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Identities on products of Genocchi numbers

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Abstract
In this paper, we discuss the properties of a class of Genocchi numbers by generating functions and Riordan arrays, we establish some identities involving Genocchi numbers, the Stirling numbers, the generalized Stirling numbers, the higher order Bernoulli numbers and Cauchy numbers. Further, we get asymptotic value of some sums relating the Genocchi numbers.

1 Introduction and preliminaries

The Genocchi numbers G_n are defined by

$$\frac{2t}{e^t+1}=\sum_{n=0}^{\infty}G_n\frac{t^n}{n!}.$$

The relationship between the Genocchi numbers G_n and the Bernoulli numbers B_n and the Euler polynomials $E_n(x)$ is

$$G_n=2\left(1-2^n\right)B_n=nE_{n-1}(0)\quad\text{for } n\geq 1.$$

They are known as follows [1].

Genocchi numbers and Genocchi polynomials are prolific in the mathematical literature, and many results on Genocchi numbers and Genocchi polynomials identities may be seen in the papers [2–4]. In this paper, we will mainly study the products of Genocchi numbers $G_n^{(k)}$ with the following forms:

$$\frac{G_n^{(k)}}{n!}=\sum_{i_1+i_2+\cdots+i_k=n}\frac{G_{i_1}G_{i_2}\cdots G_{i_k}}{i_1!i_2!\cdots i_k!}\quad(k=1,2,\ldots),$$

where n is a nonnegative positive integer $G_n^{(1)}=G_n$, and $G_0=G_0^{(k)}=0$ for $k\geq 1$, $G_n^{(k)}=0$ for $n<k$.

It is clear that

$$\left(\frac{2t}{e^t+1}\right)^k=\sum_{n=k}^{\infty}G_n^{(k)}\frac{t^n}{n!}=\sum_{n=0}^{\infty}G_n^{(k)}\frac{t^n}{n!}\quad(k=1,2,\ldots).\tag{1.1}$$

Then we have

$$\left(\frac{2}{e^t+1}\right)^k = \sum_{n=k}^{\infty} \frac{G_n^{(k)} t^{n-k}}{n!} = \sum_{n=0}^{\infty} \frac{G_{n+k}^{(k)} t^n}{(n+k)!} = \sum_{n=0}^{\infty} \frac{G_{n+k}^{(k)}}{\langle n+1\rangle_k} \frac{t^n}{n!}, \tag{1.2}$$

where $\langle n\rangle_j = n(n+1)\cdots(n+j-1)$.

The paper is organized as follows. In Section 2, we obtain some properties for $G_n^{(k)}$ by means of the method of generating function. In Section 3, we establish some identities involving $G_n^{(k)}$, the Stirling numbers, the generalized Stirling numbers, the higher order Bernoulli numbers and the Cauchy numbers. Finally, in Section 4, we give the asymptotic expansion of some sums involving $G_n^{(k)}$.

For convenience, we recall some definitions and notations involved in the paper. Stirling number of the first kind is denoted by $s(n,k,r)$, let $B_n^{(k)}$, $C_n^{(k)}$, $B_{n,k}$, $P_n^{(k)}$, stand for the higher order Bernoulli numbers, products of Cauchy numbers, Bell polynomials and the potential polynomials. The corresponding generating functions are

$$\sum_{n=k}^{\infty} s(n,k,r)\frac{t^n}{n!} = \frac{\ln^k(1+t)}{(1+t)^r k!}, \qquad \sum_{n=k}^{\infty} \left|s(n,k,r)\right| \frac{t^n}{n!} = \frac{(-\ln(1-t))^k}{(1-t)^r k!},$$

where $n = k, k+1, \ldots, k = 0, 1, 2, \ldots,$

$$s(n,k,0) = s(n,k) \text{ (Stirling numbers of the first kind)}, \tag{1.3}$$

$$\sum_{n=k}^{\infty} S(n,k,r)\frac{t^n}{n!} = \frac{e^{rt}(e^t-1)^k}{k!}, \quad n = k, k+1, \ldots, k = 0, 1, 2, \ldots,$$

