + h(x) \int_{b(x_0)}^{b(x)} \frac{f(b^{-1}(s))}{b'(b^{-1}(s))} w\big(u(s)\big)\,ds + \sum_{x_0 < x_i < x} \beta_i u^m (x_i - 0).$$

Let $f_1(x,s) = h(x) \frac{f(b^{-1}(s))}{b'(b^{-1}(s))}$ and $f_2(x,s) \equiv 0$, the inequality above is same as (1.7). Similarly, (1.5) can also be reduced to (1.7).

Consider the inequality

$$\begin{aligned}
u(x) \le a(x) &+ \int_{x_0}^x g_1(x,s) \int_{x_0}^s h_1(s,\tau) w_1\big(u(\tau)\big)\,d\tau\,ds \\
&+ \int_{x_0}^x g_2(x,s) \int_{x_0}^s h_2(s,\tau) w_2\big(u(\tau)\big)\,d\tau\,ds + \sum_{x_0 < x_i < x} \beta_i u^m (x_i - 0),
\end{aligned} \tag{2.4}$$

which looks more complicated than (1.7).

Corollary 2.1 *Suppose that* (C_1)-(C_3) *and* (C_5) *hold, and that the functions* g_j *and* h_j $(j = 1,2)$ *are both nonnegative and continuous on* $[x_0,\infty)\times[x_0,\infty)$. *If* (2.4) *holds, then for* $x \in [x_i, x_{i+1})$, $i = 0,1,2,\dots,$

$$u_i(x) \le W_2^{-1}\biggl[W_2 \circ W_1^{-1}\biggl(W_1\bigl(r_{i+1}(x)\bigr) + \int_{x_i}^{x}\int_{s}^{x}\max_{x_0\le\tau\le x} g_1(\tau,\nu)h_1(\nu,s)\,d\nu\,ds\biggr) + \int_{x_i}^{x}\int_{s}^{x}\max_{x_0\le\tau\le x} g_2(\tau,\nu)h_2(\nu,s)\,d\nu\,ds\biggr], \tag{2.5}$$

where r_{i+1} *and its related functions are defined as in Theorem* 2.1 *by replacing* $f_j(x,s)$ *with* $\int_s^x \max_{x_0\le\tau\le x} g_j(\tau,\nu)h_j(\nu,s)\,d\nu$, $j = 1,2$.

Proof Because f_j, h_j and w_j are continuous, we have

$$\begin{aligned}
&\int_{x_0}^{x} g_j(x,s)\int_{x_0}^{s} h_j(s,\tau)w_j\bigl(u(\tau)\bigr)\,d\tau\,ds \\
&\quad= \int_{x_0}^{x} w_j\bigl(u(\tau)\bigr)\int_{\tau}^{x} g_j(x,s)h_j(s,\tau)\,ds\,d\tau \\
&\quad= \int_{x_0}^{x} w_j\bigl(u(s)\bigr)\int_{s}^{x} g_j(x,\tau)h_j(\tau,s)\,d\tau\,ds \le \int_{x_0}^{x} f_j(x,s)w_j\bigl(u(s)\bigr)\,ds,
\end{aligned}$$

where $f_j(x,s) := \int_s^x \max_{x_0\le\tau\le x} g_j(\tau,\nu)h_j(\nu,s)\,d\nu$. Then (2.4) is reduced to

$$\begin{aligned}
u(x) \le{}& a(x) + \int_{x_0}^{x} f_1(x,s)w_1\bigl(u(s)\bigr)\,ds + \int_{x_0}^{x} f_2(x,s)w_2\bigl(u(s)\bigr)\,ds \\
&+ \sum_{x_0<x_i<x}\beta_i u^m(x_i - 0),
\end{aligned}$$

which is just the form of (1.7), if we take $b_j(x) = x$ for $j = 1,2$. Note that for fixed s, the function $f_j(x,s)$ is increasing in x. So $\tilde{f}_j(x,s) := \max_{t_0\le\tau\le x} f_j(\tau,s) = f_j(x,s)$. By Theorem 2.1, for $x \in [x_i, x_{i+1})$, $i = 0,1,2,\dots,$

$$u_i(x) \le W_2^{-1}\biggl[W_2 \circ W_1^{-1}\biggl(W_1\bigl(r_{i+1}(x)\bigr) + \int_{x_i}^{x}\int_{s}^{x}\max_{x_0\le\tau\le x} g_1(\tau,\nu)h_1(\nu,s)\,d\nu\,ds\biggr) + \int_{x_i}^{x}\int_{s}^{x}\max_{x_0\le\tau\le x} g_2(\tau,\nu)h_2(\nu,s)\,d\nu\,ds\biggr]. \qquad \square$$

