Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On degree of approximation of the Gauss-Weierstrass means for smooth $L^p(\mathbb{R}^n)$ functions

Melih Eryigit*

*Correspondence:
eryigit@akdeniz.edu.tr
Department of Mathematics, the Faculty of Science, Akdeniz University, Antalya, 07058, Turkey

Abstract

The notion of μ-smooth point of an $L^p(\mathbb{R}^n)$-function f is introduced in terms of some 'maximal function.' Then the connection between the order of μ-smoothness of the function f and the rate of convergence of the Gauss-Weierstrass means to f, when ε tends to 0, is obtained.

MSC: 41A25; 42B08; 26A33

Keywords: Gauss-Weierstrass integral; Gauss-Weierstrass means; approximation; inverse Fourier transform; maximal function

1 Introduction and formulations of main results

Let $\Phi \in C_0(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and $\Phi(0) = 1$. The Φ-means of the integral $\int_{\mathbb{R}^n} f(x)\,dx$ are defined as [1, p.6]

$$M_{\varepsilon,\Phi}(f) = \int_{\mathbb{R}^n} \Phi(\varepsilon x) f(x)\,dx \quad (\varepsilon > 0).$$

If $\lim_{\varepsilon\to 0^+} M_{\varepsilon,\Phi}(f) = l$, then it is said that the (divergent) integral $\int_{\mathbb{R}^n} f(x)\,dx$ is summable to l. It is possible to obtain various summability methods by choosing a suitable function Φ. For example, by letting $\Phi(x) = e^{-|x|}$, $\Phi(x) = e^{-|x|^2}$ or for $\delta > 0$, $\Phi(x) = \left\{\begin{smallmatrix} (1-|x|^2)^\delta; & |x| \le 1 \\ 0; & |x| > 1 \end{smallmatrix}\right\}$, the classical Abel, Gauss-Weierstrass and Bochner-Riesz means and corresponding summability methods are obtained. One of the important problems in classical harmonic analysis is to construct an (unknown) function f by means of its Fourier transform $\mathfrak{F}(f)$ defined as

$$\mathfrak{F}(f)(x) = \int_{\mathbb{R}^n} f(t) e^{-2\pi i x\cdot t}\,dt.$$

However, $\mathfrak{F}(f)$ needs not be integrable for some $f \in L^p(\mathbb{R}^n)$, and hence the formula

$$f(x) = \int_{\mathbb{R}^n} \mathfrak{F}(f)(t) e^{2\pi i x\cdot t}\,dt$$

becomes incorrect. To overcome this difficulty, one may apply suitable summability methods (see, *e.g.*, [1–3]).

Whenever a function Φ is radial, it is well known that [1, p.8] for the Φ-means of the convergent or divergent integral $\int_{\mathbb{R}^n} \mathfrak{F}(f)(t)e^{2\pi ix\cdot t}\,dt$, the following equality holds:

$$\int_{\mathbb{R}^n} \mathfrak{F}(f)(x)e^{2\pi ix\cdot t}\Phi(\varepsilon x)\,dx = \int_{\mathbb{R}^n} f(x)\varphi_\varepsilon(t-x)\,dx, \tag{1.1}$$

where $\varphi_\varepsilon(x) = (1/\varepsilon)^n\varphi_\varepsilon(x/\varepsilon)$ and $\varphi(x) = \mathfrak{F}(\Phi)$.

In particular, putting the function $e^{-|x|^2}$ instead of $\Phi(x)$ in (1.1), the following formula for the Gauss-Weierstrass means of the integral $\int_{\mathbb{R}^n} \mathfrak{F}(f)(t)e^{2\pi ix\cdot t}\,dt$

$$S(x,\varepsilon) = \int_{\mathbb{R}^n} f(t)\varphi_\varepsilon(x-t)\,dt \quad (\varepsilon > 0) \tag{1.2}$$

is obtained. Here, the function φ_ε is defined as

$$\varphi_\varepsilon(x) \equiv W(x,\varepsilon) = (4\pi\varepsilon)^{-(n/2)}e^{-|x|^2/4\varepsilon}, \tag{1.3}$$

and called the Gauss-Weierstrass kernel.

One of the well-known and basic results for the Gauss-Weierstrass means is the following ([4, p.5], [5, p.223]).