$$S(n,k,0) = S(n,k) \text{ (Stirling numbers of the second kind)}, \tag{1.4}$$

$$\sum_{n=0}^{\infty} B_n^{(r)} \frac{t^n}{n!} = \left(\frac{t}{e^t-1}\right)^r \quad (r \text{ is an integer}), \tag{1.5}$$

$$\sum_{n=0}^{\infty} C_n^{(k)} \frac{t^n}{n!} = \left(\frac{t}{\ln(1+t)}\right)^k, \quad k = 1, 2, \ldots, \frac{C_n^{(k)}}{n!} = \sum_{i_1+i_2+\cdots+i_k=n} \frac{C_{i_1} C_{i_2} \cdots C_{i_k}}{i_1! i_2! \cdots i_k!}, \tag{1.6}$$

$$\sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \ldots)\frac{t^n}{n!} = \frac{1}{k!}\left(\sum_{n=1}^{\infty} x_m \frac{t^m}{m!}\right)^k, \quad k = 0, 1, \ldots, \tag{1.7}$$

$$\left(\sum_{n=0}^{\infty} g_n \frac{t^n}{n!}\right)^r = 1 + \sum_{n=1}^{\infty} P_n^{(r)} \frac{t^n}{n!}, \quad g_0 = 1 \text{ (} r \text{ is a complex number) and} \tag{1.8}$$

$$P_n^{(r)} = P_n^{(r)}(g_1, g_2, \ldots, g_n) = \sum_{j=1}^{n} (k)_j B_{n,j}(g_1, g_2, \ldots), \quad P_0^{(r)} = 1,$$

$$\text{where } (k)_j = k(k-1)\cdots(k-j+1). \tag{1.9}$$

In this paper, we let $[\frac{t^n}{n!}]f(t)$ denote the coefficient of $\frac{t^n}{n!}$ in $f(t)$, where $f(t) = \sum_{n=0}^{\infty} f_n \frac{t^n}{n!}$.

An exponential Riordan array is a pair $(d(t), h(t))$ of formal power series. Where $d(t) = \sum_{n=0}^{\infty} d_n \frac{t^n}{n!}$, $h(t) = \sum_{n=1}^{\infty} h_n \frac{t^n}{n!}$.

It defines an infinite, lower triangular array $(d_{n,k})_{n,k\in N}$ according to the rule

$$d_{n,k} = \left[\frac{t^n}{n!}\right] d(t)\frac{(th(t))^k}{k!}, \quad n \in N.$$

Hence we write $(d_{n,k}) = (d(t), h(t))$.

The most important property of Riordan array is: If $(d(t), h(t))$ is an exponential Riordan array, and let $f(t)$ be the exponential generating function of the sequence $\{f_k\}_{k\in N}$, then we have

$$\sum_{k=0}^{\infty} d_{n,k} f_k = \left[\frac{t^n}{n!}\right] d(t) f\bigl(th(t)\bigr) = \left[\frac{t^n}{n!}\right] d(t)\bigl[f(y) \mid y = th(t)\bigr], \tag{1.10}$$

where we use the notation $[f(y) \mid y = g(t)]$ as a linearization of the more common one $f(y)|_{y=g(t)}$ to denote substitution $f(g(t))$.

A singularity of $f(z)$ at $|z| = w$ is called algebraic if $f(z)$ can be written as the sum of a function analytic in a neighborhood of w and a finite number of terms of the form

$$\left(1 - \frac{z}{w}\right)^{\alpha} g(z), \tag{1.11}$$

where $g(z)$ is analytic near w, $g(z) \neq 0$, and $\alpha \notin \{0, 1, 2, \ldots\}$ (see [5]).