Remark 2.2 Using the same way, we can change inequality (1.4) into the form of (1.7) with $a(x) = c$, $f_1(x,s) = q(s)$, $f_2(x,s) = g(s)\int_s^x q(\tau)\,d\tau$, $w_1(u) = u$ and $w_2(u) = u^m$.

3 Proof of Theorem 2.1

Obviously, $r_1(x)$ is positive and nondecreasing in x, and $\tilde{f}_j(x,s)$ is nonnegative and nondecreasing in x for each fixed s and $j = 1,2$. They satisfy $r_1(x) \ge a(x)$ and $\tilde{f}_j(x,s) \ge f_j(x,s)$.

We first consider $x \in [x_0, x_1)$, and we have from (1.7) and (2.2)

$$\begin{aligned}
u(x) &\le a(x) + \int_{b_1(x_0)}^{b_1(x)} f_1(x,s)w_1\bigl(u(s)\bigr)\,ds + \int_{b_2(x_0)}^{b_2(x)} f_2(x,s)w_2\bigl(u(s)\bigr)\,ds \\
&\le r_1(x) + \int_{b_1(x_0)}^{b_1(x)} f_1(x,s)w_1\bigl(u(s)\bigr)\,ds + \int_{b_2(x_0)}^{b_2(x)} f_2(x,s)w_2\bigl(u(s)\bigr)\,ds.
\end{aligned} \tag{3.1}$$

Take any fixed $T \in (x_0, x_1)$, and we investigate the following inequality

$$u(x) \leq r_1(T) + \int_{b_1(x_0)}^{b_1(x)} \tilde{f}_1(T,s) w_1\bigl(u(s)\bigr)\,ds + \int_{b_2(x_0)}^{b_2(x)} \tilde{f}_2(T,s) w_2\bigl(u(s)\bigr)\,ds \tag{3.2}$$

for $x \in [x_0, T]$, where $\tilde{f}_1$ and $\tilde{f}_2$ are defined in (2.2). Let

$$z(x) = \int_{b_1(x_0)}^{b_1(x)} \tilde{f}_1(T,s) w_1\bigl(u(s)\bigr)\,ds + \int_{b_2(x_0)}^{b_2(x)} \tilde{f}_2(T,s) w_2\bigl(u(s)\bigr)\,ds$$

and $z(x_0) = 0$. Hence, $u(x) \leq r_1(T) + z(x)$. Clearly, $z(x)$ is a nonnegative, nondecreasing and differentiable function for $x \in (x_0, T]$. Moreover, $b_j(x)$ is differentiable and nondecreasing in $x \in [x_0, T]$ for $j = 1, 2$. Thus, $b_j'(x) \geq 0$ for $x \in [x_0, T]$. Since w_1 and w_2 are nondecreasing, $z(x) + r_1(T) > 0$ and $b_j(x) \leq x$ for $x \in [x_0, T]$, we have

$$\begin{aligned}
\frac{z'(x)}{w_1(z(x) + r_1(T))} &\leq \frac{b_1'(x)\tilde{f}_1(T, b_1(x))w_1(u(b_1(x)))}{w_1(z(x) + r_1(T))} + \frac{b_2'(x)\tilde{f}_2(T, b_2(x))w_2(u(b_2(x)))}{w_1(z(x) + r_1(T))} \\
&\leq \frac{b_1'(x)\tilde{f}_1(T, b_1(x))w_1(z(b_1(x)) + r_1(T))}{w_1(z(x) + r_1(T))} \\
&\quad + \frac{b_2'(x)\tilde{f}_2(T, b_2(x))w_2(z(b_2(x)) + r_1(T))}{w_1(z(x) + r_1(T))} \\
&\leq \frac{b_1'(x)\tilde{f}_1(T, b_1(x))w_1(z(x) + r_1(T))}{w_1(z(x) + r_1(T))} \\
&\quad + \frac{b_2'(x)\tilde{f}_2(T, b_2(x))w_2(z(b_2(x)) + r_1(T))}{w_1(z(x) + r_1(T))} \\
&\leq b_1'(x)\tilde{f}_1\bigl(T, b_1(x)\bigr) + \frac{b_2'(x)\tilde{f}_2(T, b_2(x))w_2(z(b_2(x)) + r_1(T))}{w_1(z(b_2(x)) + r_1(T))}.
\end{aligned}$$