Proposition 1.1 *Let $f \in L^p(\mathbb{R}^n)$ $(1 \le p < \infty)$, and let the Gauss-Weierstrass means of f be defined as in* (1.2). *Then*

(a) $\lim_{\varepsilon\to 0} \|S(x,\varepsilon) - f\|_{L^p} = 0$;
(b) $\lim_{\varepsilon\to 0} S(x,\varepsilon) = f(x)$ *at each x belonging to the Lebesgue set of f*;
(c) $\sup_{\varepsilon>0} |S(x,\varepsilon)| \le c(Mf)(x)$, *where $(Mf)(x)$ is the Hardy-Littlewood maximal function.*

Various aspects of the Gauss-Weierstrass and Abel-Poisson type summability of the multiple Fourier series and integrals have been studied in Stein and Weiss [1], Golubov [6, 7] and Gorodetskii [8]; see also Weisz [2] and [9] and references therein.

The aim of the paper is to investigate the error of approximation of $f(x)$ by its Gauss-Weierstrass means $S(x,\varepsilon)$ as $\varepsilon \to 0$ at the so-called μ-smoothness point of f. Note that some problems of the Bochner-Riesz summability of Fourier transform of $f \in L_p(\mathbb{R}^n)$ at the Dini-like points was studied in [10]. Also, the rate of convergence of the Gauss-Weierstrass means of relevant Fourier series at some kind of smoothness points was studied in [9].

Definition 1.2 Let $\mu(r)$ be a positive function on $(0,\infty)$, and assume that $\lim_{r\to 0^+}\mu(r) = 0$. If $\psi(t,x)$, defined on $\mathbb{R}^n \times \mathbb{R}^n$, is measurable, we define its μ-maximal function by

$$(M_\mu\psi)(x) = \sup_{r>0} \frac{1}{\mu(r)r^n}\int_{|t|<r} \left|\psi(t,x)\right|\,dt. \tag{1.4}$$

Definition 1.3 Let, for a constant $\rho < 1$, a function $\mu(r)$ be a continuous and positive function on the interval $(0,\rho)$, and assume that $\lim_{r\to 0^+}\mu(r) = \mu(0) = 0$. We say that a function $f \in L^1_{loc}(\mathbb{R}^n)$ is μ-smooth of order $\mu(r)$ at $x \in \mathbb{R}^n$ if

$$D_\mu(x) = \sup_{0<r<1} \frac{1}{r^n\mu(r)}\int_{|t|<r} \left|f(x-t) - f(x)\right|\,dt < \infty. \tag{1.5}$$

The points $x \in \mathbb{R}^n$, for which (1.5) holds, are called *μ-smoothness points* of f.

Remark 1.4 Simple characterization of a μ-smoothness point is not known. However, most of the classes of 'smooth' functions in a classical sense have the μ-smoothness property. For example, if the modulus of continuity of f

$$w_f(r) = \sup_{|x|\leq r} \left\| f(\cdot - x) - f(\cdot) \right\|_\infty$$

satisfies the inequality $w_f(r) \leq c\mu(r)$ for $r \to 0$, then every point $x \in \mathbb{R}^n$ is a μ-smoothness point of f, as can easily be seen from (1.5). In particular, if f satisfies the local Lipschitz (Hölder) condition

$$\left| f(x-t) - f(x) \right| \leq c|t|^\alpha, \quad 0 < \alpha \leq 1,$$

then x is a μ-smoothness point of f, provided $\mu(r) = r^\alpha$.

From now on, we will assume that the function $\mu(r)$ is continued as a constant from $[0,\rho]$ to $[\rho,\infty)$, that is, $\mu(r) = \mu(\rho)$, $r \geq \rho$.

Now, we state the main results of the paper.

Theorem 1.5 *Let $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, be μ-smooth at $x \in \mathbb{R}^n$. Then the following estimate holds*:

$$\left| S(x,\varepsilon) - f(x) \right| \leq c_1 \int_0^\infty r^{n+1} e^{-r^2/4} \mu(\varepsilon r)\, dr + c_2 \varepsilon^{-n/2} e^{-1/4\varepsilon} \quad \left(\varepsilon \to 0^+\right), \tag{1.6}$$

where c_1 and c_2 are constants independent of ε.