Lemma 1.1 (See [5]) *Suppose that $f(z)$ is analytic for $|z| < R$, $R > 0$, and has only algebraic singularities of on $|z| = R$. Let a be the minimum of* $\mathrm{Re}(\alpha)$ *for the terms of the form at the singularity of $f(z)$ on $|z| = R$, and let w_j, α_j, and $g_j(z)$ be the w, α, and $g(z)$ for those terms of form* (1.11), *for which* $\mathrm{Re}(\alpha) = a$. *Then, as $n \to \infty$,*

$$\bigl[z^n\bigr] f(z) = \sum_j \frac{g_j(w_j) n^{-\alpha_j - 1}}{\Gamma(-\alpha_j) w_j^n} + o\bigl(R^{-n} n^{-\alpha-1}\bigr).$$

2 Properties of products of Genocchi numbers

In this section, we obtain some properties for $G_n^{(k)}$ by means of the method of generating functions and the Euler transformation.

Theorem 2.1 *Let $n \geq k \geq 2$ be any integers, then*

$$G_n^{(k)} = \sum_{j=k-1}^{n} \binom{n}{j} G_j^{(k-1)} G_{n-j},$$

where $G_n^{(1)} = G_n$.

Proof By (1.1), we have

$$\sum_{n=k}^{\infty} G_n^{(k)} \frac{t^n}{n!} = \sum_{m=k-1}^{\infty} G_m^{(k-1)} \frac{t^m}{m!} \sum_{m=1}^{\infty} G_m \frac{t^m}{m!} = \sum_{n=k-1}^{\infty} \sum_{j=k-1}^{n} \binom{n}{j} G_j^{(k-1)} G_{n-j} \frac{t^n}{n!},$$

and the comparison of the coefficients of $t^n/n!$ in the first and the last formulas completes the proof. □

Theorem 2.2 *Let $n \geq k+1$, $k \geq 1$ be any integers, then*

$$kG_n^{(k+1)} = 2knG_{n-1}^{(k)} + 2(n-k)G_n^{(k)}.$$

Proof Deriving both sides of (1.1) with respect to t, we get

$$\begin{aligned}\sum_{n=k}^{\infty} G_n^{(k)} \frac{t^{n-1}}{(n-1)!} &= k\left(\frac{2t}{e^t+1}\right)^{k-1}\left(\frac{2t}{e^t+1}\right)' \\ &= \frac{k}{t}\left(\frac{2t}{e^t+1}\right)^{k} - k\left(\frac{2t}{e^t+1}\right)^{k} + \frac{k}{2t}\left(\frac{2t}{e^t+1}\right)^{k+1}.\end{aligned}$$

Multiplying both sides by t, we have

$$\sum_{n=k}^{\infty} nG_n^{(k)} \frac{t^n}{n!} = k\sum_{n=k}^{\infty} G_n^{(k)} \frac{t^n}{n!} - k\sum_{n=k}^{\infty} G_n^{(k)} \frac{t^{n+1}}{n!} + \frac{k}{2}\sum_{n=k+1}^{\infty} G_n^{(k+1)} \frac{t^n}{n!},$$

and the comparison of the coefficients of $t^n/n!$ in the first and the last formulas completes the proof. □

Theorem 2.3 *Let $n \geq k \geq 1$ be any integers, then*

$$G_n^{(k)} = k!B_{n,k}(G_1, G_2, \ldots).$$

Proof By (1.1) and (1.7), we get

$$\begin{aligned}\sum_{n=k}^{\infty} G_n^{(k)} \frac{t^n}{n!} &= \left(\sum_{n=1}^{\infty} G_n \frac{t^n}{n!}\right)^k = \left(G_1 t + G_2\frac{t^2}{2!} + \cdots + G_n\frac{t^n}{n!} + \cdots\right)^k \frac{1}{k!}k! \\ &= k!\sum_{n=k}^{\infty} B_{n,k}(G_1, G_2, \ldots)\frac{t^n}{n!},\end{aligned}$$

and the comparison of the coefficients of $t^n/n!$ in the first and the last formulas completes the proof. □

Theorem 2.4 *Let $n \geq k \geq 1$ be any integers, then*

$$G_n^{(k)} = \sum_{i=1}^{n}\sum_{j=0}^{k}\binom{k}{j}\binom{j}{i}(-1)^{k-j}G_n^{(i)}.$$