Integrating both sides of the inequality above, from x_0 to x, we obtain

$$\begin{aligned}
W_1\bigl(z(x) + r_1(T)\bigr) &\leq W_1\bigl(r_1(T)\bigr) + \int_{x_0}^{x} b_1'(s)\tilde{f}_1\bigl(T, b_1(s)\bigr)\,ds \\
&\quad + \int_{x_0}^{x} b_2'(s)\tilde{f}_2\bigl(T, b_2(s)\bigr)\phi\bigl(z\bigl(b_2(s)\bigr) + r_1(T)\bigr)\,ds \\
&\leq W_1\bigl(r_1(T)\bigr) + \int_{b_1(x_0)}^{b_1(x)} \tilde{f}_1(T,s)\,ds + \int_{b_2(x_0)}^{b_2(x)} \tilde{f}_2(T,s)\phi\bigl(z(s) + r_1(T)\bigr)\,ds
\end{aligned}$$

for $x_0 < x \leq T$, where $\phi(x) = \frac{w_2(x)}{w_1(x)}$, or equivalently,

$$\xi(x) \leq W_1\bigl(r_1(T)\bigr) + \int_{b_1(x_0)}^{b_1(x)} \tilde{f}_1(T,s)\,ds + \int_{b_2(x_0)}^{b_2(x)} \tilde{f}_2(T,s)\phi\bigl(W_1^{-1}\bigl(\xi(s)\bigr)\bigr)\,ds \triangleq z_1(x),$$

where

$$\xi(x) = W_1\bigl(z(x) + r_1(T)\bigr).$$

It is easy to check that $\xi(x) \leq z_1(x)$, $z_1(x_0) = W_1(r_1(T))$ and $z_1(x)$ is differentiable, positive and nondecreasing on $(x_0, T]$. Since $\phi(W_1^{-1}(u))$ is nondecreasing from the assumption (C_1), we have by (2.3)

$$\begin{aligned}
&\frac{z_1'(x)}{\phi(W_1^{-1}(z_1(x)))} \\
&\quad\leq \frac{\tilde{f}_1(T, b_1(x))b_1'(x)}{\phi(W_1^{-1}(z_1(x)))} + \frac{\tilde{f}_2(T, b_2(x))\phi(W_1^{-1}(\xi(b_2(x))))b_2'(x)}{\phi(W_1^{-1}(z_1(x)))} \\
&\quad\leq \frac{\tilde{f}_1(T, b_1(x))b_1'(x)}{\phi(W_1^{-1}(z_1(x)))} + \frac{\tilde{f}_2(T, b_2(x))\phi(W_1^{-1}(z_1(b_2(x))))b_2'(x)}{\phi(W_1^{-1}(z_1(x)))} \\
&\quad\leq \frac{\tilde{f}_1(T, b_1(x))b_1'(x)}{\phi(W_1^{-1}(W_1(r_1(T)) + \int_{b_1(x_0)}^{b_1(x)} \tilde{f}_1(T, s)\,ds))} + \tilde{f}_2\big(T, b_2(x)\big)b_2'(x).
\end{aligned} \tag{3.3}$$

Note that

$$\begin{aligned}
\int_{x_0}^{x} \frac{z_1'(s)}{\phi(W_1^{-1}(z_1(s)))}\,ds &= \int_{x_0}^{x} \frac{w_1(W_1^{-1}(z_1(s)))z_1'(s)}{w_2(W_1^{-1}(z_1(s)))}\,ds = \int_{W_1^{-1}(z_1(x_0))}^{W_1^{-1}(z_1(x))} \frac{du}{w_2(u)} \\
&= W_2 \circ W_1^{-1}\big(z_1(x)\big) - W_2 \circ W_1^{-1}\big(z_1(x_0)\big) \\
&= W_2 \circ W_1^{-1}\big(z_1(x)\big) - W_2\big(r_1(T)\big).
\end{aligned}$$