Corollary 1.6 *Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, have the μ-smoothness property at x, and let $\mu(r)$ be a modulus of continuity* (*see* [11, p.40]) *on $[0,\rho]$ and continued as a constant to $[\rho,\infty)$, i.e., $\mu(r) = \mu(\rho)$, $r \geq \rho$ $(0 < \rho < 1)$. Then, under the conditions of Theorem* 1.5, *we have*

$$\left| S(x,\varepsilon) - f(x) \right| \leq c\mu(\varepsilon) \quad (\varepsilon \to 0). \tag{1.7}$$

Corollary 1.7 *Let $\alpha > 0$ and $\mu(r) = (\frac{1}{\ln\frac{1}{r}})^\alpha$, then*

$$\left| S(x,\varepsilon) - f(x) \right| \leq c\left(\frac{1}{\ln\frac{1}{\varepsilon}} \right)^\alpha \quad (\varepsilon \to 0). \tag{1.8}$$

Corollary 1.8 *Let $\alpha > 0$ and $-\infty < \beta < \infty$ be fixed parameters. If we take $\mu(r) = r^\alpha |\ln r|^\beta$ for $0 < r \leq \rho < 1$ and $\mu(r) = \mu(\rho)$ for $\rho < r < \infty$, then under the conditions of Theorem* 1.5,

$$\left| S(x,\varepsilon) - f(x) \right| \leq c\varepsilon^\alpha |\ln\varepsilon|^\beta \quad (\varepsilon \to 0). \tag{1.9}$$

In particular, for $\beta = 0$ in (1.9), *we obtain $|S(x,\varepsilon) - f(x)| \leq c\varepsilon^\alpha$ as $\varepsilon \to 0$.*

The following lemma plays a crucial role in the proof of the main results.

Lemma A (*cf.* [9, Lemma A]) *Suppose that* φ *is differentiable on* $(0,\infty)$, *and that the following limits exist*:

$$\lim_{r\to\infty} r^n\mu(r)\varphi(r) = 0 \quad \textit{and} \quad \lim_{r\to 0^+} r^n\mu(r)\varphi(r) = 0. \tag{1.10}$$

Let $\psi(t,x)$ *be measurable on* $\mathbb{R}^n \times \mathbb{R}^n$ *and* $(M_\mu\psi)(x) < \infty$, *then*

$$\int_{\mathbb{R}^n} \bigl|\psi(t,x)\varphi\bigl(|t|\bigr)\bigr|\,dt \le (M_\mu\psi)(x)\int_0^\infty r^n\mu(r)\bigl|\varphi'(r)\bigr|\,dr, \tag{1.11}$$

where $(M_\mu\psi)(x)$ *is a* μ-*maximal function defined by* (1.4) *and* φ' *is a derivative of* φ.

We need also the following lemmas on the well-known properties of the Gauss-Weierstrass kernel and the upper incomplete gamma function.

Lemma 1.9 [1, p.9] *The Gauss-Weierstrass kernel*, $W(t,\varepsilon) = (4\pi\varepsilon)^{-(n/2)}e^{-|t|^2/4\varepsilon}$, *has the following property*:

$$\int_{\mathbb{R}^n} W(t,\varepsilon)\,dt = 1 \quad (\textit{for all } \varepsilon > 0). \tag{1.12}$$

Lemma 1.10 [12, p.948] *The upper incomplete gamma function, defined as*

$$\Gamma(s,\tau) = \int_\tau^\infty u^{s-1}e^{-u}\,du \quad (s > 0, \tau > 0),$$

has the following asymptotic property:

$$\Gamma(s,\tau) = O(1)\tau^{s-1}e^{-\tau} \quad \textit{as } \tau \to \infty. \tag{1.13}$$

2 Proof of the main results

Proof of Lemma A Changing variables to polar coordinates $t \to (r,\theta)$, $0 < r < \infty$, $\theta \in S^{n-1}$ (S^{n-1} is the unite sphere of $\mathbb{R}^n$), the left side of (1.11) becomes

$$\begin{aligned} I(x) &= \int_{\mathbb{R}^n} \bigl|\psi(t,x)\varphi\bigl(|t|\bigr)\bigr|\,dt \\ &= \int_0^\infty r^{n-1}\biggl[\int_{S^{n-1}} \bigl|\psi(r\theta,x)\varphi(r)\bigr|\,d\sigma(\theta)\biggr]\,dr \\ &= \int_0^\infty r^{n-1}\bigl|\varphi(r)\bigr|\biggl[\int_{S^{n-1}} \bigl|\psi(r\theta,x)\bigr|\,d\sigma(\theta)\biggr]\,dr. \end{aligned}$$