Proof By (1.1), (1.8), (1.9) and Theorem 2.3, we have

$$\begin{aligned}\sum_{n=0}^{\infty} G_n^{(k)} \frac{t^n}{n!} &= \left(1 + \sum_{m=1}^{\infty} G_n\frac{t^n}{n!} - 1\right)^k = \sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\left(1 + \sum_{m=1}^{\infty} G_m\frac{t^m}{m!}\right)^j \\ &= \sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\sum_{n=0}^{\infty} P_n^{(j)}(G_1, G_2, \ldots)\frac{t^n}{n!}.\end{aligned}$$

Then

$$G_n^{(k)} = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} P_n^{(j)}(G_1, G_2, \ldots) = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \sum_{i=1}^{n} (j)_i B_{n,i}(G_1, G_2, \ldots)$$
$$= \sum_{i=1}^{n} \sum_{j=0}^{k} \binom{k}{j} \binom{j}{i} (-1)^{k-j} G_n^{(i)},$$

which completes the proof. □

Theorem 2.5 *Let $n \geq k \geq 1$ be any integers, then*

$$\frac{G_{n+k}^{(k)}}{k!} = \sum_{j=k}^{n+k} \binom{n+k}{j} S(j,k) 2^{n+k-j} B_{n+k-j}^{(k)}.$$

Proof By (1.2) and (1.5), we have

$$\sum_{n=0}^{\infty} \frac{G_{n+k}^{(k)}}{\langle n+1 \rangle_k} \frac{t^n}{n!} = \left(\frac{2}{e^t+1}\right)^k = \frac{k!}{t^k} \left(\frac{2t}{e^{2t}-1}\right)^k \frac{(e^t-1)^k}{k!}$$
$$= \frac{k!}{t^k} \sum_{m=0}^{\infty} B_m^{(k)} \frac{(2t)^m}{m!} \sum_{m=k}^{\infty} \frac{S(m,k) t^m}{m!}$$
$$= \frac{k!}{t^k} \sum_{n=k}^{\infty} \sum_{j=k}^{n} \binom{n}{j} S(j,k) 2^{n-j} B_{n-j}^{(k)} \frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} \sum_{j=k}^{n+k} \binom{n+k}{j} \binom{n+k}{k}^{-1} S(j,k) 2^{n+k-j} B_{n+k-j}^{(k)} \frac{t^n}{n!},$$

and the comparison of the coefficients of $t^n/n!$ in the first and the last formulas completes the proof. □

Theorem 2.6 *Let $n \geq k \geq 1$ be any integers, then*

$$\frac{G_{n+k}^{(k)}}{\langle n+1 \rangle_k} = \sum_{j=0}^{n} \frac{S(n,j) \langle k \rangle_j (-1)^j}{2^j}.$$

Proof Let $(S(n,j)) = (1, \frac{e^t-1}{t})$, and by (1.10), we have

$$\sum_{j=0}^{n} \frac{S(n,j) \langle k \rangle_j (-1)^j}{2^j} = \left[\frac{t^n}{n!}\right] \left(\frac{1}{(1+\frac{y}{2})^k} \,\Big|\, y = e^t - 1\right) = \left[\frac{t^n}{n!}\right] \left(\frac{t}{e^t+1}\right)^k = \frac{G_{n+k}^{(k)}}{\langle n+1 \rangle_k},$$

which completes the proof. □

3 Identities involving $G_n^{(k)}$, $s(n,k,r)$, $S(n,k,r)$, $B_n^{(k)}$ and $C_n^{(k)}$

In this section, by using the Riordan and generating functions method, we explore relationships between these numbers, the $G_n^{(k)}$, $s(n,k,r)$, $S(n,k,r)$, $B_n^{(k)}$ and $C_n^{(k)}$.

Theorem 3.1 *Let $n \geq k \geq 1$ be any integers, then*

$$\sum_{j=0}^{n} \binom{n}{j} G_{n-j}^{(k)} B_j^{(k)} = 2^{n-k} B_{n-k}^{(k)} (n)_k = 2^{n-k} B_{n-k}^{(k)} \sum_{j=0}^{k} s(k,j) n^j,$$

where $s(n,h)$ are the Stirling numbers of the first kind, $(n)_k = n(n-1)\cdots(n-k+1)$.