Integrating both sides of inequality (3.3), from x_0 to x, we obtain

$$\begin{aligned}
&W_2 \circ W_1^{-1}\big(z_1(x)\big) - W_2\big(r_1(T)\big) \\
&\quad= \int_{x_0}^{x} \frac{z_1'(s)}{\phi(W_1^{-1}(z_1(s)))}\,ds \\
&\quad\leq \int_{x_0}^{x} \frac{\tilde{f}_1(T, b_1(s))b_1'(s)}{\phi(W_1^{-1}(W_1(r_1(T)) + \int_{b_1(x_0)}^{b_1(s)} \tilde{f}_1(T, \tau)\,d\tau))}\,ds + \int_{x_0}^{x} \tilde{f}_2\big(T, b_2(s)\big)b_2'(s)\,ds \\
&\quad\leq W_2 \circ W_1^{-1}\left(W_1\big(r_1(T)\big) + \int_{b_1(x_0)}^{b_1(x)} \tilde{f}_1(T, s)\,ds\right) - W_2\big(r_1(T)\big) + \int_{b_2(x_0)}^{b_2(x)} \tilde{f}_2(T, s)\,ds.
\end{aligned}$$

Thus,

$$W_2 \circ W_1^{-1}\big(z_1(x)\big) \leq W_2 \circ W_1^{-1}\left(W_1\big(r_1(T)\big) + \int_{b_1(x_0)}^{b_1(x)} \tilde{f}_1(T, s)\,ds\right) + \int_{b_2(x_0)}^{b_2(x)} \tilde{f}_2(T, s)\,ds.$$

We have by (2.3)

$$\begin{aligned}
u(x) &\leq z(x) + r_1(T) \leq W_1^{-1}\big(\xi(x)\big) \leq W_1^{-1}\big(z_1(x)\big) \\
&\leq W_2^{-1}\left[W_2 \circ W_1^{-1}\left(W_1\big(r_1(T)\big) + \int_{b_1(x_0)}^{b_1(x)} \tilde{f}_1(T, s)\,ds\right) + \int_{b_2(x_0)}^{b_2(x)} \tilde{f}_2(T, s)\,ds\right].
\end{aligned}$$

Since the inequality above is true for any $x \in [x_0, T]$, we obtain

$$u(T) \leq W_2^{-1}\left[W_2 \circ W_1^{-1}\left(W_1\big(r_1(T)\big) + \int_{b_1(x_0)}^{b_1(T)} \tilde{f}_1(T, s)\,ds\right) + \int_{b_2(x_0)}^{b_2(T)} \tilde{f}_2(T, s)\,ds\right].$$

Replacing T by x yields

$$u(x) \leq W_2^{-1}\left[W_2 \circ W_1^{-1}\left(W_1\big(r_1(x)\big) + \int_{b_1(x_0)}^{b_1(x)} \tilde{f}_1(x,s)\,ds\right) + \int_{b_2(x_0)}^{b_2(x)} \tilde{f}_2(x,s)\,ds\right]. \tag{3.4}$$

This means that (2.1) is true for $x \in [x_0, x_1)$ and $i = 0$ if replace $u(x)$ with $u_0(x)$.

For $i = 1$ and $x \in [x_1, x_2)$, (1.7) becomes

$$\begin{aligned} u(x) \leq{}& r_1(x) + \int_{b_1(x_0)}^{b_1(x_1)} f_1(x,s)w_1\big(u_0(s)\big)\,ds + \int_{b_2(x_0)}^{b_2(x_1)} f_2(x,s)w_2\big(u_0(s)\big)\,ds \\ &+ \beta_1 u_0^m(x_1 - 0) + \int_{b_1(x_1)}^{b_1(x)} f_1(x,s)w_1\big(u(s)\big)\,ds + \int_{b_2(x_1)}^{b_2(x)} f_2(x,s)w_2\big(u(s)\big)\,ds \\ \leq{}& r_2(x) + \int_{b_1(x_1)}^{b_1(x)} f_1(x,s)w_1\big(u(s)\big)\,ds + \int_{b_2(x_1)}^{b_2(x)} f_2(x,s)w_2\big(u(s)\big)\,ds, \end{aligned} \tag{3.5}$$