Now, denoting

$$\lambda(t) = \int_{S^{n-1}} \bigl|\psi(t\theta,x)\bigr|\,d\sigma(\theta), \quad 0 \le t < \infty, \tag{2.1}$$

$$\Lambda(r) = \int_0^r \lambda(t)t^{n-1}\,dt, \quad 0 \le r \le \infty, \tag{2.2}$$

we get

$$I(x) = \int_0^\infty r^{n-1}\left|\varphi(r)\right|\lambda(r)\,dr = \int_0^\infty \left|\varphi(r)\right|\,d\Lambda(r)$$
$$= \left|\varphi(r)\right|\Lambda(r)\big|_0^\infty - \int_0^\infty \Lambda(r)\operatorname{sgn}\varphi(r)\varphi'(r)\,dr.$$

Using (1.10) and considering the inequality

$$\Lambda(r) = \int_0^r \lambda(t)t^{n-1} = \int_{|t|\le r}\left|\psi(t,x)\right|dx \le r^n\mu(r)(M_\mu\psi)(x), \tag{2.3}$$

we have

$$\left|\varphi(r)\right|\Lambda(r)\big|_0^\infty = 0.$$

Thus,

$$I(x) = -\int_0^\infty \Lambda(r)\operatorname{sgn}\varphi(r)\varphi'(r)\,dr$$
$$\le \int_0^\infty \Lambda(r)\left|\varphi'(r)\right|dr \overset{(2.3)}{\le} (M_\mu\psi)(x)\int_0^\infty r^n\mu(r)\left|\varphi'(r)\right|dr.$$

That completes the proof. □

Proof of Theorem 1.5 Let us fix x, a μ-smoothness point of f, and consider the difference

$$\left|S(x,\varepsilon) - f(x)\right| = \left|\int_{\mathbb{R}^n}\left[f(x-t) - f(x)\right]W(t,\varepsilon)\,dt\right|$$
$$\le \int_{|t|\le 1}\left|f(x-t) - f(x)\right|W(t,\varepsilon)\,dt + \int_{|t|>1}\left|f(x-t) - f(x)\right|W(t,\varepsilon)\,dt$$
$$= A_1(\varepsilon) + A_2(\varepsilon). \tag{2.4}$$

In order to estimate $A_1(\varepsilon)$, we let

$$\psi(t,x) = \begin{cases} f(x-t) - f(x), & |t| \le 1; \\ 0, & |t| > 1, \end{cases}$$

and then

$$A_1(\varepsilon) = \int_{\mathbb{R}^n} W(t,\varepsilon)\left|\psi(t,x)\right|dt, \qquad W(t,\varepsilon) = (4\pi\varepsilon)^{-(n/2)}e^{-|t|^2/4\varepsilon}. \tag{2.5}$$

Now, by Lemma A, taking $\varphi(|t|) = (4\pi\varepsilon)^{-n/2}e^{-|t|^2/4\varepsilon}$, we have

$$A_1(\varepsilon) \le (M_\mu\psi)(x)\int_0^\infty r^n\mu(r)\left|\varphi'(r)\right|dr, \quad \text{where } \varphi(r) = (4\pi\varepsilon)^{-(n/2)}e^{-r^2/4\varepsilon}. \tag{2.6}$$

Since f is μ-smooth at the point $x \in \mathbb{R}^n$, we have $(M_\mu \psi)(x) \equiv D_\mu(x) < \infty$ (see (1.5)). So we get

$$A_1(\varepsilon) \leq c_1 \int_0^\infty r^n \mu(r) \left|\varphi'(r)\right| dr \leq c \int_0^\infty r^{n+1} e^{-r^2/4} \mu(\varepsilon r)\, dr. \tag{2.7}$$