Proof By (1.1), (1.5), we have

$$\begin{aligned}
\sum_{j=0}^{n} \binom{n}{j} G_{n-j}^{(k)} B_j^{(k)} &= \left[\frac{t^n}{n!}\right] \left(\frac{2t}{e^t+1}\right)^k \left(\frac{t}{e^t-1}\right)^k = \left[\frac{t^n}{n!}\right] \left(\frac{2t^2}{e^{2t}-1}\right)^k \\
&= \left[\frac{t^n}{n!}\right] \sum_{m=0}^{\infty} B_m^{(k)} \frac{2^m t^{m+k}}{m!} = 2^{n-k} B_{n-k}^{(k)} (n)_k = 2^{n-k} B_{n-k}^{(k)} \sum_{j=0}^{k} s(k,j) n^j,
\end{aligned}$$

which yields Theorem 3.1. □

Theorem 3.2 *Let $n \geq k \geq 1$ be any integers, then*

$$\begin{aligned}
&\sum_{j=0}^{n} \sum_{h=j}^{n} \binom{n}{h} s(h,j) C_{n-j}^{(i)} G_j^{(k)} \\
&= \begin{cases} n! \sum_{j=0}^{n-k} \binom{k+j-1}{j} \frac{G_{n-k-j}^{(i-k)}}{(n-k-j)!} (-\frac{1}{2})^j, & i > k, \\ n! \binom{n-1}{k-1} (-\frac{1}{2})^{n-k}, & i = k, \\ n!(k-i)! \sum_{j=0}^{n-k} \binom{k+j-1}{j} s(n-i-j, k-i) (-\frac{1}{2})^j, & i < k. \end{cases}
\end{aligned}$$

Proof Since

$$\sum_{h=j}^{n} \binom{n}{h} s(h,j) C_{n-j}^{(i)} = \left[\frac{t^n}{n!}\right] \frac{t^i}{\ln^i(1+t)} \frac{\ln^j(1+t)}{j!},$$

then by $(\sum_{h=j}^{n} \binom{n}{h} s(h,j) C_{n-j}^{(i)}) = (\frac{t^i}{\ln^i(1+t)}, \frac{\ln(1+t)}{t})$.

From (1.10), we have

$$\begin{aligned}
\sum_{j=0}^{n} \sum_{h=j}^{n} \binom{n}{h} s(h,j) C_{n-j}^{(i)} G_j^{(k)} &= \left[\frac{t^n}{n!}\right] \frac{t^i}{\ln^i(1+t)} \left[\left(\frac{2y}{e^y+1}\right)^k \Big| y = \ln(1+t)\right] \\
&= \left[\frac{t^n}{n!}\right] \frac{t^i}{\ln^i(1+t)} \frac{\ln^k(1+t)}{(1+\frac{t}{2})^k} \\
&= \left[\frac{t^n}{n!}\right] \begin{cases} \frac{t^i}{(\ln(1+t))^{i-k}(1+\frac{t}{2})^k}, & i > k, \\ \frac{t^k}{(1+\frac{t}{2})^k}, & i = k, \\ \frac{t^i(\ln(1+t))^{k-i}}{(1+\frac{t}{2})^k}, & i < k, \end{cases}
\end{aligned}$$

which completes the proof. □

By the proof of Theorem 3.2, we can get the following corollary.

Corollary 3.3 *Let $i \geq 1$, $j \geq 0$ be any integer, then*

$$\sum_{h=j}^{n} \binom{n}{h} s(h,j) C_{n-j}^{(i)} = \begin{cases} \binom{n}{j} C_{n-j}^{(i-j)}, & i > j, \\ \delta_{n,i}, & i = k, \\ \binom{n}{i} \binom{j}{i}^{-1} s(n-i, j-i)(-1)^{n-j}, & i < k. \end{cases}$$

Theorem 3.4 *Let $n \geq k \geq 1$, be any integer, let $r > 0$ be a real number, then*

$$\sum_{j=0}^{n} s(n,j,r) G_j^{(k)} = \sum_{j=0}^{n-k} \binom{n}{j} \frac{(-1)^j s(n-j,k,r)\langle k\rangle_j k!}{2^j}.$$