where the definition of $r_2(x)$ is given in (2.2). Note that the estimate of $u_0(x)$ is known. Equation (3.5) is same as (3.1) if replace $r_1(x)$ and x_0 by $r_2(x)$ and x_1. Thus, by (3.4), we have

$$u(x) \leq W_2^{-1}\left[W_2 \circ W_1^{-1}\left(W_1\big(r_2(x)\big) + \int_{b_1(x_1)}^{b_1(x)} \tilde{f}_1(x,s)\,ds\right) + \int_{b_2(x_1)}^{b_2(x)} \tilde{f}_2(x,s)\,ds\right]. \tag{3.6}$$

This implies that (2.1) is true for $x \in [x_1, x_2)$ and $i = 1$ if replace $u(x)$ by $u_1(x)$.

Assume that (2.1) is true for $x \in [x_i, x_{i+1})$, *i.e.*,

$$u_i(x) \leq W_2^{-1}\left[W_2 \circ W_1^{-1}\left(W_1\big(r_{i+1}(x)\big) + \int_{b_1(x_i)}^{b_1(x)} \tilde{f}_1(x,s)\,ds\right) + \int_{b_2(x_i)}^{b_2(x)} \tilde{f}_2(x,s)\,ds\right] \tag{3.7}$$

for $x \in [x_i, x_{i+1})$.

For $x \in [x_{i+1}, x_{i+2})$, (1.7) becomes

$$\begin{aligned} u(x) \leq{}& a(x) + \int_{b_1(x_0)}^{b_1(x)} f_1(x,s)w_1\big(u(s)\big)\,ds + \int_{b_2(x_0)}^{b_2(x)} f_2(x,s)w_2\big(u(s)\big)\,ds \\ &+ \sum_{x_0 < x_{i+1} < x} \beta_{i+1} u^m(x_{i+1} - 0) \\ \leq{}& r_1(x) + \sum_{k=0}^{i} \int_{b_1(x_k)}^{b_1(x_{k+1})} f_1(x,s)w_1\big(u_k(s)\big)\,ds + \sum_{k=0}^{i} \int_{b_2(x_k)}^{b_2(x_{k+1})} f_2(x,s)w_2\big(u_k(s)\big)\,ds \\ &+ \sum_{k=0}^{i} \beta_{k+1} u_k^m(x_{k+1} - 0) + \int_{b_1(x_{i+1})}^{b_1(x)} f_1(x,s)w_1\big(u(s)\big)\,ds \\ &+ \int_{b_2(x_{i+1})}^{b_2(x)} f_2(x,s)w_2\big(u(s)\big)\,ds \\ \leq{}& r_{i+2}(x) + \int_{b_1(x_{i+1})}^{b_1(x)} f_1(x,s)w_1\big(u(s)\big)\,ds + \int_{b_2(x_{i+1})}^{b_2(x)} f_2(x,s)w_2\big(u(s)\big)\,ds, \end{aligned} \tag{3.8}$$

where we use the fact that the estimate of $u(x)$ is already known for $x \in [x_0, x_{i+1})$ by the assumption (3.7). Again (3.8) is same as (3.1) if replace $r_1(x)$ and x_0 by $r_{i+2}(x)$ and x_{i+1}.

Thus, by (3.4), we have

$$u(x) \le W_2^{-1}\left[W_2 \circ W_1^{-1}\left(W_1\big(r_{i+2}(x)\big) + \int_{b_1(x_{i+1})}^{b_1(x)} \tilde{f}_1(x,s)\,ds\right) + \int_{b_2(x_{i+1})}^{b_2(x)} \tilde{f}_2(x,s)\,ds\right].$$

This yields that (2.1) is true for $x \in [x_{i+1}, x_{i+2})$ if replace $u(x)$ by $u_{i+1}(x)$. By induction, we know that (2.1) holds for $x \in [x_i, x_{i+1})$ for any nonnegative integer i. This completes the proof of Theorem 2.1.