To estimate $A_2(\varepsilon)$, we first apply Hölder's inequality for $p > 1$ and observe that

$$\begin{aligned} A_2(\varepsilon) &\leq \left|f(x)\right| \int_{|t|>1} W(t,\varepsilon)\, dt \\ &\quad + \left(\int_{|t|>1} \left|f(x-t)\right|^p dt\right)^{1/p} \left(\int_{|t|>1} \left|W(t,\varepsilon)\right|^q dt\right)^{1/q} \quad \left(\frac{1}{p} + \frac{1}{q} = 1\right). \end{aligned} \tag{2.8}$$

Let us estimate the first term on the right of (2.8). Changing variables to polar coordinates yields

$$\begin{aligned} \int_{|t|>1} W(t,\varepsilon)\, dt &= k_1 \int_1^\infty r^{n-1} \left[\int_{S^{n-1}} \varepsilon^{-n/2} e^{-r^2/4\varepsilon}\, d\sigma(\theta)\right] dr \\ &= k_2 \int_1^\infty r^{n-1} \varepsilon^{-n/2} e^{-r^2/4\varepsilon} \\ &= k_3 \int_{(1/2\sqrt{\varepsilon})}^\infty r^{n-1} e^{-r^2}\, dr \quad (\text{we set } u = r^2,\ du = 2r\, dr) \\ &= k_3 \int_{(1/4\varepsilon)}^\infty u^{n/2-1} e^{-u}\, du \\ &= k_3 \Gamma\left(\frac{n}{2}, \frac{1}{4\varepsilon}\right), \end{aligned}$$

where $\Gamma(s,\tau)$ is the upper incomplete gamma function. Now, using asymptotic formula (1.13), we get

$$\int_{|t|>1} W(t,\varepsilon)\, dt = O\left(\varepsilon^{1-\frac{n}{2}} e^{-1/4\varepsilon}\right) \quad \text{as } \varepsilon \to 0^+. \tag{2.9}$$

The same is true for the second term of (2.8):

$$\begin{aligned} \left(\int_{|t|>1} \left|W(t,\varepsilon)\right|^q dt\right)^{1/q} &= k_4 \left(\int_1^\infty r^{n-1} \left[\int_{S^{n-1}} \left(\varepsilon^{-n/2} e^{-r^2/4\varepsilon}\right)^q d\sigma(\theta)\right] dr\right)^{1/q} \\ &= k_5 \varepsilon^{\frac{n}{2q}-\frac{n}{2}} \left(\int_{(\sqrt{q}/2\sqrt{\varepsilon})}^\infty r^{n-1} e^{-r^2}\, dr\right)^{1/q} \\ &= k_5 \varepsilon^{\frac{n}{2q}-\frac{n}{2}} \left(\int_{(q/4\varepsilon)}^\infty u^{n/2-1} e^{-u}\, du\right)^{1/q}. \end{aligned}$$

Now, using formula (1.13), we get

$$\left(\int_{|t|>1} \left|W(t,\varepsilon)\right|^q dt\right)^{1/q} = O\left(\varepsilon^{\frac{1}{q}-\frac{n}{2}} e^{-1/4\varepsilon}\right) \quad \text{as } \varepsilon \to 0^+. \tag{2.10}$$

Collecting estimates (2.9) and (2.10), and taking into account that

$$\left(\int_{|t|>1} \left|f(x-t)\right|^p\right)^{1/p} \le \|f\|_p < \infty,$$

we have

$$A_2(\varepsilon) = O\left(\varepsilon^{-n/2} e^{-1/4\varepsilon}\right) \quad \text{as } \varepsilon \to 0^+. \tag{2.11}$$

By (2.7) and (2.11) we have shown that inequality (1.6) holds, as desired.

To complete the proof, we have to show that the conditions of Lemma A are satisfied; that is, for $\varphi(r) = (4\pi\varepsilon)^{-n/2} e^{-r^2/4\varepsilon}$,

$$\lim_{r\to\infty} r^n \mu(r)\varphi(r) = 0 \quad \text{and} \quad \lim_{r\to 0^+} r^n \mu(r)\varphi(r) = 0.$$

But this is obvious. □

Proof of Corollary 1.6 Let $\mu(r)$, $r \in [0,\infty)$ be a modulus of continuity, *i.e.*, (a) $\mu(r) \to 0$ as $r \to 0^+$; (b) $\mu(r)$ is non-negative and non-decreasing on $(0,\infty)$; (c) $\mu(r)$ is continuous and subadditive $(0,\infty)$.