Proof By (1.3) and (1.10), we have

$$\begin{aligned} \sum_{j=0}^{n} s(n,j,r) G_j^{(k)} &= \left[\frac{t^n}{n!}\right] \frac{1}{(1+t)^r} \left[\left(\frac{2y}{e^y+1}\right)^k \Bigm| y = \ln(1+t)\right] \\ &= \left[\frac{t^n}{n!}\right] \frac{\ln^k(1+t)}{(1+t)^r} \frac{1}{(1+\frac{t}{2})^k} \\ &= k! \left[\frac{t^n}{n!}\right] \sum_{m=k}^{\infty} s(m,k,r) \frac{t^m}{m!} \sum_{m=0}^{\infty} \binom{k+m-1}{m} \left(-\frac{1}{2}\right)^m t^m \\ &= \sum_{j=0}^{n-k} \binom{n}{j} \frac{(-1)^j s(n-j,k,r)\langle k\rangle_j k!}{2^j}. \end{aligned}$$

□

Theorem 3.5 *Let $n \geq k \geq 1$ be any integers, let $r > 0$ be a real number, then*

$$\sum_{j=0}^{n} \left|s(n,j,r)\right| G_j^{(k)} = \sum_{j=0}^{n-k} \binom{n}{j} \frac{|s(n-j,k,r-1)|\langle k\rangle_j k!}{2^j}.$$

The proof of Theorem 3.5 is similar to that of Theorem 3.4, and is omitted here.

Theorem 3.6 *Let $n \geq k \geq 1$ be any integers, let $r > 0$ be a real number, then*

$$\sum_{j=0}^{n} s(n,j,r) \frac{G_{k+j}^{(k)}}{\langle j+1\rangle_k} = n!(-1)^n \sum_{j=0}^{n} \binom{r+n-j-1}{n-j} \binom{k+j-1}{j} \frac{1}{2^j}.$$

Proof For $(s(n,j,r)) = (\frac{1}{(1+t)^r}, \frac{\ln(1+t)}{t})$.
By (1.2), (1.3) and (1.10), we have

$$\begin{aligned} \sum_{j=0}^{n} s(n,j,r) \frac{G_{k+j}^{(k)}}{\langle j+1\rangle_k} &= \left[\frac{t^n}{n!}\right] \frac{1}{(1+t)^r} \left[\left(\frac{2}{e^y+1}\right)^k \Bigm| y = \ln(1+t)\right] \\ &= \left[\frac{t^n}{n!}\right] \frac{1}{(1+t)^r} \frac{1}{(1+\frac{t}{2})^k} = n!(-1)^n \sum_{j=0}^{n} \binom{r+n-j-1}{n-j} \binom{k+j-1}{j} \frac{1}{2^j}, \end{aligned}$$

which completes the proof. □

Theorem 3.7 *Let $n \geq k \geq 1$ be any integers, let $r \geq 0$ be a real number, then*

$$\sum_{j=0}^{n} \bigl|s(n,j,r)\bigr| \frac{G_{k+j}^{(k)}}{\langle j+1\rangle_k} = \begin{cases} n! \sum_{j=0}^{n} \binom{r-k+n-j-1}{n-j} \binom{k+j-1}{j} 2^{k-j}, & r > k, \\ \frac{\langle k\rangle_n}{2^{n-k}}, & r = k, \\ n! \sum_{j=0}^{k-r} \binom{k-r}{j} \binom{k-1+n-j}{n-j} \frac{1}{2^{n-k-j}}, & r < k. \end{cases}$$