4 Applications

Consider the following impulsive differential equation

$$\begin{aligned} &\frac{dy}{dx} = F(x,y), \quad x \neq x_i, \\ &\Delta y|_{x=x_i} = I_i(y), \end{aligned} \tag{4.1}$$

where $y \in \mathbf{R}^n$, $F : \mathbf{R}^{n+1} \to \mathbf{R}^n$, $I_i : \mathbf{R}^n \to \mathbf{R}^n$ $(i = 1, 2, \ldots)$, $x \ge x_0 \ge 0$, $\lim_{i\to\infty} x_i = \infty$, $x_{i-1} < x_i$ for all $i = 1, 2, \ldots$.

Assume that

(1) $\|F(x,y)\| \le h_1(x)\|y\| + h_2(x)e^{\|y\|}$, where h_1, h_2 are nonnegative and continuous on $[x_0, \infty)$;

(2) $\|I_i(y)\| \le \beta_i \|y\|^m$, where β_i and m are nonnegative constants.

The solution of (4.1) with an initial value $y(x_0) = y_0$ is given by

$$y(x) = y_0 + \int_{x_0}^{x} F(s,y)\,ds + \sum_{x_0 < x_i < x} I_i\big(y(x_i - 0)\big), \tag{4.2}$$

which implies that

$$\big\|y(x)\big\| \le \|y_0\| + \int_{x_0}^{x} \big(h_1(s)\|y\| + h_2(s)e^{\|y\|}\big)\,ds + \sum_{x_0 < x_i < x} \beta_i \big\|y(x_i - 0)\big\|^m. \tag{4.3}$$

Let

$$\begin{aligned} &u(x) = \big\|y(x)\big\|, \qquad a(x) \equiv \|y_0\|, \qquad b_1(x) = x, \qquad b_2(x) = x, \\ &f_1(x,s) = h_1(s), \qquad f_2(x,s) = h_2(s), \qquad w_1(u) = u, \qquad w_2(u) = e^u, \end{aligned}$$

so (4.3) is same as (1.7). It is easy to obtain for any positive constants $\tilde{u}_1$ and $\tilde{u}_2$

$$\begin{aligned} &r_1(x) \equiv \|y_0\|, \qquad \tilde{f}_1(x,s) = h_1(s), \qquad \tilde{f}_2(x,s) = h_2(s), \qquad W_1(u) = \int_{\tilde{u}_1}^{u} \frac{dz}{w_1(z)} = \ln\frac{u}{\tilde{u}_1}, \\ &W_1^{-1}(u) = \tilde{u}_1 e^u, \qquad W_2(u) = \int_{\tilde{u}_2}^{u} \frac{dz}{w_2(u)} = e^{-\tilde{u}_2} - e^{-u}, \qquad W_2^{-1}(u) = -\ln\big(e^{-\tilde{u}_2} - u\big), \\ &r_{i+1}(x) = \|y_0\| + \sum_{k=1}^{i} \int_{x_{k-1}}^{x_k} h_1(s)u_{k-1}(s)\,ds + \sum_{k=1}^{i} \int_{x_{k-1}}^{x_k} h_2(s)e^{u_{k-1}(s)}\,ds \\ &\qquad + \sum_{k=1}^{i} \beta_k u_{k-1}^m(x_k - 0). \end{aligned}$$

Thus, for any nonnegative integer i and $x \in (x_i, x_{i+1})$

$$u_i(x) \leq -\ln\left(e^{-r_{i+1}(x)e^{\int_{x_i}^{x} h_1(s)\,ds}} - \int_{x_i}^{x} h_2(s)\,ds\right),$$

provided that

$$e^{-r_{i+1}(x)e^{\int_{x_i}^{x} h_1(s)\,ds}} - \int_{x_i}^{x} h_2(s)\,ds > 0.$$

Remark 4.1 From (4.3), we know that $w_2(u) = e^u$. Clearly, $w_2(2u) = e^{2u} \leq w_2(2)w_2(u) = e^2e^u$ does not hold for large $u > 0$. Thus, $w_2(u)$ does not belong to the class $\wp$. Again $w_2(\frac{u}{2}) = e^{\frac{u}{2}} \geq \frac{1}{2}w_2(u) = \frac{1}{2}e^u$ does not hold for large $u > 0$, so $w_2(u)$ does not belong to the class $\jmath$. Hence, the results in [22] can not be applied to inequality (4.3).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors read and approved the final manuscript.

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