It follows from the subadditivity of $\mu(r)$ that

$$\mu(\varepsilon r) \le (1+r)\mu(\varepsilon) \quad \text{for all } \varepsilon, r > 0.$$

By employing this in (1.6), we get

$$\begin{aligned} \left|S(x,\varepsilon) - f(x)\right| &\le c_1 \mu(\varepsilon) \int_0^\infty (1+r) r^{n+1} e^{-r^2}\, dr + c_2 \varepsilon^{-n/2} e^{-1/4\varepsilon} \\ &\le c_3 \mu(\varepsilon) + c_2 \varepsilon^{-n/2} e^{-1/4\varepsilon}. \end{aligned} \tag{2.12}$$

Now, since the function $\mu(r)$ is a modulus of continuity, it cannot tend to zero too rapidly as $\varepsilon \to 0$, that is, for instance, if $\frac{\mu(\varepsilon)}{\varepsilon} \to 0$ as $\varepsilon \to 0$, then $\mu(\varepsilon) \equiv 0$. Therefore

$$\varepsilon^{-n/2} e^{-1/4\varepsilon} \le c_4 \mu(\varepsilon), \quad \varepsilon \to 0$$

for some constant c_4. Taking into account this in (2.12), we obtain

$$\left|S(x,\varepsilon) - f(x)\right| \le c\mu(\varepsilon), \quad \varepsilon \to 0,$$

where the constant c does not depend on $\varepsilon > 0$. □

Proof of Corollary 1.7 Let us show that for some $0 < \rho < 1$ the function

$$\mu(r) = \left\{ \begin{array}{ll} 0, & r = 0 \\ (1/\ln 1/r)^\alpha, & 0 < r < \rho < 1 \\ (1/\ln 1/\rho)^\alpha, & \rho \le r < \infty \end{array} \right\} \quad (0 < \alpha < \infty)$$

is a modulus of continuity, *i.e.*, it is continuous, non-decreasing, subadditive on $[0,\infty)$ and tends to zero as $r \to 0^+$. The continuity and $\lim_{r\to 0}\mu(r) = 0$ are obvious. To prove the other properties, it suffices to show that (see [11], p.41)

$$\mu'(r) \geq 0 \quad \text{and} \quad \bigl(\mu(r)/r\bigr)' \leq 0 \quad (0 < r < \rho).$$

Simple calculations show that the above inequalities are fulfilled if one takes $\rho = e^{-\alpha}$. □

Proof of Corollary 1.8 Let us substitute the function

$$\mu(r) = \left\{\begin{array}{ll} 0, & r = 0 \\ r^\alpha |\ln r|^\beta, & 0 < r \leq \rho \\ \rho^\alpha |\ln \rho|^\beta, & r > \rho \end{array}\right\}$$

in (1.6), where $\alpha > 0$ and $\beta \in (-\infty,\infty)$ are given numbers and $0 < \rho < 1$.

If $\beta \geq 0$, we have, for sufficiently small $\varepsilon > 0$,

$$\mu(\varepsilon r) \leq \varepsilon^\alpha |\ln\varepsilon|^\beta r^\alpha \left(1 + \frac{|\ln r|}{|\ln\varepsilon|}\right)^\beta \leq \varepsilon^\alpha |\ln\varepsilon|^\beta r^\alpha \bigl(1 + |\ln r|\bigr)^\beta.$$

By making use of this estimate in (1.6), we have for $\varepsilon \ll 1$ that

$$\begin{aligned} \bigl|S(x,\varepsilon) - f(x)\bigr| &\leq c\varepsilon^\alpha |\ln\varepsilon|^\beta \int_0^\infty r^{n+1} e^{-r^2/4} r^\alpha (1 + \ln r)^\beta \, dr + O\bigl(\varepsilon^{-n/2} e^{-1/4\varepsilon}\bigr) \\ &\leq c_1 \varepsilon^\alpha |\ln\varepsilon|^\beta + c_2 \varepsilon^{-n/2} e^{-1/4\varepsilon} \leq c_3 \varepsilon^\alpha |\ln\varepsilon|^\alpha, \quad \varepsilon \to 0. \end{aligned}$$