Proof Since $(|s(n,j,r)|) = (\frac{1}{(1-t)^r}, \frac{-\ln(1-t)}{t})$, then by (1.2) and (1.10), we have

$$\begin{aligned} \sum_{j=0}^{n} \bigl|s(n,j,r)\bigr| \frac{G_{k+j}^{(k)}}{\langle j+1\rangle_k} &= \left[\frac{t^n}{n!}\right] \frac{1}{(1-t)^r} \left[\left(\frac{2}{e^y+1}\right)^k \Bigm| y = -\ln(1-t) \right] \\ &= n!\bigl[t^n\bigr] \frac{1}{(1-t)^{r-k}} \frac{1}{(1-\frac{t}{2})^k} \\ &= \begin{cases} n! \sum_{j=0}^{n} \binom{r-k+n-j-1}{n-j} \binom{k+j-1}{j} 2^{k-j}, & r > k, \\ \frac{\langle k\rangle_n}{2^{n-k}}, & r = k, \\ n! \sum_{j=0}^{k-r} \binom{k-r}{j} \binom{k-1+n-j}{n-j} \frac{1}{2^{n-k-j}}, & r < k, \end{cases} \end{aligned}$$

which completes the proof. □

Theorem 3.8 *Let $n \geq k \geq 1$ be any integers, and let $r > 0$ be a real number, then*

$$\sum_{j=0}^{n} S(n,j,r) \frac{(-1)^j \langle k\rangle_j}{2^j} = \sum_{j=0}^{n} \binom{n}{j} \frac{G_{k+j}^{(k)} r^{n-j}}{\langle j+1\rangle_k}.$$

Proof Since $(S(n,j,r)) = (e^{rt}, \frac{e^t-1}{t})$, then by (1.3), (1.8) and (1.10), we have

$$\begin{aligned} \sum_{j=0}^{n} S(n,j,r) \frac{(-1)^j \langle k\rangle_j}{2^j} &= \left[\frac{t^n}{n!}\right] e^{rt} \left[\left(\frac{1}{1+\frac{y}{2}}\right)^k \Bigm| y = e^t - 1 \right] \\ &= n!\bigl[t^n\bigr] e^{rt} \left(\frac{2}{e^t+1}\right)^k = \sum_{j=0}^{n} \binom{n}{j} \frac{G_{k+j}^{(k)} r^{n-j}}{\langle j+1\rangle_k}. \end{aligned}$$

□

Theorem 3.9 *Let $n \geq 1$ be any integer, and let $r > 0$ be a real number, then*

$$\sum_{j=0}^{n} s(n,j,r) \frac{2^j G_{k+j}^{(k)}}{\langle j+1\rangle_k} = 2^k n! \sum_{j=0}^{\infty} \sum_{i=0}^{n} \binom{r+n-i-1}{n-i} \binom{k+i+j-1}{i+j} \binom{2(i+j)}{i} (-1)^{n-i}.$$

Proof Since $(s(n,j,r)) = (\frac{1}{(1+t)^r}, \frac{\ln(1+t)}{t})$, then by (1.2) and (1.10), we have

$$\begin{aligned} &\sum_{j=0}^{n} s(n,j,r) \frac{2^j G_{k+j}^{(k)}}{\langle j+1\rangle_k} \\ &\quad = \left[\frac{t^n}{n!}\right] \frac{1}{(1+t)^r} \left[\left(\frac{2}{e^{2y}+1}\right)^k \Bigm| y = \ln(1+t) \right] = \left[\frac{t^n}{n!}\right] \frac{1}{(1+t)^r} \frac{2^k}{(1+(1+t)^2)^k} \end{aligned}$$

$$= 2^k n! [t^n] \sum_{i=0}^{\infty} \binom{r+i-1}{i} (-1)^i t^i \sum_{i=0}^{\infty} \left(\sum_{j=0} \binom{k+i+j-1}{i+j} \binom{2(i+j)}{i} \right) x^i$$

$$= 2^k n! \sum_{j=0}^{\infty} \sum_{i=0}^{n} \binom{r+n-i-1}{n-i} \binom{k+i+j-1}{i+j} \binom{2(i+j)}{i} (-1)^{n-i},$$

which completes the proof. □

4 Asymptotics

In this section, we give the asymptotic expansion of certain sums involving $G_n^{(k)}$.

Theorem 4.1 *Let $r > 0$ be a real number, and suppose that $n \to \infty$, then*

$$\sum_{j=0}^{n} s(n,j,r) \frac{G_{k+j}^{(k)}}{\langle j+1 \rangle_k} \sim \frac{(-1)^n 2^k n^{n+r}}{\Gamma(r) e^n} \sqrt{\frac{2\pi}{n}}.$$

Proof From the proof of Theorem 3.6, we know

$$\sum_{j=0}^{n} s(n,j,r) \frac{G_{k+j}^{(k)}}{\langle j+1 \rangle_k} = \left[\frac{t^n}{n!} \right] \frac{1}{(1+t)^r} \frac{1}{(1+\frac{t}{2})^k}.$$

In Lemma 1.1, let $f(t) = \frac{1}{(1+t)^r}$, $g(t) = \frac{1}{(1+\frac{t}{2})^k}$. Obviously, $f(t)$ is analytic for $|t| < 1$, and it has only algebraic singularity on $|t| = 1$, namely at $t = -1$, so by Lemma 1.1 and Stirling's formula $n! \sim e^{-n} n^n \sqrt{2\pi n}$, we have

$$\sum_{j=0}^{n} s(n,j,r) \frac{G_{k+j}^{(k)}}{\langle j+1 \rangle_k} \sim \frac{(-1)^n 2^k n^{n+r}}{\Gamma(r) e^n} \sqrt{\frac{2\pi}{n}},$$

which completes the proof. □

Theorem 4.2 *Let $k \geq 1$ be any integer, let $r > 0$ be a real number, and suppose that $n \to \infty$, then*

$$\sum_{j=0}^{n} \left| s(n,j,r) \right| \frac{G_{k+j}^{(k)}}{\langle j+1 \rangle_k} \sim \begin{cases} \frac{2^k n^{n+r+k}}{\Gamma(r-k) e^n} \sqrt{\frac{2\pi}{n}}, & r > k, \\ \frac{n^{n+k}}{\Gamma(k) e^n} \sqrt{\frac{2\pi}{n}}, & r = k, \\ \frac{n^{n+k} (-1)^{k-r}}{\Gamma(k) 2^n} \sqrt{\frac{2\pi}{n}}, & r < k. \end{cases}$$

Proof By Lemma 1.1 and Stirling's formula, we have

$$\sum_{j=0}^{n} \left| s(n,j,r) \right| \frac{G_{k+j}^{(k)}}{\langle j+1 \rangle_k} = n! 2^k [t^n] \begin{cases} \frac{1}{(1-t)^{r-k}} \frac{1}{(1-\frac{t}{2})^k}, & r > k, \\ \frac{1}{(1-\frac{t}{2})^k}, & r = k, \\ \frac{(1-t)^{k-r}}{(1-\frac{t}{2})^k}, & r < k \end{cases}$$

$$\sim \begin{cases} \frac{2^k n^{n+r+k}}{\Gamma(r-k) e^n} \sqrt{\frac{2\pi}{n}}, & r > k, \\ \frac{n^{n+k}}{\Gamma(k) e^n} \sqrt{\frac{2\pi}{n}}, & r = k, \\ \frac{n^{n+k} (-1)^{k-r}}{\Gamma(k) 2^n} \sqrt{\frac{2\pi}{n}}, & r < k, \end{cases}$$

which completes the proof. □

Theorem 4.3 *Let $r > 0$ be a real number, as $n \to \infty$, and suppose that*

$$\sum_{j=0}^{n} s(n,j,r)\frac{2^j G_{k+j}^{(k)}}{\langle j+1\rangle_k} \sim \frac{(-1)^n 2^k n^{n+r}}{\Gamma(r)e^n}\sqrt{\frac{2\pi}{n}}.$$

Proof By Lemma 1.1 and Stirling's formula, we have

$$\sum_{j=0}^{n} s(n,j,r)\frac{2^j G_{k+j}^{(k)}}{\langle j+1\rangle_k} = n!\left[t^n\right]\frac{1}{(1+t)^r}\frac{2^k}{(2+2t+t^2)^k} \sim \frac{(-1)^n 2^k n^{n+r}}{\Gamma(r)e^n}\sqrt{\frac{2\pi}{n}},$$

which completes the proof. □

Competing interests
The author declares that they have no competing interests.

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