Let now $\beta < 0$. By setting $\delta = -\beta > 0$, we have for $\varepsilon \ll 1$

$$\begin{aligned} \mu(\varepsilon r) &= \varepsilon^\alpha |\ln\varepsilon|^\beta r^\alpha \left|\frac{\ln\varepsilon r}{\ln\varepsilon}\right|^\beta = \varepsilon^\alpha |\ln\varepsilon|^\beta r^\alpha \left|\frac{\ln\varepsilon}{\ln\varepsilon r}\right|^\delta \\ &= \varepsilon^\alpha |\ln\varepsilon|^\beta r^\alpha \left|1 - \frac{\ln r}{\ln\varepsilon r}\right|^\delta \leq \varepsilon^\alpha |\ln\varepsilon|^\beta r^\alpha \left(1 + \frac{|\ln r|}{|\ln\varepsilon r|}\right)^\delta. \end{aligned}$$

Since $\varepsilon r < \rho < 1$, it follows that

$$\mu(\varepsilon r) \leq \varepsilon^\alpha |\ln\varepsilon|^\beta r^\alpha \left(1 + \frac{|\ln r|}{|\ln\rho|}\right)^\delta.$$

Using this in (1.6), we get

$$\begin{aligned} \bigl|S(x,\varepsilon) - f(x)\bigr| &\leq c\varepsilon^\alpha |\ln\varepsilon|^\beta \int_0^\infty r^{n+1} e^{-r^2/4} r^\alpha \left(1 + \frac{|\ln r|}{|\ln\rho|}\right)^\delta e^{-r} \, dr + c_2 \varepsilon^{-n/2} e^{-1/4\varepsilon} \\ &\leq c\varepsilon^\alpha |\ln\varepsilon|^\beta, \end{aligned}$$

which is the desired result. □

Competing interests
The author declares that they have no competing interests.

Acknowledgements
This paper was supported by the Scientific Research Project Administration Unit of Akdeniz University and TUBITAK (Turkey).

Received: 13 May 2013 Accepted: 23 August 2013 Published: 11 September 2013

References

1. Stein, EM, Weiss, G: Introduction to Fourier Analysis on Euclidean Spaces. Princeton University Press, Princeton (1971)
2. Weisz, F: Summability of Multi-dimensional Fourier Series and Hardy Spaces, Mathematics and Its Applications. Kluwer Academic, Dordrecht (2002)
3. Weisz, F: Restricted summability of Fourier transforms and local Hardy spaces. Acta Math. Sin. Engl. Ser. **26**(9), 1627-1640 (2010)
4. Stein, EM: Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton University Press, Princeton (1993)
5. Rubin, B: Fractional Integrals and Potentials. Pitman Monographs and Surveys in Pure and Applied Mathematics. Longman, Harlow (1996)
6. Golubov, BI: On the rate of convergence of integrals of Gauss-Weierstrass type for functions of several variables. Math. USSR, Izv. **17**, 455-475 (1981)
7. Golubov, BI: On the summability method of Abel-Poisson type for multiple Fourier integrals. Math. USSR Sb. **36**, 213-229 (1980)
8. Gorodetskii, VV: Summation of formal Fourier series by methods of Gauss-Weierstrass type. Ukr. Math. J. **41**, 715-717 (1989)
9. Sezer, S, Aliev, IA: On the Gauss-Weierstrass summability of multiple trigonometric series at μ-smoothness points. Acta Math. Sin. Engl. Ser. **27**(4), 741-746 (2011)
10. Aliev, IA: On the Bochner-Riesz summability and restoration of μ-smooth functions by means of their Fourier transforms. Fract. Calc. Appl. Anal. **2**(3), 265-277 (1999)
11. DeVore, RA, Lorentz, GG: Constructive Approximation. Springer, Berlin (1993)
12. Gradshteyn, IS, Ryzhik, IM: Table of Integrals, Series, and Products, 5th edn. Academic Press, San Diego (1994)

doi:10.1186/1029-242X-2013-428
Cite this article as: Eryigit: On degree of approximation of the Gauss-Weierstrass means for smooth $L^p(\mathbb{R}^n)$ functions. *Journal of Inequalities and Applications* 2013 **2013**:428.